MATH 403/503 L2

Def A congruence relation on a monoid M is an equivalence relation $\equiv$ such that $\forall x, y, z \in M$:

$$x \equiv y \Rightarrow (xz \equiv yz \text{ and } zx \equiv zy)$$

Ex Let $n \in \mathbb{Z}_{>0}$. On the monoid $\mathbb{Z} = (\mathbb{Z}, +, 0)$ we have the relation $\equiv_n$ given by

$$x \equiv_n y \overset{\text{def}}{\Longleftrightarrow} n \mid (x - y).$$

$\equiv_n$ is a congruence relation since

$$x \equiv_n y \Rightarrow x + z \equiv_n y + z$$

Ex Let $\varphi : M \to N$ be a monoid homomorphism. Define a relation $\equiv_\varphi$ on M by:

$$x \equiv_\varphi y \overset{\text{def}}{\Longleftrightarrow} \varphi(x) = \varphi(y)$$

Then $x \equiv_\varphi y \Rightarrow \varphi(xz) = \varphi(x)\varphi(z) = \varphi(y)\varphi(z)$
$= \varphi(yz) \Rightarrow xz \equiv_\varphi yz$

Similarly $x \equiv_\varphi y \Rightarrow zx \equiv_\varphi zy \quad \forall x, y, z \in M.$

Thus $\equiv_\varphi$ is a congruence relation on M.

In fact, every congruence relation on a monoid M arises this way. To see this, we must define quotient monoids.

Def Let $\equiv$ be a congruence relation on a monoid M. The quotient monoid $M/\equiv$ is the set of equivalence classes in M w.r.t. $\equiv$, equipped with the operation

$$[x]\cdot[y] = [x\cdot y] \qquad (*)$$

and identity element $[e]$.

Proposition If $[x]=[\tilde{x}]$ and $[y]=[\tilde{y}]$ then $[x\cdot y]=[\tilde{x}\cdot\tilde{y}]$, so the operation $(*)$ is well-defined.

Proof $[x]=[\tilde{x}] \iff x\equiv\tilde{x}$
$[y]=[\tilde{y}] \iff y\equiv\tilde{y}$

$x\equiv\tilde{x} \Rightarrow xy=\tilde{x}y$
$y\equiv\tilde{y} \Rightarrow \tilde{x}y\equiv\tilde{x}\tilde{y}$ $\quad\Rightarrow xy\equiv\tilde{x}\tilde{y} \Rightarrow [x\cdot y]=[\tilde{x}\cdot\tilde{y}]$ $\blacksquare$

One can check $M/_{\equiv}$ is indeed a monoid.

<u>Proposition</u> If $\equiv$ is a congruence relation on a monoid M, there is a canonical monoid homomorphism

$$\pi : M \longrightarrow M/_{\equiv}$$

given by $\pi(x) = [x] \quad \forall x \in M$.

<u>Proof</u> $\pi(x \cdot y) = [x \cdot y] = [x] \cdot [y] = \pi(x)\pi(y)$

$\pi(e) = [e] = e_{M/\equiv}$. ∎

<u>Remark</u> The congruence $\equiv$ coincides with $\equiv_\pi$. Indeed:

$$x \equiv y \iff [x] = [y] \iff \pi(x) = \pi(y).$$

Congruences on Groups

Ex Let $\equiv$ be a congruence relation on a group G. Then $\forall x,y \in G$:

$$x \equiv y \iff xy^{-1} \equiv e$$
$$\iff e \equiv x^{-1}y$$

Define

$$N_{\equiv} = \{x \in G : x \equiv e\}$$

Then $N_{\equiv}$ is a subgroup of G:

- If $x, y \in N_{\equiv}$ then $x \equiv e$, $y \equiv e$ hence $x \equiv y$ (by symm., trans. of $\equiv$)

  $\Rightarrow xy^{-1} \equiv e \Rightarrow xy^{-1} \in N_{\equiv}$.
- $e \equiv e \Rightarrow e \in N_{\equiv}$

Furthermore, for any $g \in G$, $x \in N_{\equiv}$ we have:

$x \equiv e$ since $x \in N_{\equiv}$

$gx \equiv ge = g$ since $\equiv$ congruence

$gxg^{-1} \equiv gg^{-1} = e$ —— " ——

$\Rightarrow gxg^{-1} \in N_{\equiv}$.

Def A subgroup N of a group G is *normal* if $gxg^{-1} \in N \ \forall g \in G \ \forall x \in N$.

Notation $N \leq G$ N is a subgroup of G

$N \trianglelefteq G$ N is a normal subgrp of G

Exercise Let $N \trianglelefteq G$, G a group. Define $x \equiv y \iff xy^{-1} \in N$. Prove that $\equiv$ is a congruence relation on G.

Proposition Let G be a group and $N \leq G$. TFAE:

1) $N \trianglelefteq G$
2) $\forall g \in G$: $gN = Ng$
3) $\forall g \in G$: $gNg^{-1} = N$

Proof. 1) $\Rightarrow$ 2): Let $x \in gN$. Thus $x = gn$ for some $n \in N$. We have $x = (gng^{-1})g$. Since $N \trianglelefteq G$, $gng^{-1} \in N$. Thus $x \in Ng$. Therefore $gN \subseteq Ng$. Similarly, $Ng \subseteq gN$.

2) $\Rightarrow$ 3) Mult. by g^{-1} from the right.

3) $\Rightarrow$ 1) Trivial. ∎

<u>Example</u> Dihedral group:

$G = D_4 = \{ \underset{\overset{\shortparallel}{e}}{r^0}, r, r^2, r^3, \underset{\overset{\shortparallel}{s}}{sr^0}, sr, sr^2, sr^3 \}$

Relations:

$$\begin{cases} s^2 = e \\ rs = sr^{-1} \\ r^4 = e \end{cases} \quad (r^n = e \text{ in } D_n)$$

Let

$H = \{ e, r^2, s, sr^2 \}$

One checks $H \leq G$. Also:

$$rH = \{ r, r^3, \underset{\substack{\shortparallel \\ sr^{-1} = sr^3}}{rs}, \underset{\substack{\shortparallel \\ sr^{-1}r^2 = sr}}{rsr^2} \} = \{ r, r^3, sr^3, sr \}$$

$Hr = \{ r, r^3, sr, sr^3 \} = rH$

Similarly $gH = Hg \quad \forall g \in D_4$

So $H \trianglelefteq D_4$.