MATH 403/503 L12

<u>Def</u> G group, p prime

a) A group of order p^α, some $\alpha \geq 0$ is a p-group. Subgroups of G which are p-groups are <u>p-subgroups</u>

b) If $|G| = p^\alpha m$, $p \nmid m$, then a subgroup of order p^α is a Sylow p-subgroup of G.

c) The set of Sylow p-subgroups of G is denoted $Syl_p(G)$. The number of Sylow p-subgrp. of G is denoted $n_p(G)$, or just n_p when G is clear.

Th (Sylow's Th)

Let G be a group of order $p^\alpha m$, p prime $p \nmid m$

a) $Syl_p(G) \neq \emptyset$

b) If $P \in Syl_p(G)$ and Q is any ~~sub~~ p-subgroup of G, then $Q \leq gPg^{-1}$, for some $g \in G$

c) $n_p(G) \equiv 1 \pmod p$ and $n_p(G) | m$.
(In fact $n_p(G) = [G : N_G(P)]$.)

<u>Theorem</u> (Cauchy's Thm; Abelian case)
G finite abelian group
Suppose p is a prime
s.t. $p \mid |G|$. Then G has
an element of order p.

<u>Proof</u> Induction on $|G|$. If
$|G| = 1$ the conclusion is vacuously
true since no p divides $|G|$.
Suppose $|G| = m$ and the
result is true for all abelian
groups of order less than m.
Let $x \neq e$ in G, put $H = \langle x \rangle$.
$|G| = |G/H| \cdot |H|$ by Lagrange's Th.

If p divides $|H|$ then some power of x has order p.

Otherwise, p divides $|G/H|$.
Then, by the induction hypothesis, G/H contains an element, gH, say, of order p. That is,

$$(gH)^p = eH \Rightarrow g^p \in H = \langle x \rangle$$

so $g^p = x^a$, some $a \geq 0$.
Let $l = |H| =$ order of x.
Since $p \nmid l$, $g^l \neq e$.
But $(g^l)^p = (x^l)^a = e$
So g^l has order p. ∎

The following technical Lemma will be used when proving Sylow's Theorem:

<u>Lem</u> Let $P \in Syl_p(G)$. If Q is any p-subgroup of G, then $Q \cap N_G(P) = Q \cap P$.

<u>Proof</u> Let $H = N_G(P) \cap Q$. Since $P \leq N_G(P)$, H contains $Q \cap P$.

Conversely, WTS $H \leq Q \cap P$.

Since $H \leq Q$ by definition, it remains to prove $H \leq P$. We do this by showing PH is a p-subgroup of G containing both P and H; but $P \leq PH$ and P is a p-subgroup of G of maximal order. So $P = PH$, hence $H \leq P$ ■

To show PH is a p-subgroup of G, first note that since $H \leq N_G(P)$, we have $PH \leq G$ (proof as in proof of 2^{nd} iso. th.)

So $PH/H \cong P/H \cap P \Rightarrow |PH| = \dfrac{|P| \cdot |H|}{|H \cap P|}$.

Since $|P|, |H|$ are powers of p, PH is a p-group. $\blacksquare$

Example of Application of Sylow's Thm:
Show that any group G of order 42 has a normal subgroup $\neq \{1\}, G$.

Sol. $|G| = 42 = 2 \cdot 3 \cdot 7$. By Sylow's Thm:

$n_7 \equiv 1 \pmod 7$

and $n_7 \mid 2 \cdot 3 = 6$

So $n_7 \in \{1, \cancel{2}, \cancel{3}, \cancel{6}\}$ only 1 is $\equiv 1 \pmod 7$

So G has a unique subgroup P of order 7 (a Sylow 7-subgroup). Since $|gPg^{-1}| = \underbrace{|P|}_{=7}$ $\forall g \in G$ we have $P \trianglelefteq G$. $\blacksquare$