MATH 403/503 L10

Orbit - Stabilizer Theorem & Class Equation.

<u>Example</u>
Let G be a group and $H \leq G$. Then G acts on G/H by

$$g.(g_1H) = (gg_1)H$$

<u>Def</u> Let G be a group acting on a set X and let $x \in X$. The <u>stabilizer</u> of x is

$$G_x = \{g \in G \mid g.x = x\}$$

<u>Example</u> Let G act on itself by conjugation. Then for $x \in G$:

$$\begin{aligned} G_x &= \{g \in G \mid g.x = x\} \\ &= \{g \in G \mid gxg^{-1} = x\} \\ &= \{g \in G \mid gx = xg\} \end{aligned}$$

This set is called the <u>centralizer of x in G</u>.

<u>Example</u> Let G act on the set $\mathrm{Sub}(G)$ of all subgroups of G by conjugation. Then

$$\begin{aligned} G_H &= \{ g \in G \mid g . H = H \} = \\ H \in \mathrm{Sub}(G) \quad &= \{ g \in G \mid gHg^{-1} = H \} \\ &=: N_G(H) \end{aligned}$$

the <u>normalizer</u> of H in G

$$\begin{tikzcd} G \arrow[d, dash] \\ N_G(H) \arrow[d, dash] \\ H \end{tikzcd}$$

<u>Theorem</u> (Orbit-Stabilizer Theorem)
Let G be a group acting on a set X and let $x \in X$.

Then
$$|O_x| = [G : G_x]$$
i.e. the size of the orbit of x equals the index of the stabilizer of x.

<u>Proof</u> We aim to define a function

$$f : G/G_x \longrightarrow \mathcal{O}_x$$

by $f(gG_x) = g.x$.

First we show this is <u>well-defined</u>.
Suppose $gG_x = hG_x$. Then $g^{-1}h \in G_x$
So $(g^{-1}h).x = x$. Therefore

$$g.x = g.((g^{-1}h).x) = (gg^{-1}h).x = h.x$$

so $f(gG_x) = f(hG_x)$.

Next, f is <u>injective</u>: Suppose $f(gG_x) = f(hG_x)$
That is, $g.x = h.x$. Then

$$x = e.x = (g^{-1}g).x = g^{-1}.(g.x) = g^{-1}.(h.x) = (g^{-1}h).x$$

hence $g^{-1}h \in G_x$. Therefore $gG_x = hG_x$.

Lastly, f is <u>surjective</u>: Let $y \in \mathcal{O}_x$.
By def. of $\mathcal{O}_x$, $y = g.x$ for some $g \in G$.
So $y = f(gG_x)$.

We have shown there is a bijection $f : G/G_x \to \mathcal{O}_x$. Therefore $|G/G_x| = |\mathcal{O}_x|$ ∎

We can apply the Orbit-Stabilizer to many different situations. An important application is to the case when G acts on itself by conjugation:

<u>Theorem</u> (Class Equation)
Let G be a finite group. Then

$$|G| = |Z(G)| + \sum_{i=1}^{k} [G : C_G(g_i)]$$

where $\{g_i\}_{i=1}^{k}$ is a complete set of representatives for the non-trivial conjugacy classes in G.

<u>Proof</u> When G acts on itself by conjugation, the orbits are the conjugacy classes. A conjugacy class $Cl(g)$ is trivial (= a singleton) iff g belongs to the center. Since the orbits form a partition we get

$$|G| = |Z(G)| + \sum_{i=1}^{k} |Cl(g_i)|$$

where $\{g_i\}_{i=1}^{k}$ is a complete set of rep's for the nontrivial conjugacy classes. By the Orbit-Stabilizer Theorem,

$$|Cl(g_i)| = [G : C_G(g_i)]$$

∎

Def A <u>p-group</u> (p = a prime) is a group G with $|G| = p^a$, for some positive integer a.

Theorem Any p-group has nontrivial center.

Proof By the class equation

$$|G| = |Z(G)| + \sum_{i=1}^{k} [G : C_G(g_i)].$$

$|G| = p^a$, some $a > 0$. Also, by Lagrange's Theorem, $|C_G(g_i)|$ divides $|G|$. So $[G : C_G(g_i)] = \frac{|G|}{|C_G(g_i)|} = p^{b_i}$ for some $b_i > 0$ (since $[G : C_G(g_i)] = |Cl(g_i)| > 1$)

So we get

$$\underbrace{p^a}_{\text{divisible by } p} = |Z(G)| + \underbrace{\sum_{i=1}^{k} p^{b_i}}_{\text{divisible by } p}$$

$\Rightarrow p$ divides $|Z(G)|$. So $|Z(G)| > 1$. ∎