

MATH 403/503 L10

Orbit - Stabilizer Theorem
& Class Equation.

Example

Let G be a group and $H \leq G$.

Then G acts on G/H by

$$g \cdot (g_1 H) = (gg_1)H$$

Def Let G be a group acting on a set X and let $x \in X$. The stabilizer of x is

$$G_x = \{g \in G \mid g \cdot x = x\}$$

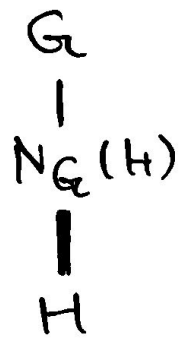
Example Let G act on itself by conjugation. Then for $x \in G$:

$$\begin{aligned} G_x &= \{g \in G \mid g \cdot x = x\} \\ &= \{g \in G \mid gxg^{-1} = x\} \\ &= \{g \in G \mid gx = xg\} \end{aligned}$$

This set is called the centralizer of x in G .

Example Let G act on the set $\text{Sub}(G)$ of all subgroups of G by conjugation. Then

$$\begin{aligned}
G_H &= \{g \in G \mid g \cdot H = H\} = \\
H \in \text{Sub}(G) &= \{g \in G \mid gHg^{-1} = H\} \\
&=: N_G(H) \text{ the } \underline{\text{normalizer}} \\
&\quad \text{of } H \text{ in } G
\end{aligned}$$



Theorem (Orbit-Stabilizer Theorem)

Let G be a group acting on a set X and let $x \in X$.

Then $|O_x| = [G : G_x]$

i.e. the size of the orbit of x equals the index of the stabilizer of x .

Proof We aim to define a function

$$f: G/G_x \rightarrow \mathcal{O}_x$$

$$\text{by } f(gG_x) = g \cdot x.$$

First we show this is well-defined. Suppose $gG_x = hG_x$. Then $g^{-1}h \in G_x$

so $(g^{-1}h) \cdot x = x$. Therefore

$$g \cdot x = g \cdot ((g^{-1}h) \cdot x) = (gg^{-1}h) \cdot x = h \cdot x$$

$$\text{so } f(gG_x) = f(hG_x).$$

Next, f is injective: Suppose $f(gG_x) = f(hG_x)$

That is, $g \cdot x = h \cdot x$. Then

$$x = e \cdot x = (g^{-1}g) \cdot x = g^{-1} \cdot (g \cdot x) = g^{-1} \cdot (h \cdot x) = (g^{-1}h) \cdot x$$

hence $g^{-1}h \in G_x$. Therefore $gG_x = hG_x$.

Lastly, f is surjective: Let $y \in \mathcal{O}_x$.

By def. of \mathcal{O}_x , $y = g \cdot x$ for some $g \in G$.

$$\text{so } y = f(gG_x).$$

We have shown there is a bijection

$$f: G/G_x \rightarrow \mathcal{O}_x. \text{ Therefore } |G/G_x| = |\mathcal{O}_x|$$



We can apply the Orbit-Stabilizer to many different situations. An important application is to the case when G acts on itself by conjugation!

Theorem (Class Equation)

Let G be a finite group. Then

$$|G| = |Z(G)| + \sum_{i=1}^k [G : C_G(g_i)]$$

where $\{g_i\}_{i=1}^k$ is a complete set of representatives for the non-trivial conjugacy classes in G .

Proof When G acts on itself by conjugation, the orbits are the conjugacy classes. A conjugacy class $\mathcal{C}(g)$ is trivial (=a singleton) iff g belongs to the center. Since the orbits form a partition we get

$$|G| = |Z(G)| + \sum_{i=1}^k |\mathcal{C}(g_i)|$$

where $\{g_i\}_{i=1}^k$ is a complete set of rep's for the non-trivial conjugacy classes.

By the Orbit-Stabilizer Theorem,

$$|\mathcal{C}(g_i)| = [G : C_G(g_i)]$$

Def A p-group ($p = \text{a prime}$) is
a group G with $|G| = p^a$, for
some positive integer a , 5

Theorem Any p -group has nontrivial
center.

Proof By the class equation

$$|G| = |Z(G)| + \sum_{i=1}^k [G : C_G(g_i)].$$

$|G| = p^a$, some $a > 0$. Also, by
Lagrange's Theorem, $|C_G(g_i)|$ divides
 $|G|$. So $[G : C_G(g_i)] = \frac{|G|}{|C_G(g_i)|} = p^{b_i}$

for some $b_i > 0$ (since $[G : C_G(g_i)] =$
 $= |C_G(g_i)| > 1$)

So we get

$$\underbrace{p^a}_{\text{divisible by } p} = |Z(G)| + \underbrace{\sum_{i=1}^k p^{b_i}}_{\text{divisible by } p}$$

$\Rightarrow p$ divides $|Z(G)|$. So $|Z(G)| > 1$, ■