Math 403/503 Spring 2024

Homework 8, due April 10

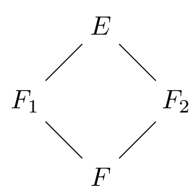
1. Show that each of the following numbers is algebraic over $\mathbb{Q}$.

   (a) $\alpha = \sqrt[3]{\sqrt{1/3} - 5}$

   (b) $\beta = \sqrt{2} - \sqrt[3]{5}$

2. Let E/F be a field extension. Prove that if $\alpha \in E$ is transcendental over F, then α^k is transcendental over F for any positive integer k.

3. Let E/F be a field extension and let F_1 and F_2 be subfields of E containing F. Let $d_i = [E : F_i]$ for $i = 1, 2$. If d_1 and d_2 are relatively prime, show that $[E : F]$ is at least $d_1 d_2$.



4. Find the minimal polynomial of the number over $\mathbb{Q}$:

   (a) $\gamma = 1 + \sqrt[3]{2}$

   (b) $z = \cos\theta + i \sin\theta$ for $\theta = 2\pi/p$ with p prime. (*Hint:* Consider z^p; use Euler's formula.)

5. Find a basis for the field extension. What is the degree of the extension?

   (a) $\mathbb{Q}(\sqrt{8})$ over $\mathbb{Q}(\sqrt{2})$

   (b) $\mathbb{Q}(\sqrt{3}, \sqrt{5}, \sqrt{7})$ over $\mathbb{Q}$

6. Consider the field extension $\mathbb{Q}(\sqrt[4]{3}, i)$ over $\mathbb{Q}$.

   (a) Find a basis for the field extension $\mathbb{Q}(\sqrt[4]{3}, i)$ over $\mathbb{Q}$. Conclude that $[\mathbb{Q}(\sqrt[4]{3}, i) : \mathbb{Q}] = 8$.

   (b) Find all subfields F of $\mathbb{Q}(\sqrt[4]{3}, i)$ such that $[F : \mathbb{Q}] = 2$.

   (c) Find all subfields F of $\mathbb{Q}(\sqrt[4]{3}, i)$ such that $[F : \mathbb{Q}] = 4$.