Math 403/503 Spring 2024

In this class, by

- a *ring* R we mean a *ring with identity* 1_R;
- a *subring* $S \subseteq R$ we mean *subring containing the identity* 1_R;
- a *ring homomorphism* $\varphi : R \to T$ we mean *ring homomorphism sending* 1_R *to* 1_T.

Homework 7, due March 27

1. The ring of *Gaussian integers*, $\mathbb{Z}[i] = \{a + bi \in \mathbb{C} \mid a, b \in \mathbb{Z}\}$, is a UFD. Factor each of the following elements in $\mathbb{Z}[i]$ into a product of irreducibles.
   (a) 5
   (b) $6 + 8i$
   (c) $1 + 3i$
2. Prove that the field of fractions of the Gaussian integers, $\mathbb{Z}[i]$, is isomorphic to
$$\mathbb{Q}(i) = \{p + qi : p, q \in \mathbb{Q}\}.$$
3. Let D be an integral domain. Define a relation on D by $a \sim b$ iff a and b are associates in D. Prove that $\sim$ is an equivalence relation on D.
4. An ideal I of a commutative ring R is said to be *finitely generated* if there exist elements $a_1, a_2, \ldots, a_n$ in R such that every element r in I can be written $r = a_1r_1 + \cdots + a_nr_n$ for some $r_1, \ldots, r_n$ in R. Prove that R satisfies the ascending chain condition if and only if every ideal of R is finitely generated.
5. Show that $\mathbb{Z}[\sqrt{-5}]$ is not a unique factorization domain.
6. Prove that $\mathbb{Z}[x]$ cannot be a Euclidean domain.
7. Let D be a Euclidean domain with Euclidean valuation ν. If a and b are associates in D, prove that $\nu(a) = \nu(b)$.