Math 403/503 Spring 2024

In this class, by

- a *ring* R we mean a *ring with identity* 1_R;
- a *subring* $S \subseteq R$ we mean *subring containing the identity* 1_R;
- a *ring homomorphism* $\varphi : R \to T$ we mean *ring homomorphism sending* 1_R *to* 1_T.

Homework 6, due March 20

1. Let R be a commutative ring. The *radical* of an ideal $I \subseteq R$, denoted $\sqrt{I}$, is defined by

$$\sqrt{I} = \{a \in R \mid a^n \in I \text{ for some integer } n > 0\}.$$

   Show that $\sqrt{I}$ is an ideal of R.

2. Let R be a commutative ring. The *nilradical* of R is defined as $\mathcal{N}(R) = \sqrt{(0)}$. In other words, $\mathcal{N}(R)$ is the set of all nilpotent elements of R:

$$\mathcal{N}(R) = \{a \in R \mid a^n = 0 \text{ for some integer } n > 0\}.$$

   Show that $\mathcal{N}(R)$ is equal to the intersection of all prime ideals of R. (*Hint:* If $a \in R$ is not nilpotent, the set of all ideals not intersecting $\{a^n \mid n \geq 0\}$ has a maximal element; show it is a prime ideal.)

3. Let R be a commutative ring. Show that R is a field if and only if R has exactly two ideals ($\{0\}$ and R itself). (This shows that fields are precisely the commutative simple rings.)

4. Prove the Third Isomorphism Theorem for rings: Let R be a ring and I and J be ideals of R, where $J \subseteq I$. Then

$$R/I \cong \frac{R/J}{I/J}.$$

5. Show that if R is an integral domain, then $R[x]$ is an integral domain. Conclude that if F is a field, then the ring $F[x_1, x_2, \ldots, x_n]$ of polynomials in n variables is an integral domain.

6. Show that $x^p - x \in \mathbb{Z}_p[x]$ has p distinct zeros in Z_p, for any prime p. Conclude that

$$x^p - x = x(x-1)(x-2)\cdots(x-(p-1)).$$

7. Which of the following polynomials in $\mathbb{Q}[x]$ are irreducible?

(a) $x^4 - 2x^3 + 2x^2 + x + 4$

(b) $3x^5 - 4x^3 - 6x^2 + 6$

(c) $x^4 - 5x^3 + 3x - 2$

(d) $5x^5 - 6x^4 - 3x^2 + 9x - 15$