Math 403/503 Spring 2024

Homework 3, due February 7

Updated on Feb 4, 4:30pm. The assumption that N is normal was missing from Problem 1.
Updated on Feb 7, 10:30am. In Problem 5, in the definition of $N_G(H)$, it should say $gHg^{-1} = H$.

1. The *commutator subgroup* of a group G, denoted $[G, G]$ is the intersection of all normal subgroups of G which contain $xyx^{-1}y^{-1}$ for all $x, y \in G$. Let G be a group and N be a **normal** subgroup. Show that G/N is abelian if and only if $[G, G] \subseteq N$.

2. What is the size of the conjugacy class $\mathrm{Cl}_{S_8}(\sigma)$ if $\sigma = (123)(456)(78)$?

3. Let G be a group. Let $\mathrm{Aut}(G)$ be the set of all automorphisms $\psi : G \to G$.
   (a) Show that $\mathrm{Aut}(G)$ is a subgroup of the symmetric group S_G.
   (b) For each $g \in G$, let $\phi_g : G \to G$ be the conjugation map $\phi_g(x) = gxg^{-1}$. Show that $\phi_g \in \mathrm{Aut}(G)$ for each $g \in G$.
   (c) Show that the function $\phi : G \to \mathrm{Aut}(G)$ given by $\phi(g) = \phi_g$ is a group homomorphism with kernel equal to the *center* of G, defined as $Z(G) = \{z \in G \mid zg = gz \, \forall g \in G\}$.

4. Let G be a finite group. Suppose $H \leq G$, $N \trianglelefteq G$ and suppose $|H|$ and $|N|$ are coprime (i.e. have no common prime factors). Prove that $HN/N \cong H$. (Hint: Use Lagrange's Theorem and 2nd Isomorphism Theorem.)

5. Let G be a group and $H \leq G$. The *normalizer of H in G* is defined as $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$.
   (a) Show that $N_G(H) \leq G$ and $H \trianglelefteq N_G(H)$.
   (b) Show that if B is any subgroup of G such that $H \trianglelefteq B$, then $B \leq N_G(H)$. (In other words, the normalizer is the largest subgroup of G containing H as a normal subgroup. Hence the name.)