Math 403/503 Spring 2024

Practice Problems for Final Exam

1. If N is a normal subgroup of a group G, and H is a subgroup of G containing N, show that N is normal subgroup of H. Is H/N a normal subgroup of G/N?

2. Show that if A is any abelian group of order 24, and $H \leq A$ is a subgroup of order 4 then A/H is cyclic.

3. Let $H \leq G$ and let G act on the set G/H of left cosets by left multiplication (also called translation). Show that all stabilizer subgroups are conjugate to H.

4. Prove or disprove: the alternating group A_4 is isomorphic to the dihedral group of order 12.

5. Prove that if $N \trianglelefteq G$ then $\frac{G\times G}{\{1\}\times N} \cong G \times (G/N)$.

6. List the conjugacy classes of S_4 with odd number of elements.

7. Show that the center of S_n is trivial, when $n > 2$.

8. How many abelian groups are there of order 48, up to isomorphism? List one group from each isomorphism class.

9. How many ways can one paint the sides of a square, using a palette of red, green, blue? (Two ways to paint are considered the same if one can rotate or reflect one square to look like the other.)

10. Show that if G is a non-abelian group of order $2p$, where p is an odd prime, then G is isomorphic to the dihedral group of order $2p$. (Hint: Use Cauchy's and Sylow's Theorem.)

11. (a) State the Third Isomorphism Theorem for groups.
    (b) State the Orbit-Stabilizer Theorem.
    (c) State the Class Equation.
    (d) State Cauchy's Theorem.
    (e) State Sylow's Theorem (three parts).

12. Find all zero-divisors in $\mathbb{Z}_4[x]/(x^2)$.

13. For each of the following rings, determine whether it is a PID, UFD, integral domain, or neither:
    (a) $\mathbb{R}[x]/(x^3+1)$
    (b) $\mathbb{Z}[x]/(x^2+1)$
    (c) $\mathbb{Q}\times\mathbb{Z}$
    (d) $M_2(\mathbb{Z})$
    (e) $\mathbb{Z}[x_1, x_2, x_3]$
    (f) $\mathbb{Q}[x,y]/(x+y)$

14. (a) State the First Isomorphism Theorem for rings.
    (b) State Eisenstein's criterion.
    (c) State the relationship between prime ideals and integral domains; maximal ideals and fields.
    (d) State the implications among fields, Euclidean domains, PIDs, UFDs, integral domains.
    (e) State the universal property of the field of fractions of an integral domain. Use it to construct an injective ring homomorphism $\phi : \mathbb{Q}(x) \to \mathbb{R}$ such that $\phi(x) = \pi$.

15. Show that if $\alpha, \beta \in \mathbb{R}_c$ are constructible numbers then any solution to the equation $x^2 + \alpha x + \beta = 0$ is a constructible number.

16. Let $f(x) = x^4 - 4x^2 + 2$. Show that $f(x)$ is irreducible over $\mathbb{Q}$. Let E be a splitting field for $f(x)$ over $\mathbb{Q}$. Find the Galois group $\mathrm{Gal}(E/\mathbb{Q})$.

17. Let E/F be a field extension of degree p, where p is prime. Show that for any element $\alpha \in E$ such that $\alpha \notin F$, we have $E = F(\alpha)$.

18. Show that the Galois group of the polynomial $x^3 - 2 \in \mathbb{Q}[x]$ is isomorphic to the symmetric group S_3. (The Galois group of a polynomial $f(x) \in F[x]$ is $\mathrm{Gal}(E/F)$ where E is the splitting field for $f(x)$ over F.)

19. Show that $x^4 + x^3 + x^2 + x + 1$ is irreducible in $\mathbb{Q}[x]$.

20. (a) State the Dimension Formula.
    (b) State the Fundamental Theorem of Field Theory.
    (c) State the characterization of the field $\mathbb{R}_c$ of constructible numbers.
    (d) State the Fundamental Theorem of Galois Theory.