Some Solutions to Practice Problems.

1 $N \trianglelefteq G \quad N \leq H \leq G$

Since $gNg^{-1} \subseteq N \ \forall g \in G$

we have $gNg^{-1} \subseteq N \ \forall g \in H$.

so $N \trianglelefteq H$.

H/N is not always normal in G/N:

Take $N = \{e\}$, $G = S_3$, $H = \langle (12) \rangle$

H is a subgroup which is not normal.

By the Correspondence Theorem

(or direct calc) H/N is not normal

in G/N.

b. $$\begin{aligned} G_{xH} &= \{ g \in G : g.(xH) = xH \} \\ &= \{ g \in G : gxH = xH \} \\ &= \{ g \in G : x^{-1}gxH = H \} \\ &= \{ g \in G : x^{-1}gx \in H \} \\ &= \{ g \in G : g \in xHx^{-1} \} = xHx^{-1} \end{aligned}$$

Thus the stabilizer of xH equals xHx^{-1} which is conjugate to H. So every stabilizer is conjugate to H.

4. The dihedral group of order 12 contains an element of order 6.
A_4 consists of elements of the form

	order
(1)	1
$(12)(34)$	2
(123)	3

Therefore A_4 has no element of order 6. So the groups are not isomorphic.

5. Define

$$\varphi : G \times G \longrightarrow G \times (G/N)$$

by $$\varphi(g_1, g_2) = (g_1, g_2 N).$$

Check φ is a group homomorphism that is surjective and has kernel $\{1\} \times N$. Use 1^{st} Isomorphism Theorem (for groups).

	Partition	Ex	# elts
6.	$4 = 4$	(1234)	$\frac{1}{4} 4! = 6$
	$= 3+1$	(123)	$4 \cdot 2 = 8$
	$= 2+2$	$(12)(34)$	3
	$= 2+1+1$	(12)	$\binom{4}{2} = 6$
	$= 1+1+1+1$	(1)	1
			$1+6+3+8+6 = 24$

Only 2 conjugacy classes have an odd # of elements:

$\{(1)\}$, $\{(12)(34), (13)(24), (14)(23)\}$.

7. Suppose $\sigma \in Z(S_n)$ where $n > 2$..

Write σ as product of disjoint cycles:

$\sigma = (a_1 \cdots a_k)(b_1 \cdots b_\ell) \cdots$

Let k be the largest length of a cycle in σ.

<u>Case $k=2$</u>: If σ has only one cycle, WLOG $\sigma = (12)$. But then $\tau\sigma\tau^{-1} = (13) \neq \sigma$ where $\tau = (23)$

If σ has at least two disj. cycles WLOG $\sigma = (12)(34)$. Then let $\tau = (23)$

$\tau\sigma\tau^{-1} = (13)(24) \neq \sigma$.

<u>Case $k > 2$</u>: Let $\tau = (a_1 a_2)$. Then

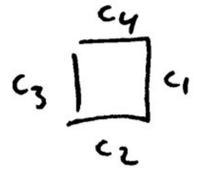
$\tau\sigma\tau^{-1} = (a_2 a_1 a_3 \cdots a_k)(b_1 b_2 \cdots) \cdots$

$\neq \sigma$.

Thus the only possibility is $k = 1$

$\Rightarrow \sigma = (1)$. Thus $Z(S_n)$ is trivial.

9. $X = \{$ all possible ways to paint without rotating $\}$



$|X| = 3 \cdot 3 \cdot 3 \cdot 3 = 81$

$G = D_8$ = dihedral group of order 8
= symmetries of the square $\circlearrowright X$

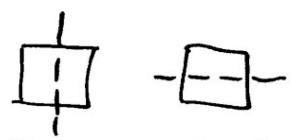
$G \backslash X$ = set of orbits $\quad |G\backslash X| = ?$

Burnside's Counting Theorem:

$$|G\backslash X| = \frac{1}{|G|} \sum_{g \in G} |X^g| \qquad X^g = \{x \in X : g . x = x\}$$

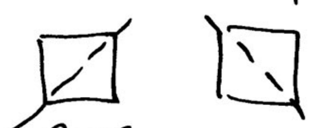
$|X^g|$:

Reflection through side:



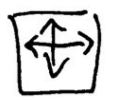
$c_1 = c_3 \quad c_2 = c_4$ $\qquad 3^3 + 3^3$

Reflection through corners



$c_1 = c_4$
$c_2 = c_3$ $\qquad 3^2 + 3^2$

Rotation $180°$



3^2

$c_2 = c_4$
$c_1 = c_3$

Rotation $90°, 270°$ $\quad c_1 = c_2 = c_3 = c_4 \qquad 3^1 + 3^1$

Identity $\qquad 3^4$

$$|G\backslash X| = \frac{1}{8}\left(3^3 + 3^3 + 3^2 + 3^2 + 3^2 + 3 + 3 + 3^4\right) = \frac{168}{8} = \boxed{21}$$

field $\Rightarrow$ ED $\Rightarrow$ PID $\Rightarrow$ UFD $\Rightarrow$ ID $\Rightarrow$ comm ring

13. a) $\mathbb{R}[x]/(x^3+1)$

neither, since x^3+1 is reducible$/\mathbb{R}$
($x+1$ divides x^3+1 so $x+1+(x^3+1)$
is a zero-divisor in $\mathbb{R}[x]/(x^3+1)$)

b) $\mathbb{Z}[x]/(x^2+1) \cong \mathbb{Z}[i]$ Gaussian integers
UFD, PID (because actually a ED.)
ID.

c) $\mathbb{Q}\times\mathbb{Z}$ has zero-divisors $(1,0)\cdot(0,1)=(0,0)$
so not I.D. (nor PID, UFD)

d) $M_2(\mathbb{Z})$ noncommutative, so neither

e) $\mathbb{Z}[x_1, x_2, x_3]$ is a UFD but not PID
Recall R UFD $\Rightarrow R[x]$ UFD; $\mathbb{Z}$ UFD.
$\mathbb{Z}[x_1, x_2, x_3] \supset (2, x_1)$ not principal

) $\mathbb{Q}[x,y]/(x+y) \cong \mathbb{Q}[x]$ via

$$f(x,y)+(x+y) \longmapsto f(x,-x)$$

(Details: ① $\mathbb{Q}[x,y] \to \mathbb{Q}(x)$, $f(x,y) \mapsto f(x,-x)$
surj ring hom. with kernel $(x-y)$
② Iso thm.)

So $\dfrac{\mathbb{Q}[x,y]}{(x+y)}$ is a PID, UFD, ID since $\mathbb{Q}[x]$ is.

a sol

since

$\frac{1}{4}\alpha^2 - \beta \in \mathbb{R}_c$.

ome $r \geq 0$.

ich satisfies (is a root of)

$\mathbb{Q}(\gamma)[y]$

$: \mathbb{Q}(\gamma)] \leq 2$ i.e. $= 1$ or 2

$\mathbb{Q}] = 2^r$ or 2^{r+1}

$\mathbb{R}_c$ (by characterization

$x \in \mathbb{R}_c$ since $\mathbb{R}_c$ is

15. $\alpha, \beta \in \mathbb{R}_c$. Suppose $x \in \mathbb{R}$ is a sol to $x^2 + \alpha x + \beta = 0$.

$$\left(x + \tfrac{1}{2}\alpha\right)^2 - \tfrac{1}{4}\alpha^2 + \beta = 0$$

Since $\mathbb{R}_c$ is a field, $\gamma := \frac{1}{4}\alpha^2 - \beta \in \mathbb{R}_c$.

Thus $[\mathbb{Q}(\gamma):\mathbb{Q}] = 2^r$, some $r \geq 0$.

Therefore $x + \frac{1}{2}\alpha$ which satisfies (is a root of) the pol $y^2 - \gamma \in \mathbb{Q}(\gamma)[y]$

gives $[\mathbb{Q}(x + \frac{1}{2}\alpha) : \mathbb{Q}(\gamma)] \leq 2$ i.e. $= 1$ or 2

$\Rightarrow [\mathbb{Q}(x + \frac{1}{2}\alpha) : \mathbb{Q}] = 2^r$ or 2^{r+1}

$\Rightarrow x + \frac{1}{2}\alpha \in \mathbb{R}_c$ (by characterization of $\mathbb{R}_c$) $\Rightarrow x \in \mathbb{R}_c$ since $\mathbb{R}_c$ is a field.

16. $f(x) = x^4 - 4x^2 + 2 = (x^2-2)^2 - 2$

Roots: $x^2 - 2 = \pm\sqrt{2}$

$$x^2 = 2 \pm \sqrt{2}$$

$$x = \pm\sqrt{2 \pm \sqrt{2}}$$

$f(x)$ is irr by Eisenstein $p = 2$.

let $\alpha = \sqrt{2+\sqrt{2}}$, $\beta = \sqrt{2-\sqrt{2}}$

Note $\alpha\beta = \sqrt{4-2} = \sqrt{2}$.

So $E = \mathbb{Q}(\alpha, \beta) = \mathbb{Q}(\sqrt{2})(\alpha)$ $\quad (\beta = \frac{\sqrt{2}}{\alpha})$

$\pm\alpha$ are the roots of $x^2 - (2+\sqrt{2}) \in \mathbb{Q}(\sqrt{2})[x]$.

Since E is the splitting field, there is an automorphism $\sigma : E \to E$ such that

$\sigma(\alpha) = -\alpha$ and $\sigma|_{\mathbb{Q}(\sqrt{2})} = \mathrm{Id}$

Then $\sigma(\beta) = \sigma\left(\frac{\sqrt{2}}{\alpha}\right) = \frac{\sqrt{2}}{-\alpha} = -\beta$.

Consider $\tilde{\tau} : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$, $\tilde{\tau}(\sqrt{2}) = -\sqrt{2}$

$\tilde{\tau}|_{\mathbb{Q}} = \mathrm{Id}$.

Since $\tilde{\tau}\left(x^2 - (2+\sqrt{2})\right) = x^2 - (2-\sqrt{2})$, $\tilde{\tau}$ extends to an automorphism $\tau : E \to E$ such that

$\tau(\alpha) = \beta = \frac{\sqrt{2}}{\alpha}$ and $\tau(\sqrt{2}) = -\sqrt{2}$,

Then $\tau^2(\alpha) = \frac{\tau(\sqrt{2})}{\tau(\alpha)} = \frac{-\sqrt{2}}{\sqrt{2}/\alpha} = -\alpha$, $\tau^2(\sqrt{2}) = \sqrt{2}$

So $\tau^2 = \sigma$. So $\mathrm{Gal}(E/\mathbb{Q}) = \langle \tau \rangle \cong \mathbb{Z}_4$.

19.

$$f(x) = x^4 + x^3 + x^2 + x + 1 \in \mathbb{Q}[x]$$

$$f(x+1) = x^4 + 4x^3 + 6x^2 + 4x + 1$$

$$x^3 + 3x^2 + 3x + 1$$

$$x^2 + 2x + 1$$

$$x + 1$$

$$+ 1 =$$

$$= x^4 + 5x^3 + 10x^2 + 10x + 5$$

irr $p = 5$ Eisenstein.

Explanation/Alternative sol:

$$f(x) = \frac{x^5 - 1}{x - 1} \Rightarrow f(x+1) = \frac{(x+1)^5 - 1}{x} =$$

$$= \frac{1}{x}\left(x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 - 1\right)$$

$$= x^4 + 5x^3 + 10x^2 + 10x + 5$$