AN ELEMENTARY PROOF OF THE 'STRANGE FORMULA' OF FREUDENTHAL AND DE VRIES

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1. Introduction

Let $\mathbb{G}$ be a semi-simple, compact, connected Lie group, and $\mathbf{g}$ its Lie algebra. Fix a Cartan subalgebra $\mathbf{h}$ and a fundamental Weyl chamber in $\mathbf{h}$. We can now define positive roots $\mathfrak{R}^+$ and we set $\rho = \frac{1}{2}\sum_{\alpha\in\mathfrak{R}^+}\alpha$. The 'strange formula' of Freudenthal and de Vries is then

THEOREM 1.1 (see [**5**]). $\frac{1}{24}\dim\mathbb{G} = (\rho, \rho)$, *where* $(,)$ *is the Killing form on* $\mathbf{g}$.

The theorem was first proved in [**5**] by considering a Taylor expansion of the Weyl character formula and the authors asked whether the formula could be proved by more algebraic or elementary means. Because of its connection with Macdonald's identities, the formula has been proved by very ingenious methods, such as the comparison of two formulae for the heat kernel [**3**]. An inductive proof based on the study of root system embeddings is outlined in [**2**], and in [**4**] it is proved by studying the spin representation $\mathrm{Spin}(\mathbf{g}) \to SO(\mathbf{S})$ where $\mathbf{S}$ is the space of spinors on the vector space $\mathbf{g}$. The proof in this note is, I hope, in the spirit of that requested in [**5**].

2. Preliminaries

All basic facts used can be found in [**1**]. Throughout the notation $\{\alpha_1, \dots, \alpha_r\}$ will indicate a basis of positive simple roots, and the corresponding fundamental weights $\{\omega_1, \dots, \omega_r\}$ are defined by the conditions that $\langle\omega_i, 2\alpha_j\rangle = \langle\alpha_j, \alpha_j\rangle\delta_{ij}\ \forall i, j$, where $\langle,\rangle$ is an invariant inner product, which we will normalize so that $\langle\tilde{\alpha}, \tilde{\alpha}\rangle = 2$, where $\tilde{\alpha}$ is the highest root. For the sake of completeness we include a different proof of the following known result [**7**].

PROPOSITION 2.1. $\sum_{\alpha\in\mathfrak{R}^+}\langle\alpha, \gamma\rangle\alpha = g\gamma$ *for all* $\gamma \in \mathbf{h}^*$, *where* g *is the eigenvalue of the Casimir element of* $\mathbf{g}$ *in its adjoint representation. If* $\mathbf{g}$ *is simply laced, then* g *is also the Coxeter number of* $\mathbf{g}$.

Proof. For each pair $(i, x) \in \{1, \dots, r\} \times \mathbb{R}$, let $\mathfrak{R}_{i,x} = \{\alpha \in \mathfrak{R} : \langle\omega_i, \alpha\rangle = x\}$. Now observe that for $j \neq i$ the simple reflection s_j in the hyperplane $\alpha_j = 0$ permutes the elements of $\mathfrak{R}_{i,x}$, so that $\langle\sum_{\alpha\in\mathfrak{R}_{i,x}}\alpha, \alpha_j\rangle = \langle s_j(\sum_{\alpha\in\mathfrak{R}_{i,x}}\alpha), s_j\alpha_j\rangle = \langle\sum_{\alpha\in\mathfrak{R}_{i,x}}\alpha, s_j\alpha_j\rangle = -\langle\sum_{\alpha\in\mathfrak{R}_{i,x}}\alpha, \alpha_j\rangle$ and therefore $\langle\sum_{\alpha\in\mathfrak{R}_{i,x}}\alpha, \alpha_j\rangle = 0$. Thus $\sum_{\alpha\in\mathfrak{R}_{i,x}}\alpha$ and $\sum_{\alpha\in\mathfrak{R}_{i,x}}\langle\alpha, \omega_i\rangle\alpha$ are multiples of ω_i. Summing over all positive values of x we see that $\sum_{\alpha\in\mathfrak{R}^+}\langle\alpha, \omega_i\rangle\alpha = g_i\omega_i$, and therefore $2g_i\langle\omega_i, \omega_i\rangle = \sum_{\alpha\in\mathfrak{R}}\langle\alpha, \omega_i\rangle\langle\alpha, \omega_i\rangle$ for some $g_i \in \mathbb{R}$. Since for the Killing form $(u, \upsilon) = \sum_{\alpha\in\mathfrak{R}}(\alpha, u)(\alpha, \upsilon)\ \forall u, \upsilon, \in \mathbf{h}^*$, and by irreducibility of the root system $\langle,\rangle = 2\mathrm{g}(,)$ for some $g \in \mathbb{R}$ we see that $g_i = g$ for all i. We have now established that $\sum_{\alpha\in\mathfrak{R}^+}\langle\alpha, \gamma\rangle\alpha = g\gamma$ for all $\gamma \in \mathbf{h}^*$, since it holds on a basis, and it only remains to identify the constant g. By repeating the above arguement we first observe that $\sum_{\langle\alpha,\omega_i\rangle>0}\alpha = \frac{\langle 2\rho,\omega_i\rangle}{\langle\omega_i,\omega_i\rangle}\omega_i$. Now choose $\gamma = \tilde{\alpha}$ above, and use the fact that $\frac{2\langle\alpha,\tilde{\alpha}\rangle}{\langle\tilde{\alpha},\tilde{\alpha}\rangle} = 0$ or 1 unless $\alpha = \tilde{\alpha}$, to obtain in the case $\tilde{\alpha} = c\omega_i$ ($\mathfrak{R}$ not of type A) that

$$g\tilde{\alpha} = \sum_{\alpha>0}\langle\alpha,\tilde{\alpha}\rangle\alpha = \left(\sum_{\langle\alpha,\omega_i\rangle>0}\alpha\right) + \tilde{\alpha} = \frac{\langle 2\rho,\omega_i\rangle}{\langle\omega_i,\omega_i\rangle}\omega_i + \tilde{\alpha} = (\langle\rho,\tilde{\alpha}\rangle + 1)\tilde{\alpha}.$$

Therefore, $g = \langle\rho,\tilde{\alpha}\rangle + 1 = (\langle\rho+\tilde{\alpha},\rho+\tilde{\alpha}\rangle - \langle\rho,\rho\rangle)/2$, which is the eigenvalue of the Casimir operator of $\mathbf{g}$ in its adjoint representation. The argument for the root system of type A is similar: just replace $c\omega_i$ by $\omega_1 + \omega_r$. Using the fact [**1**] that $\sum_{\alpha>0} 2(\langle\alpha,\gamma\rangle/\langle\alpha,\alpha\rangle)\alpha = h\gamma$ for all $\gamma \in \mathbf{h}^*$, where h is the Coxeter number of $\mathbf{g}$, we see that g and h are equal when $\mathbf{g}$ is simply laced, that is, when all the roots have the same length, equal to $\sqrt{2}$ with respect to our normalized inner product.

3. The simply laced case

In this section we will assume that $\mathbf{g}$ is simply laced. We will use the following standard description [**6**] of the exponents of $\mathbf{g}$. For $\alpha \in \mathfrak{R}^+$ let $ht(\alpha)$ denote the height of α, that is, the sum of the coefficients of α relative to the basis of positive simple roots, and let n_k be the number of positive roots of height k. Then $n_k - n_{k+1}$ is the number of times k occurs as an exponent m_k of $\mathbf{g}$. We will also use the well-known fact [**6**] that the number of positive roots is $\frac{r}{2}h$, and therefore $\dim\mathbf{g} = (h+1)r$. Now if $2\rho = \sum_{i=1}^r a_i\alpha_i$, using our observations above, and the alternative description $\rho = \sum_{i=1}^r \omega_i$, we have that $24(\rho,\rho) = \frac{6}{2h}\langle 2\rho, 2\rho\rangle = \frac{6}{h}\sum_{i=1}^r a_i$. The formula now becomes

THEOREM 3.1. *Let $2\rho = \sum_{i=1}^r a_i\alpha_i$, and let h denote the Coxeter number of* $\mathbf{g}$, *then $\sum_{i=1}^r a_i = \frac{r}{6}h(h+1)$.*

Proof. By Proposition 2.1 $\sum_{\alpha\in\mathfrak{R}^+}\langle\alpha,2\rho\rangle\langle\alpha,2\rho\rangle = h\langle 2\rho,2\rho\rangle$, and therefore $\sum_{\alpha\in\mathfrak{R}^+} 4ht(\alpha)^2 = h\langle 2\rho,2\rho\rangle = 2h\sum_{i=1}^r a_i$, or $\sum_{\alpha\in\mathfrak{R}^+} ht(\alpha)^2 = \frac{h}{2}\sum_{i=1}^r a_i$. Using the definition of the exponents of $\mathbf{g}$ in terms of the heights, and the formula for the sum of the squares of the positive integers up to and including each exponent, we obtain that $\frac{h}{2}\sum_{i=1}^r a_i = \sum_{i=1}^r \frac{1}{6}m_i(m_i+1)(2m_i+1) = \sum_{i=1}^r \frac{1}{3}m_i(m_i+1)(m_i-1) + \sum_{i=1}^r \frac{1}{2}m_i(m_i+1) = \sum_{i=1}^r \frac{1}{3}m_i(m_i+1)(m_i-1) + \sum_{i=1}^r a_i$. For the purpose of notation only here, we denote $\frac{1}{3}m_i(m_i+1)(m_i-1)$ by $\chi(m_i)$ because of Witt's formula, so that we have $(\frac{h}{2}-1)\sum_{i=1}^r a_i = \sum_{i=1}^r \chi(m_i)$. We now use the duality $m_i + m_{r+1-i} = h$ and observe that $\chi(m_i) + \chi(h-m_i) = \chi(h) - hm_i(h-m_i)$ to obtain (in the case where r is even) that $(\frac{h}{2}-1)\sum_{i=1}^r a_i = \frac{r}{2}\chi(h) - h\sum_{i=1}^{\frac{r}{2}} m_i(h-m_i)$. Since $\frac{r}{2}h^2 = \sum_{i=1}^{\frac{r}{2}}\{m_i + (h-m_i)\}^2 = \sum_{i=1}^{\frac{r}{2}}\{m_i^2 + (h-m_i)^2 + 2m_i(h-m_i)\} = \sum_{i=1}^r m_i^2 + 2\sum_{i=1}^{\frac{r}{2}} m_i(h-m_i)$, we have that $(\frac{h}{2}-1)\sum_{i=1}^r a_i = \frac{r}{2}\chi(h) - \frac{r}{4}h^3 + \frac{h}{2}\sum_{i=1}^r m_i^2$. In the case where the rank of $\mathbf{g}$ is odd this last equation still remains valid; however, it is arrived at as follows: $(\frac{h}{2}-1)\sum_{i=1}^r a_i = \frac{r-1}{2}\chi(h) - h\sum_{i=1}^{\frac{r-1}{2}} m_i(h-m_i) + \chi\left(\frac{h}{2}\right)$. Since $\sum_{i=1}^{\frac{r-1}{2}} m_i(h-m_i) = \frac{r-1}{4}h^2 - \frac{1}{2}\sum_{i=1}^r m_i^2 + \frac{1}{2}\left(\frac{h}{2}\right)^2$ we obtain that $\left(\frac{h}{2}-1\right)\sum_{i=1}^r a_i = \frac{r-1}{2}\chi(h) - \frac{r-1}{4}h^3 + \frac{h}{2}\sum_{i=1}^r m_i^2 - \frac{1}{8}h^3 + \chi\left(\frac{h}{2}\right) = \frac{r}{2}\chi(h) - \frac{r}{4}h^3 + \frac{h}{2}\sum_{i=1}^r m_i^2$. Now since the exponents sum to the number of positive roots, we see that $\left(\frac{h}{2}-1\right)\sum_{i=1}^r a_i = \frac{r}{2}\chi(h) - \frac{r}{4}h^3 + h(\sum_{i=1}^r a_i - \frac{rh}{4})$. Now recalling that $\chi(h) = \frac{1}{3}(h^3-h)$ we obtain that $\frac{1}{2}\sum_{i=1}^r a_i(h+2) = \frac{r}{12}h(h+1)(h+2)$, and the theorem is proved.

4. The non-simply laced case

Throughout this section ρ^* will denote half the sum of the positive dual roots $\check{\alpha} = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$. Our proof of the formula in the non-simply laced case proceeds along the same lines as that of the simply laced case initially. Our starting point is again Proposition 2.1 and the observation that $\langle\alpha, \rho^*\rangle = ht(\alpha)$.

Proof of Theorem 1.1 . We may assume that $\mathbf{g}$ is not simply laced. By Proposition 2.1 $g\langle\rho^*, \rho^*\rangle = \sum_{\alpha\in\mathfrak{R}^+}\langle\alpha, \rho^*\rangle\langle\alpha, \rho^*\rangle = \sum_{\alpha\in\mathfrak{R}^+} ht(\alpha)^2$. Much as in the simply laced case we obtain that $g\langle\rho^*, \rho^*\rangle = \sum_{i=1}^r \chi(m_i)+\sum_{i=1}^r a_i = (h+1)\sum_{i-1}^r a_i - \frac{r}{12}h(h+1)(h+2)$. We now use the fact [**7**] that there exists a Weyl group element $w \in W$ such that $g\rho^* = (h+1)\rho - w(\rho)$, so that $g^2\langle\rho^*, \rho^*\rangle = \langle(h+1)\rho - w(\rho), (h+1)\rho - w(\rho)\rangle = \{(h+1)^2+1\}\langle\rho, \rho\rangle - 2(h+1)\langle\rho, (h+1)\rho - g\rho^*\rangle$. Now using the fact that $\langle 2\rho, \rho^*\rangle = \sum_{i=1}^r a_i$, we obtain $g\langle\rho^*, \rho^*\rangle = (h+1)\sum_{i=1}^r a_i - \frac{h}{g}(h+2)\langle\rho, \rho\rangle$. Comparing this with our earlier expression for $g\langle\rho^*, \rho^*\rangle$ we obtain that $\frac{1}{g}h(h+2)\langle\rho, \rho\rangle = \frac{1}{12}rh(h+1)(h+2)$, and therefore $\frac{24}{2g}\langle\rho, \rho\rangle = r(h+1) = \dim\mathbf{g}$, and the theorem is proved.

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References

1. N. Bourbaki, *Groupes et Algèbres de Lie*, Hermann, Paris, 1968, Chs. 4–6.

2. R. Carles, Méthode récurrente pour la classification des systèmes de racines réduits et irréductibles, *C. R. Acad. Sci. Paris* **276** (1973), 355–358.

3. H. D. Fegan, The heat equation on a compact Lie group, *Trans. Amer. Math. Soc.* **246** (1978) 339–357.

4. H. D. Fegan and B. Steer, On the 'strange formula' of Freudenthal and de Vries, *Math. Proc. Camb. Phil. Soc.* **105** (1989) 249–252.

5. H. Freudenthal and H. de Vries, *Linear Lie Groups*, Academic Press, New York, 1969.

6. B. Kostant, The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group, *Amer. J. Math.* **81** (1959) 973–1032.

7. I. G. Macdonald, Affine root systems and Dedekind's η-function, *Invent. Math.* **15** (1972) 91–143.