# 5

# *Integration on Manifolds*

## *MANIFOLDS*

If $U$ and $V$ are open sets in $\mathbf{R}^n$, a differentiable function $h: U \to V$ with a differentiable inverse $h^{-1}: V \to U$ will be called a **diffeomorphism.** ("Differentiable" henceforth means "$C^\infty$".)

A subset $M$ of $\mathbf{R}^n$ is called a **$k$-dimensional manifold** (in $\mathbf{R}^n$) if for every point $x \in M$ the following condition is satisfied:

($M$) There is an open set $U$ containing $x$, an open set $V \subset \mathbf{R}^n$, and a diffeomorphism $h: U \to V$ such that

$$\begin{aligned} h(U \cap M) &= V \cap (\mathbf{R}^k \times \{0\}) \\ &= \{y \in V: y^{k+1} = \cdots = y^n = 0\}. \end{aligned}$$

In other words, $U \cap M$ is, "up to diffeomorphism," simply $\mathbf{R}^k \times \{0\}$ (see Figure 5-1). The two extreme cases of our definition should be noted: a point in $\mathbf{R}^n$ is a 0-dimensional manifold, and an open subset of $\mathbf{R}^n$ is an $n$-dimensional manifold.

One common example of an $n$-dimensional manifold is the

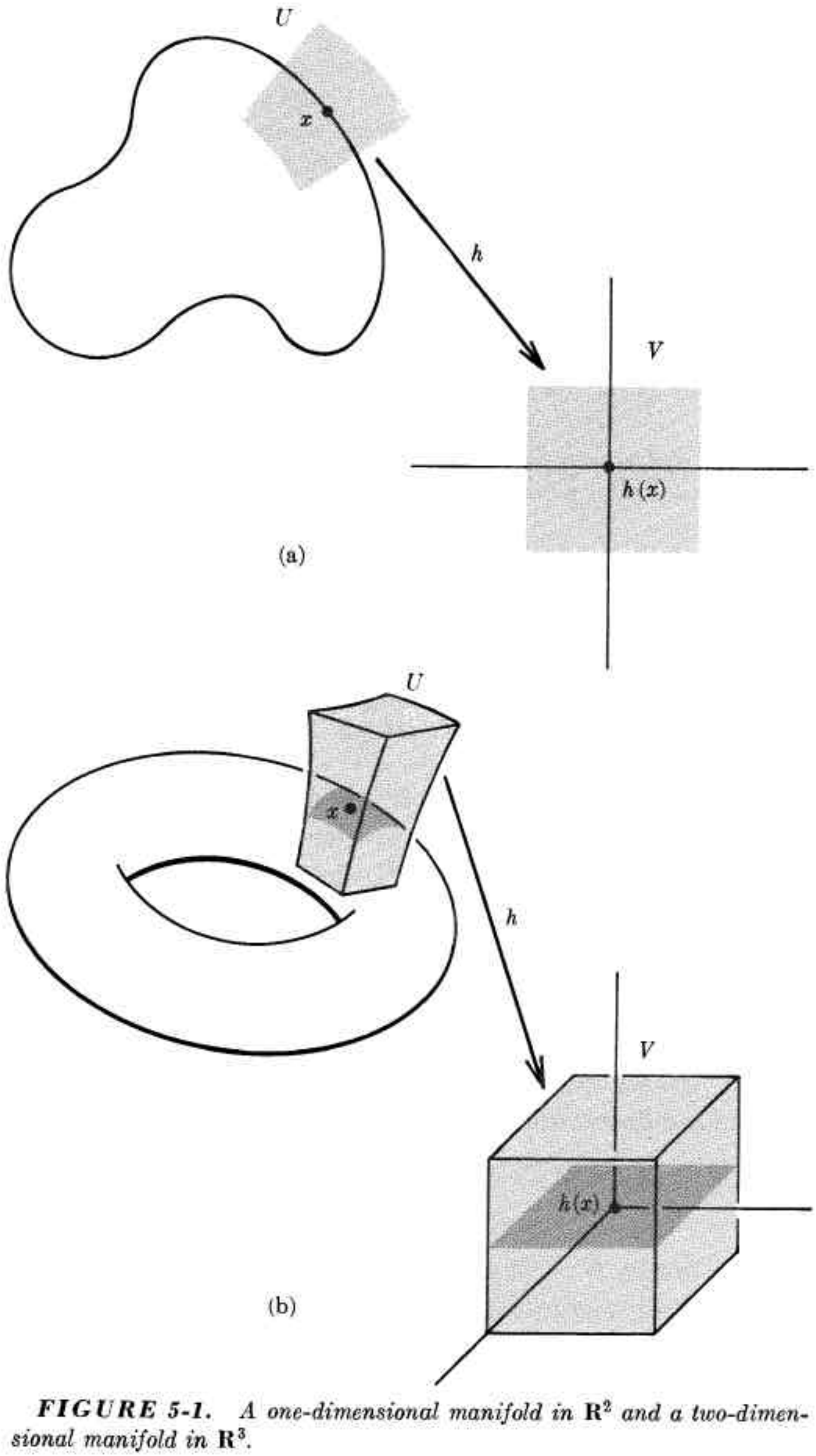



***FIGURE 5-1.*** *A one-dimensional manifold in* $\mathbf{R}^2$ *and a two-dimensional manifold in* $\mathbf{R}^3$.

***n*-sphere** $S^n$, defined as $\{x \in \mathbf{R}^{n+1}: |x| = 1\}$. We leave it as an exercise for the reader to prove that condition $(M)$ is satisfied. If you are unwilling to trouble yourself with the details, you may instead use the following theorem, which provides many examples of manifolds (note that $S^n = g^{-1}(0)$, where $g: \mathbf{R}^{n+1} \to \mathbf{R}$ is defined by $g(x) = |x|^2 - 1$).

***5-1 Theorem.*** *Let $A \subset \mathbf{R}^n$ be open and let $g: A \to \mathbf{R}^p$ be a differentiable function such that $g'(x)$ has rank $p$ whenever $g(x) = 0$. Then $g^{-1}(0)$ is an $(n - p)$-dimensional manifold in $\mathbf{R}^n$.*

***Proof.*** This follows immediately from Theorem 2-13. ▌

There is an alternative characterization of manifolds which is very important.

***5-2 Theorem.*** *A subset $M$ of $\mathbf{R}^n$ is a $k$-dimensional manifold if and only if for each point $x \in M$ the following "coordinate condition" is satisfied:*

*$(C)$ There is an open set $U$ containing $x$, an open set $W \subset \mathbf{R}^k$, and a* 1-1 *differentiable function $f: W \to \mathbf{R}^n$ such that*

*(1) $f(W) = M \cap U$,*
*(2) $f'(y)$ has rank $k$ for each $y \in W$,*
*(3) $f^{-1}: f(W) \to W$ is continuous.*

[Such a function $f$ is called a **coordinate system** around $x$ (see Figure 5-2).]

***Proof.*** If $M$ is a $k$-dimensional manifold in $\mathbf{R}^n$, choose $h: U \to V$ satisfying $(M)$. Let $W = \{a \in \mathbf{R}^k: (a,0) \in h(M)\}$ and define $f: W \to \mathbf{R}^n$ by $f(a) = h^{-1}(a,0)$. Clearly $f(W) = M \cap U$ and $f^{-1}$ is continuous. If $H: U \to \mathbf{R}^k$ is $H(z) = (h^1(z), \ldots, h^k(z))$, then $H(f(y)) = y$ for all $y \in W$; therefore $H'(f(y)) \cdot f'(y) = I$ and $f'(y)$ must have rank $k$.

Suppose, conversely, that $f: W \to \mathbf{R}^n$ satisfies condition $(C)$. Let $x = f(y)$. Assume that the matrix $(D_j f^i(y))$, $1 \le i, j \le k$ has a non-zero determinant. Define $g: W \times \mathbf{R}^{n-k} \to \mathbf{R}^n$ by

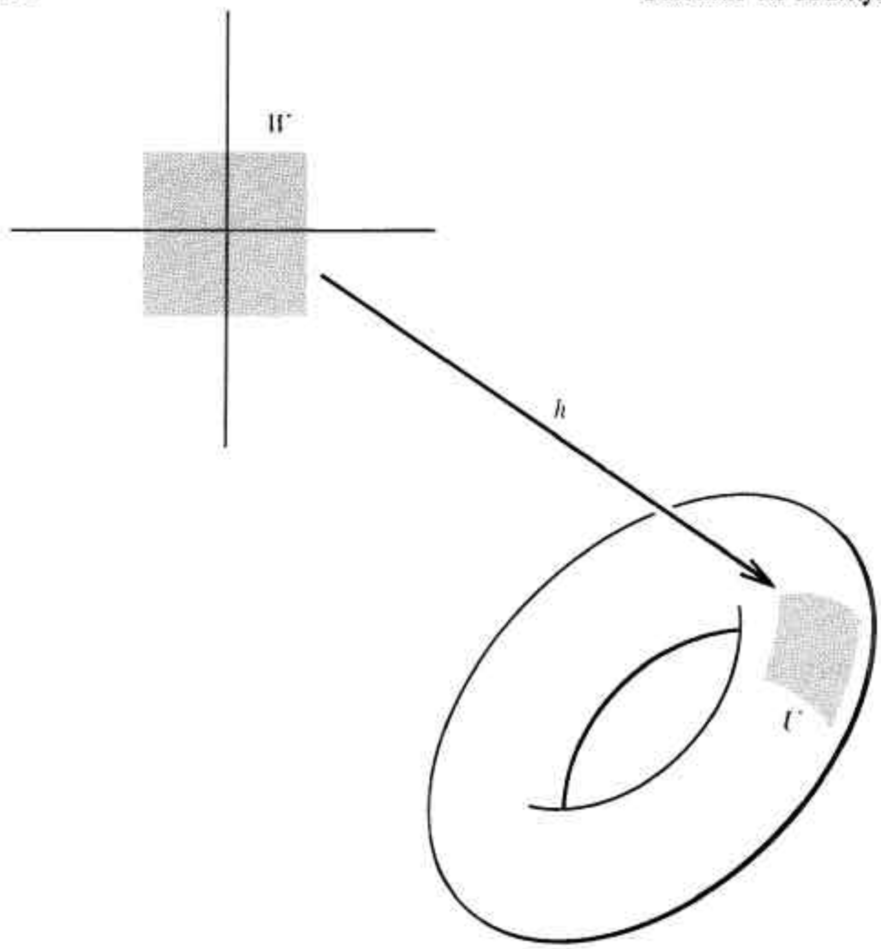



*FIGURE 5-2*

$g(a,b) = f(a) + (0,b)$. Then $\det g'(a,b) = \det (D_j f^i(a))$, so $\det g'(y,0) \neq 0$. By Theorem 2-11 there is an open set $V_1'$ containing $(y,0)$ and an open set $V_2'$ containing $g(y,0) = x$ such that $g\colon V_1' \to V_2'$ has a differentiable inverse $h\colon V_2' \to V_1'$. Since $f^{-1}$ is continuous, $\{f(a)\colon (a,0) \in V_1'\} = U \cap f(W)$ for some open set $U$. Let $V_2 = V_2' \cap U$ and $V_1 = g^{-1}(V_2)$. Then $V_2 \cap M$ is exactly $\{f(a)\colon (a,0) \in V_1\} = \{g(a,0)\colon (a,0) \in V_1\}$, so

$$\begin{aligned} h(V_2 \cap M) = g^{-1}(V_2 \cap M) &= g^{-1}(\{g(a,0)\colon (a,0) \in V_1\}) \\ &= V_1 \cap (\mathbf{R}^k \times \{0\}). \quad \blacksquare \end{aligned}$$

One consequence of the proof of Theorem 5-2 should be noted. If $f_1\colon W_1 \to \mathbf{R}^n$ and $f_2\colon W_2 \to \mathbf{R}^n$ are two coordinate systems, then

$$f_2^{-1} \circ f_1 \colon f_1^{-1}(f_2(W_2)) \to \mathbf{R}^k$$

is differentiable with non-singular Jacobian. If fact, $f_2^{-1}(y)$ consists of the first $k$ components of $h(y)$.

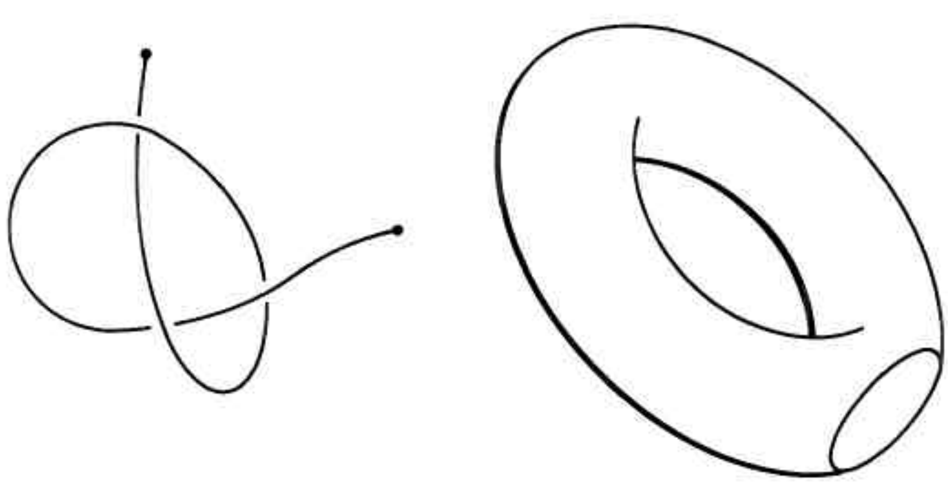
The **half-space** $\mathbf{H}^k \subset \mathbf{R}^k$ is defined as $\{x \in \mathbf{R}^k \colon x^k \geq 0\}$. A subset $M$ of $\mathbf{R}^n$ is a ***k*-dimensional manifold-with-boundary** (Figure 5-3) if for every point $x \in M$ either condition $(M)$ or the following condition is satisfied:

$(M')$ There is an open set $U$ containing $x$, an open set $V \subset \mathbf{R}^n$, and a diffeomorphism $h \colon U \to V$ such that

$$\begin{aligned} h(U \cap M) &= V \cap (\mathbf{H}^k \times \{0\}) \\ &= \{y \in V \colon y^k \geq 0 \text{ and } y^{k+1} = \cdots = y^n = 0\} \end{aligned}$$

and $h(x)$ has $k$th component $= 0$.

It is important to note that conditions $(M)$ and $(M')$ cannot both hold for the same $x$. In fact, if $h_1 \colon U_1 \to V_1$ and $h_2 \colon U_2 \to V_2$ satisfied $(M)$ and $(M')$, respectively, then $h_2 \circ h_1{}^{-1}$ would be a differentiable map that takes an open set in $\mathbf{R}^k$, containing $h(x)$, into a subset of $\mathbf{H}^k$ which is not open in $\mathbf{R}^k$. Since $\det (h_2 \circ h_1{}^{-1})' \neq 0$, this contradicts Problem 2-36. The set of all points $x \in M$ for which condition $M'$ is satisfied is called the **boundary** of $M$ and denoted $\partial M$. This



(a) (b)

***FIGURE 5-3.*** *A one-dimensional and a two-dimensional manifold-with-boundary in* $\mathbf{R}^3$.

must not be confused with the boundary of a set, as defined in Chapter 1 (see Problems 5-3 and 5-8).

***Problems.*** **5-1.** If $M$ is a $k$-dimensional manifold-with-boundary, prove that $\partial M$ is a $(k-1)$-dimensional manifold and $M - \partial M$ is a $k$-dimensional manifold.

**5-2.** Find a counterexample to Theorem 5-2 if condition (3) is omitted. *Hint:* Wrap an open interval into a figure six.

**5-3.** (a) Let $A \subset \mathbf{R}^n$ be an open set such that boundary $A$ is an $(n-1)$-dimensional manifold. Show that $N = A \cup$ boundary $A$ is an $n$-dimensional manifold-with-boundary. (It is well to bear in mind the following example: if $A = \{x \in \mathbf{R}^n: |x| < 1 \text{ or } 1 < |x| < 2\}$ then $N = A \cup$ boundary $A$ is a manifold-with-boundary, but $\partial N \neq$ boundary $A$.)

(b) Prove a similar assertion for an open subset of an $n$-dimensional manifold.

**5-4.** Prove a partial converse of Theorem 5-1: If $M \subset \mathbf{R}^n$ is a $k$-dimensional manifold and $x \in M$, then there is an open set $A \subset \mathbf{R}^n$ containing $x$ and a differentiable function $g: A \to \mathbf{R}^{n-k}$ such that $A \cap M = g^{-1}(0)$ and $g'(y)$ has rank $n - k$ when $g(y) = 0$.

**5-5.** Prove that a $k$-dimensional (vector) subspace of $\mathbf{R}^n$ is a $k$-dimensional manifold.

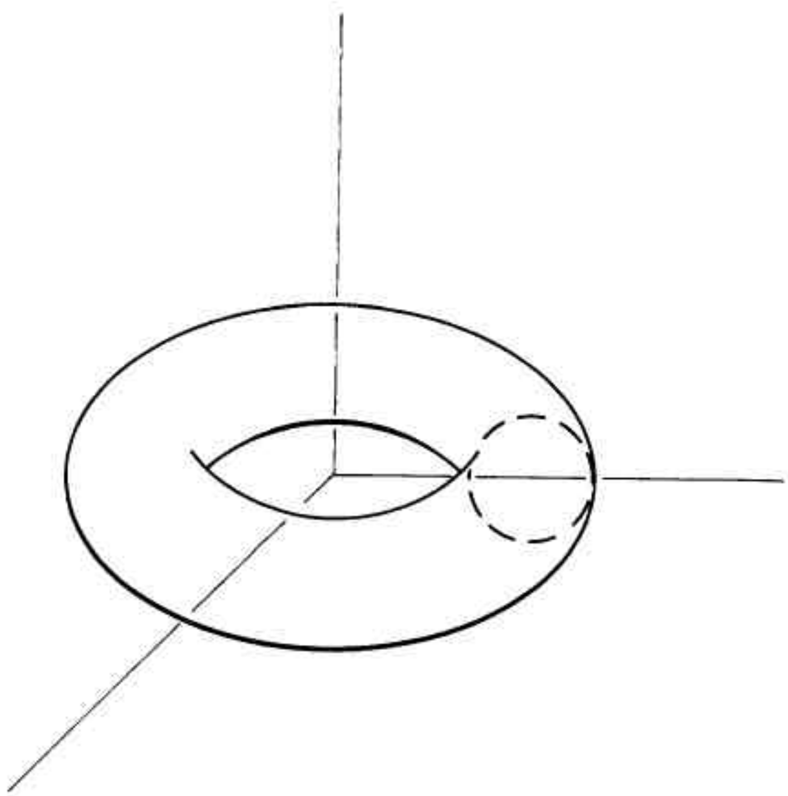


***FIGURE 5-4***

**5-6.** If $f: \mathbf{R}^n \to \mathbf{R}^m$, the **graph** of $f$ is $\{(x,y): y = f(x)\}$. Show that the graph of $f$ is an $n$-dimensional manifold if and only if $f$ is differentiable.

**5-7.** Let $\mathbf{K}^n = \{x \in \mathbf{R}^n: x^1 = 0 \text{ and } x^2, \ldots, x^{n-1} > 0\}$. If $M \subset \mathbf{K}^n$ is a $k$-dimensional manifold and $N$ is obtained by revolving $M$ around the axis $x^1 = \cdots = x^{n-1} = 0$, show that $N$ is a $(k+1)$-dimensional manifold. Example: the torus (Figure 5-4).

**5-8.** (a) If $M$ is a $k$-dimensional manifold in $\mathbf{R}^n$ and $k < n$, show that $M$ has measure 0.

(b) If $M$ is a closed $n$-dimensional manifold-with-boundary in $\mathbf{R}^n$, show that the boundary of $M$ is $\partial M$. Give a counterexample if $M$ is not closed.

(c) If $M$ is a compact $n$-dimensional manifold-with-boundary in $\mathbf{R}^n$, show that $M$ is Jordan-measurable.

## FIELDS AND FORMS ON MANIFOLDS

Let $M$ be a $k$-dimensional manifold in $\mathbf{R}^n$ and let $f: W \to \mathbf{R}^n$ be a coordinate system around $x = f(a)$. Since $f'(a)$ has rank $k$, the linear transformation $f_*: \mathbf{R}^k{}_a \to \mathbf{R}^n{}_x$ is 1-1, and $f_*(\mathbf{R}^k{}_a)$ is a $k$-dimensional subspace of $\mathbf{R}^n{}_x$. If $g: V \to \mathbf{R}^n$ is another coordinate system, with $x = g(b)$, then

$$g_*(\mathbf{R}^k{}_b) = f_*(f^{-1} \circ g)_*(\mathbf{R}^k{}_b) = f_*(\mathbf{R}^k{}_a).$$

Thus the $k$-dimensional subspace $f_*(\mathbf{R}^k{}_a)$ does not depend on the coordinate system $f$. This subspace is denoted $M_x$, and is called the **tangent space** of $M$ at $x$ (see Figure 5-5). In later sections we will use the fact that there is a natural inner product $T_x$ on $M_x$, induced by that on $\mathbf{R}^n{}_x$: if $v,w \in M_x$ define $T_x(v,w) = \langle v,w \rangle_x$.

Suppose that $A$ is an open set containing $M$, and $F$ is a differentiable vector field on $A$ such that $F(x) \in M_x$ for each $x \in M$. If $f: W \to \mathbf{R}^n$ is a coordinate system, there is a unique (differentiable) vector field $G$ on $W$ such that $f_*(G(a)) = F(f(a))$ for each $a \in W$. We can also consider a function $F$ which merely assigns a vector $F(x) \in M_x$ for each $x \in M$; such a function is called a **vector field on** $M$. There is still a unique vector field $G$ on $W$ such that $f_*(G(a)) = F(f(a))$ for $a \in W$; we *define* $F$ to be differentiable if $G$ is differentiable. Note that our definition does not depend on the coordinate

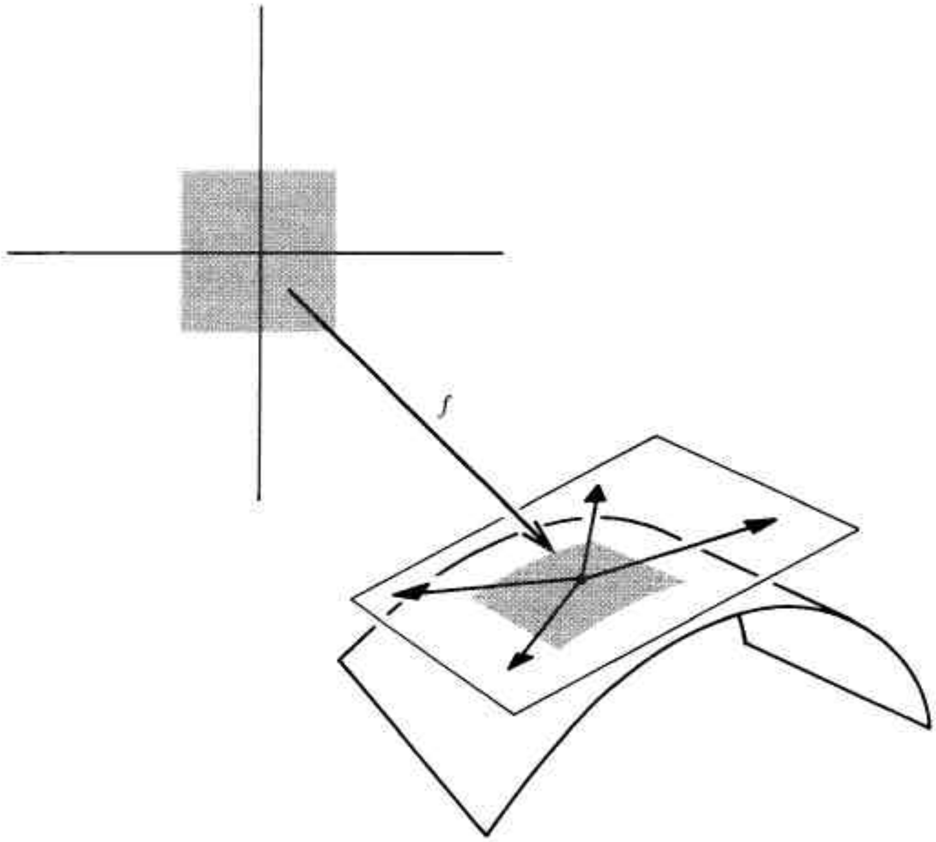



*FIGURE 5-5*

system chosen: if $g: V \to \mathbf{R}^n$ and $g_*(H(b)) = F(g(b))$ for all $b \in V$, then the component functions of $H(b)$ must equal the component functions of $G(f^{-1}(g(b)))$, so $H$ is differentiable if $G$ is.

Precisely the same considerations hold for forms. A function $\omega$ which assigns $\omega(x) \in \Lambda^p(M_x)$ for each $x \in M$ is called **a $p$-form on** $M$. If $f: W \to \mathbf{R}^n$ is a coordinate system, then $f^*\omega$ is a $p$-form on $W$; we *define* $\omega$ to be differentiable if $f^*\omega$ is. A $p$-form $\omega$ on $M$ can be written as

$$\omega = \sum_{i_1 < \cdots < i_p} \omega_{i_1, \ldots, i_p} \, dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

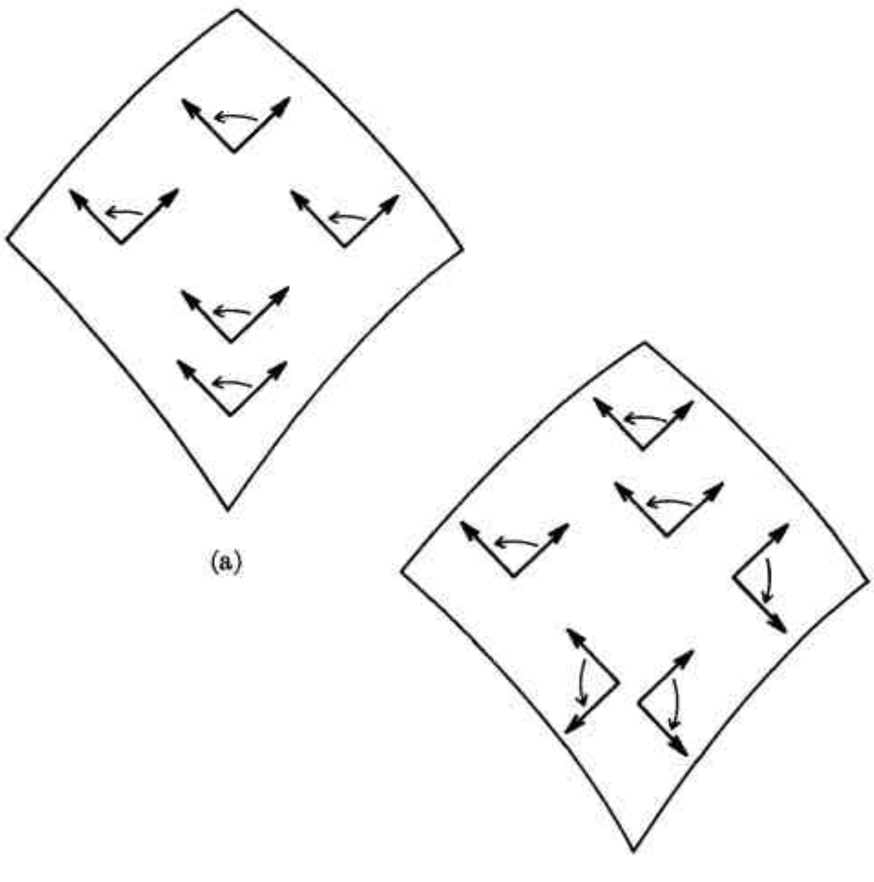
Here the functions $\omega_{i_1, \ldots, i_p}$ are defined only on $M$. The definition of $d\omega$ given previously would make no sense here, since $D_j(\omega_{i_1, \ldots, i_p})$ has no meaning. Nevertheless, there is a reasonable way of defining $d\omega$.

**5-3 Theorem.** *There is a unique $(p+1)$-form $d\omega$ on $M$ such that for every coordinate system $f\colon W \to \mathbf{R}^n$ we have*

$$f^*(d\omega) = d(f^*\omega).$$

***Proof.*** If $f\colon W \to \mathbf{R}^n$ is a coordinate system with $x = f(a)$ and $v_1, \ldots, v_{p+1} \in M_x$, there are unique $w_1, \ldots, w_{p+1}$ in $\mathbf{R}^k{}_a$ such that $f_*(w_i) = v_i$. Define $d\omega(x)(v_1, \ldots, v_{p+1}) = d(f^*\omega)(a)(w_1, \ldots, w_{p+1})$. One can check that this definition of $d\omega(x)$ does not depend on the coordinate system $f$, so that $d\omega$ is well-defined. Moreover, it is clear that $d\omega$ has to be defined this way, so $d\omega$ is unique. ▌

It is often necessary to choose an orientation $\mu_x$ for each tangent space $M_x$ of a manifold $M$. Such choices are called **consistent** (Figure 5-6) provided that for every coordinate



(a) (b)

***FIGURE 5-6.*** *(a) Consistent and (b) inconsistent choices of orientations.*

system $f\colon W \to \mathbf{R}^n$ and $a,b \in W$ the relation

$$[f_*((e_1)_a), \ldots, f_*((e_k)_a)] = \mu_{f(a)}$$

holds if and only if

$$[f_*((e_1)_b), \ldots, f_*((e_k)_b)] = \mu_{f(b)}.$$

Suppose orientations $\mu_x$ have been chosen consistently. If $f\colon W \to \mathbf{R}^n$ is a coordinate system such that

$$[f_*((e_1)_a), \ldots, f_*((e_k)_a)] = \mu_{f(a)}$$

for one, and hence for every $a \in W$, then $f$ is called **orientation-preserving.** If $f$ is *not* orientation-preserving and $T\colon \mathbf{R}^k \to \mathbf{R}^k$ is a linear transformation with $\det T = -1$, then $f \circ T$ *is* orientation-preserving. Therefore there is an orientation-preserving coordinate system around each point. If $f$ and $g$ are orientation-preserving and $x = f(a) = g(b)$, then the relation

$$[f_*((e_1)_a), \ldots, f_*((e_k)_a)] = \mu_x = [g_*((e_1)_b), \ldots, g_*((e_k)_b)]$$

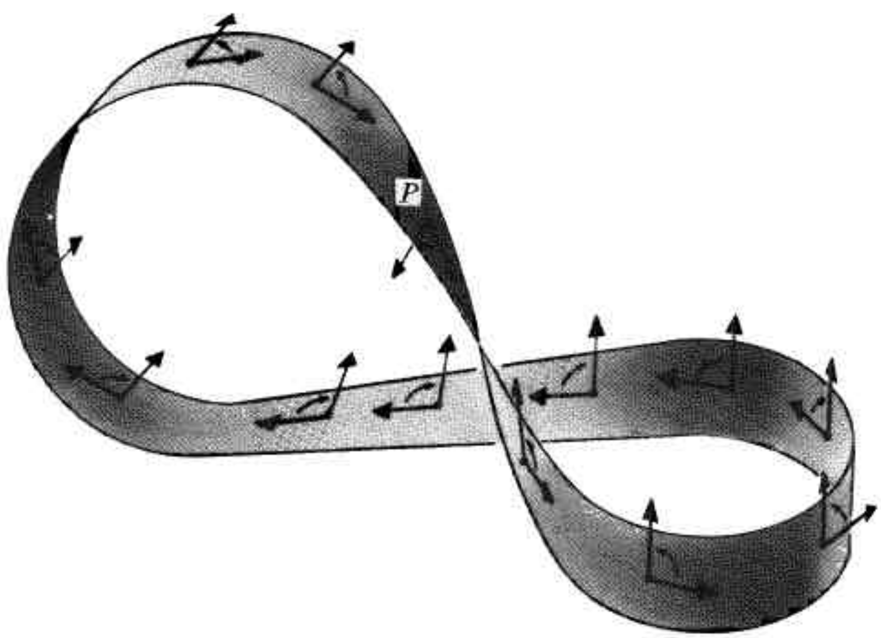



***FIGURE 5-7.*** *The Möbius strip, a non-orientable manifold. A basis begins at $P$, moves to the right and around, and comes back to $P$ with the wrong orientation.*

implies that

$$[(g^{-1} \circ f)_*((e_1)_a), \ldots, (g^{-1} \circ f)_*((e_k)_a)] = [(e_1)_b, \ldots, (e_k)_b],$$

so that $\det (g^{-1} \circ f)' > 0$, an important fact to remember.

A manifold for which orientations $\mu_x$ can be chosen consistently is called **orientable,** and a particular choice of the $\mu_x$ is called an **orientation** $\mu$ of $M$. A manifold together with an orientation $\mu$ is called an **oriented** manifold. The classical example of a non-orientable manifold is the Möbius strip. A model can be made by gluing together the ends of a strip of paper which has been given a half twist (Figure 5-7).

Our definitions of vector fields, forms, and orientations can be made for manifolds-with-boundary also. If $M$ is a $k$-dimensional manifold-with-boundary and $x \in \partial M$, then $(\partial M)_x$ is a $(k-1)$-dimensional subspace of the $k$-dimensional vector space $M_x$. Thus there are exactly two unit vectors in $M_x$ which are perpendicular to $(\partial M)_x$; they can be distinguished as follows (Figure 5-8). If $f\colon W \to \mathbf{R}^n$ is a coordinate system with $W \subset H^k$ and $f(0) = x$, then only one of these unit vectors is $f_*(v_0)$ for some $v_0$ with $v^k < 0$. This unit vector is called the **outward unit normal** $n(x)$; it is not hard to check that this definition does not depend on the coordinate system $f$.

Suppose that $\mu$ is an orientation of a $k$-dimensional manifold-with-boundary $M$. If $x \in \partial M$, choose $v_1, \ldots, v_{k-1} \in (\partial M)_x$ so that $[n(x), v_1, \ldots, v_{k-1}] = \mu_x$. If it is also true that $[n(x), w_1, \ldots, w_{k-1}] = \mu_x$, then both $[v_1, \ldots, v_{k-1}]$ and $[w_1, \ldots, w_{k-1}]$ are the same orientation for $(\partial M)_x$. This orientation is denoted $(\partial\mu)_x$. It is easy to see that the orientations $(\partial\mu)_x$, for $x \in \partial M$, are consistent on $\partial M$. Thus if $M$ is orientable, $\partial M$ is also orientable, and an orientation $\mu$ for $M$ determines an orientation $\partial\mu$ for $\partial M$, called the **induced orientation.** If we apply these definitions to $\mathbf{H}^k$ with the usual orientation, we find that the induced orientation on $\mathbf{R}^{k-1} = \{x \in \mathbf{H}^k\colon x^k = 0\}$ is $(-1)^k$ times the usual orientation. The reason for such a choice will become clear in the next section.

If $M$ is an *oriented* $(n-1)$-dimensional manifold in $\mathbf{R}^n$, a substitute for outward unit normal vectors can be defined,

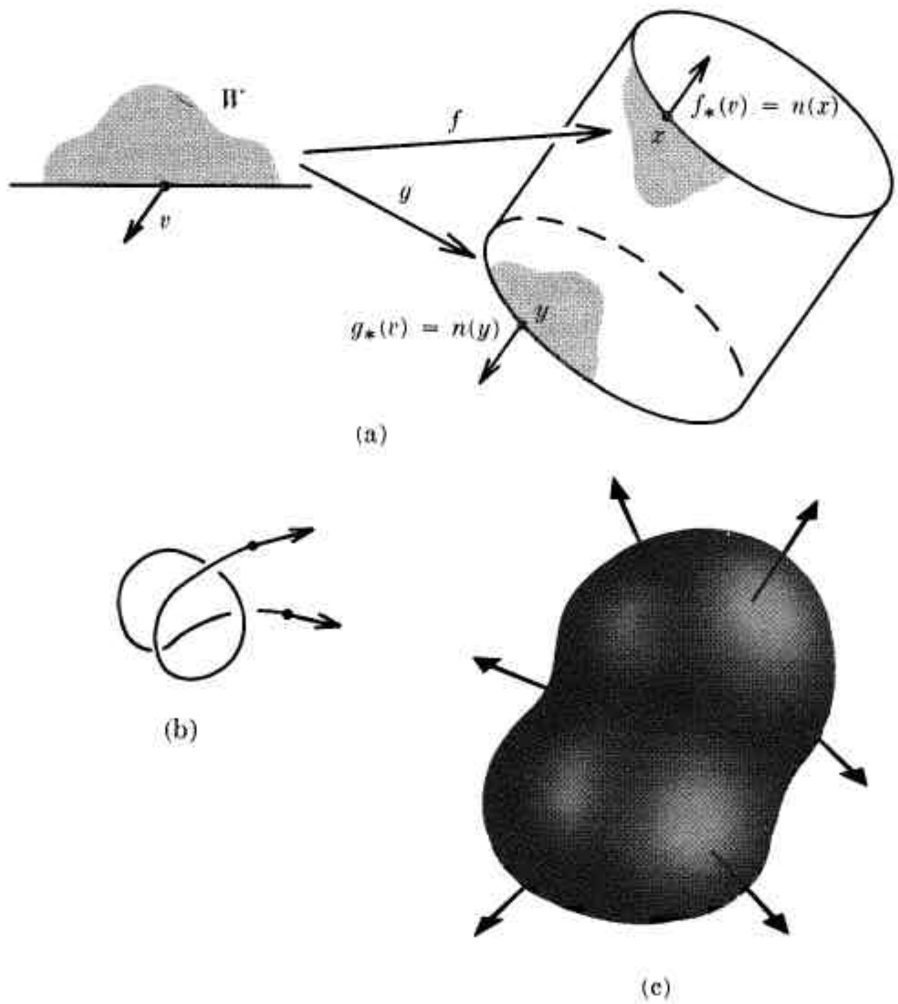



***FIGURE 5-8.*** *Some outward unit normal vectors of manifolds-with-boundary in* $\mathbf{R}^3$.

even though $M$ is not necessarily the boundary of an $n$-dimensional manifold. If $[v_1, \ldots, v_{n-1}] = \mu_x$, we choose $n(x)$ in $\mathbf{R}^n{}_x$ so that $n(x)$ is a unit vector perpendicular to $M_x$ and $[n(x), v_1, \ldots, v_{n-1}]$ is the usual orientation of $\mathbf{R}^n{}_x$. We still call $n(x)$ the outward unit normal to $M$ (determined by $\mu$). The vectors $n(x)$ vary continuously on $M$, in an obvious sense. Conversely, if a continuous family of unit normal vectors $n(x)$ is defined on all of $M$, then we can determine an orientation of $M$. This shows that such a continuous choice of normal vectors is impossible on the Möbius strip. In the paper model of the Möbius strip the two sides of the paper (which has thickness) may be thought of as the end points of the unit normal vectors in both directions. The impossibility of choosing normal vectors continuously is reflected by the famous property of the paper model. The paper model is one-sided (if you start to paint it on one side you end up painting it all over); in other words, choosing $n(x)$ arbitrarily at one point, and then by the continuity requirement at other points, eventually forces the opposite choice for $n(x)$ at the initial point.

***Problems.*** **5-9.** Show that $M_x$ consists of the tangent vectors at $t$ of curves $c$ in $M$ with $c(t) = x$.

**5-10.** Suppose $\mathcal{C}$ is a collection of coordinate systems for $M$ such that (1) For each $x \in M$ there is $f \in \mathcal{C}$ which is a coordinate system around $x$; (2) if $f,g \in \mathcal{C}$, then $\det (f^{-1} \circ g)' > 0$. Show that there is a unique orientation of $M$ such that $f$ is orientation-preserving if $f \in \mathcal{C}$.

**5-11.** If $M$ is an $n$-dimensional manifold-with-boundary in $\mathbf{R}^n$, define $\mu_x$ as the usual orientation of $M_x = \mathbf{R}^n{}_x$ (the orientation $\mu$ so defined is the **usual orientation** of $M$). If $x \in \partial M$, show that the two definitions of $n(x)$ given above agree.

**5-12.** (a) If $F$ is a differentiable vector field on $M \subset \mathbf{R}^n$, show that there is an open set $A \supset M$ and a differentiable vector field $\tilde{F}$ on $A$ with $\tilde{F}(x) = F(x)$ for $x \in M$. *Hint:* Do this locally and use partitions of unity.

(b) If $M$ is closed, show that we can choose $A = \mathbf{R}^n$.

**5-13.** Let $g\colon A \to \mathbf{R}^p$ be as in Theorem 5-1.

(a) If $x \in M = g^{-1}(0)$, let $h\colon U \to \mathbf{R}^n$ be the essentially unique diffeomorphism such that $g \circ h(y) = (y^{n-p+1}, \ldots, y^n)$ and $h(0) = x$. Define $f\colon \mathbf{R}^{n-p} \to \mathbf{R}^n$ by $f(a) = h(0,a)$. Show that $f_*$ is 1-1 so that the $n - p$ vectors $f_*((e_1)_0), \ldots, f_*((e_{n-p})_0)$ are linearly independent.

(b) Show that orientations $\mu_x$ can be defined consistently, so that $M$ is orientable.

(c) If $p = 1$, show that the components of the outward normal at $x$ are some multiple of $D_1g(x), \ldots, D_ng(x)$.

**5-14. If $M \subset \mathbf{R}^n$ is an orientable $(n - 1)$-dimensional manifold, show that there is an open set $A \subset \mathbf{R}^n$ and a differentiable $g\colon A \to \mathbf{R}^1$ so that $M = g^{-1}(0)$ and $g'(x)$ has rank 1 for $x \in M$. *Hint:* Problem 5-4 does this locally. Use the orientation to choose consistent local solutions and use partitions of unity.**

**5-15.** Let $M$ be an $(n - 1)$-dimensional manifold in $\mathbf{R}^n$. Let $M(\varepsilon)$ be the set of end points of normal vectors (in both directions) of length $\varepsilon$ and suppose $\varepsilon$ is small enough so that $M(\varepsilon)$ is also an