Math 620X Lecture Notes

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1.63	May 1, 2022	JH	Lecture 39 on Hopf algebras
1.631	May 3, 2022	JH	Corrected loop alg realization
1.65	May 5, 2022	JH	Lecture 40,41 on Hopf Algebras

Contents

Introduction

Geometry is a good way of dealing with infinite sets. Thus, to study infinite groups, we want to impose some kind of geometric structure on the underlying sets. There are many ways to do this: topological groups, algebraic groups, group schemes, etc. The nicest kind of geometric object, is that of a manifold. Their properties most closely resemble our everyday intuition about curves and surfaces.

A *Lie*[i] *group* is a group which is also a manifold. Tangent spaces may be viewed as linear approximations of manifolds. The tangent space of a Lie group (at the identity element) can be given extra structure (coming from the group operation) making it into a *Lie algebra*. In other words, *Lie algebras are linear approximations of Lie groups.*

Therefore, to properly understand the origin of Lie algebras we must first understand something about Lie groups, and therefore we begin by studying manifolds.

1 Lecture 1

1.1 Manifolds

Reference: Spivak, *Calculus on Manifolds*, Ch. 5.

Throughout, we say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is *differentiable (of class $\mathcal{C}^\infty$)* if all of the partial derivatives of f exist to any order. A function $F : \mathbb{R}^n \to \mathbb{R}^m$, $F(x) = (F_1(x), \ldots, F_m(x))$ is differentiable if all the component functions F_i are. Another word for differentiable is *smooth.*

Definition 1.1. Let $U, V \subset \mathbb{R}^n$ be open sets. Then a map $f \colon U \to V$ is a *diffeomorphism* if f is differentiable and invertible, and f^{-1} is differentiable.

Definition 1.2 (Manifold in $\mathbb{R}^n$)**.** A subset M of $\mathbb{R}^n$ is a k-dimensional *manifold* if $\forall x \in M$ the following condition holds: $\exists$ open sets $U_x, V_x \subset \mathbb{R}^n$ and a diffeomorphism $h_x \colon U_x \to V_x$ such that $x \in U_x$ and (see Figure 1)

$$h_x(U_x \cap M) = V_x \cap (\mathbb{R}^k \times \{0\}) = \{y \in V_x \mid y_{k+1} = \cdots = y_n = 0\}.$$

Remark 1.3.

1. As we have defined it, the manifold (given as a certain subset of $\mathbb{R}^n$) does not depend on the choices of (U_x, V_x, h_x) at each x. Such triples just have to exist.

2. The restriction of h_x to $U_x \cap M$ followed by projection π_k to the first k components provides each point in this portion of the manifold with *coordinates*: $\phi_k = \pi_k \circ h_x(a) = (a_1, a_2, \ldots, a_k)$ for some $a_k \in \mathbb{R}$. We call the pair $(U_x \cap M, \phi_x)$ a *coordinate chart.*

3. An abstract manifold is defined differently, (as a second countable topological space with a smooth atlas). However, the *Whitney Embedding Theorem* says that any abstract manifold of dimension m can be embedded into $\mathbb{R}^{2m}$. Thus the "concrete" definition given here is equivalent to the abstract one.

Example 1.4. Any open subset of $\mathbb{R}^n$ is an n-dimensional manifold.

Example 1.5. Any singleton $\{x\}$ is a zero-dimensional manifold.

[i] "Lie" (pronounced *LEE*) refers to the Norwegian mathematician Sophus Lie, who himself called them *continuous groups.*

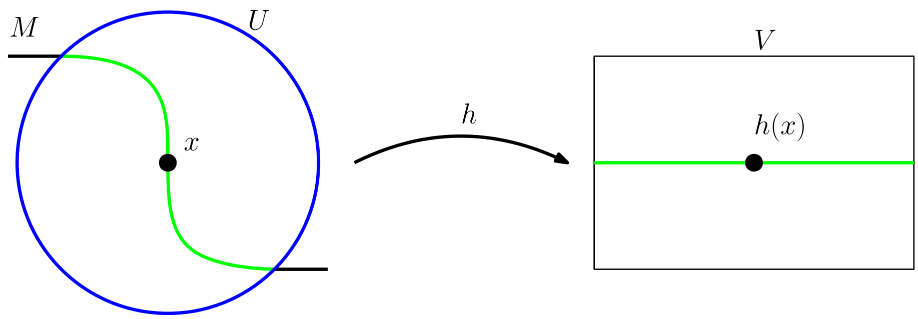


Figure 1: Manifold visualization

Example 1.6. Any linear subspace V of $\mathbb{R}^n$ is a k-dimensional manifold, where $k = \dim_{\mathbb{R}} V$.

Exercise 1.7. Prove that if M is a k-dimensional manifold in $\mathbb{R}^m$ and N is an l-dimensional manifold in $\mathbb{R}^n$ then $M \times N$ is naturally a $(k+l)$-dimensional manifold in $\mathbb{R}^{m+n}$.

How do we prove that something more interesting is a manifold?

Theorem 1.8 (Implicit Function Theorem)**.** *Let $n \geqslant p \geqslant 0$ be integers. Let $A \subset \mathbb{R}^n$ be an open subset and $g\colon A \to \mathbb{R}^p$ be a differentiable function such that $g'(x) = \left(\dfrac{\partial g_i}{\partial x_j}\right)_{ij}$ has rank p whenever $g(x) = 0$. Then $g^{-1}(\{0\})$ is an $(n-p)$-dimensional manifold.*

For a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ we write $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ for the standard Euclidean norm.

Example 1.9. The n-sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid |x|^2 = 1\}$ is a manifold. To prove it, let $g\colon \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $g(x) = |x|^2 - 1$. Then $g'(x) = \nabla g(x) = \begin{bmatrix} 2x_1 & 2x_2 & \cdots & 2x_{n+1} \end{bmatrix}$. If $g'(x) = 0$ (the zero vector) then $x_i = 0$ for all i, hence $g(x) = -1$. So whenever $g(x) = 0$ we have $g'(x) \neq 0$ (thus, viewed as a $1 \times (n+1)$-matrix, has rank 1). By Theorem 1.8, $g^{-1}(\{0\}) = S^n$ is an n-dimensional manifold.

Exercise 1.10. The *special linear group*, denoted $SL_2(\mathbb{R})$, is the set of all real 2×2 matrices of determinant one. Show that $SL_2(\mathbb{R})$ is a 3-dimensional manifold. (The set of all real 2×2-matrices can be identified with $\mathbb{R}^4$.) Can you generalize this to $SL_3(\mathbb{R})$? $SL_n(\mathbb{R})$?

Definition 1.11. A *morphism of manifolds* (or *differentiable map*, or *smooth map*)

$$f\colon M \to N,$$

where M and N are manifolds of dimension k and ℓ respectively, is a function such that $\forall x \in M$, the function (see Figure 2)

$$\widetilde{f}_x = h_{f(x)} \circ f \circ h_x^{-1}\colon V_x \cap \left(\mathbb{R}^k \times \{0\}\right) \to V_{f(x)} \cap \left(\mathbb{R}^\ell \times \{0\}\right)$$

is differentiable. If furthermore f is invertible and its inverse is a morphism of manifolds, then f is an *isomorphism of manifolds* (or *diffeomorphism*). When such an f exists, M and N are called *diffeomorphic*.

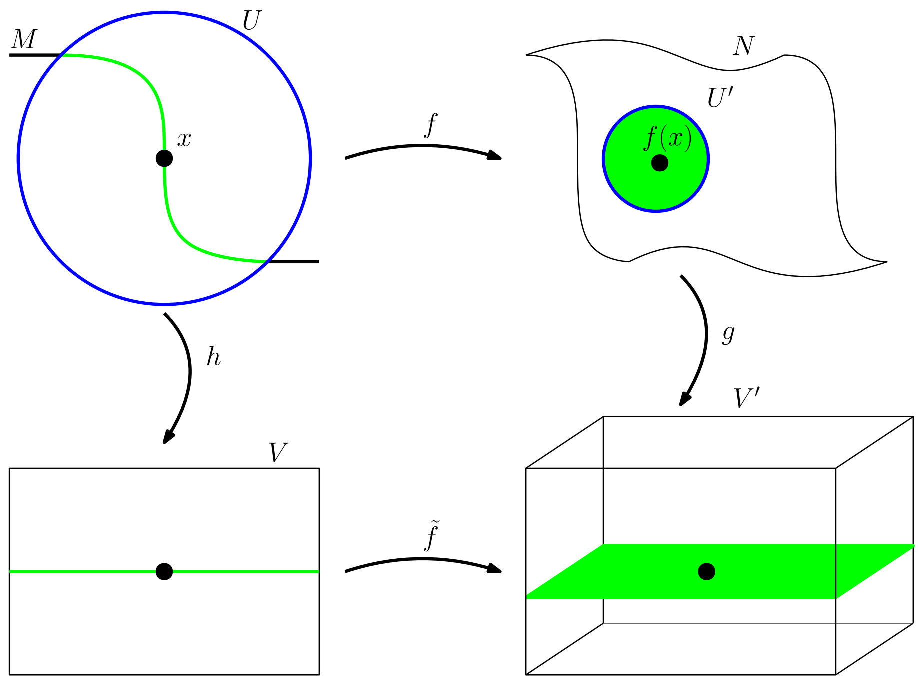


Figure 2: Morphism visualization

Definition 1.12. A *Lie group* G is a group which is also a manifold such that the maps,

$$G \times G \to G \text{ by } (g, h) \mapsto gh$$

and

$$G \to G \text{ by } g \mapsto g^{-1}$$

are differentiable maps (i.e morphisms of manifolds). A map $\phi\colon G \to H$ is a *morphism of Lie groups* if it is a morphism of manifolds and a group homomorphism.

Example 1.13. $(\mathbb{R}^n, +)$ is a Lie group.

Example 1.14. $S^1 = U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$ is a 1-dimensional (real) Lie group with respect to multiplication.

Example 1.15. $GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$ is an open subset of $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$, so it is a manifold. Matrix multiplication (respectively, matrix inverse) is given by polynomials (respectively, rational functions) of the entries and are therefore smooth maps. Thus $GL(n, \mathbb{R})$ is a Lie group.

Why is it open? $\det\colon \mathbb{R}^{n^2} \to \mathbb{R}$ is continuous which implies that $\det^{-1}(\mathbb{R}\backslash\{0\})$ is open in $\mathbb{R}^{n^2}$.

Example 1.16. $SU(2) = \{A \in M_2(\mathbb{C}) \mid AA^* = A^*A = I, \text{ and } \det(A) = 1\}$ is a 3-dimensional Lie group diffeomorphic to S^3 (see Example 2.5(5) in Kirillov, Jr.).

Remark 1.17. For the definition of *complex Lie Groups*, replace "differentiable" with "complex analytic". Unless otherwise emphasized, any theorem about Lie groups holds in both the real and complex cases.

2 Lecture 2

2.1 Connectedness

Let M be a manifold. Define a binary relation $\sim$ on M by $\forall x, y \in M$:

$$x \sim y \Leftrightarrow \exists \text{ continuous map } \gamma\colon [0,1] \to M \text{ with } \gamma(0) = x, \gamma(1) = y$$

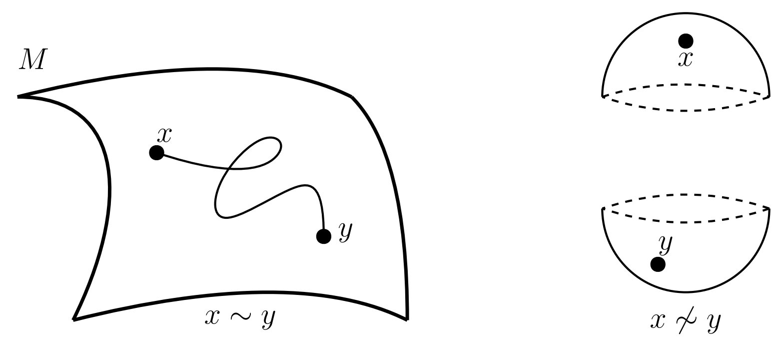


Figure 3: An example and nonexample

If $x \sim y$ we say that x *is connected to* y.

Exercise 2.1. Prove $\sim$ is an equivalence relation.

The equivalence classes

$$[x] = \{y \in M \mid y \sim x\}$$

are *connected components*. The set of equivalence classes of $M/\sim$ is denoted by $\pi_0(M)$. If $x \sim y$ for all $x, y \in M$ we say that M is *connected*.

Example 2.2. The n-sphere S^n is a connected manifold for any $n \geqslant 1$.

Proof. Let $x, y \in S^n$, $x \neq y$. We will show $x \sim y$. By transitivity of connectedness, it suffices to do this in the case $x + y \neq 0$, that is, for non-antipodal pairs of points. For such x, y the straight line in $\mathbb{R}^{n+1}$ through x and y does not pass through the origin. Define $\gamma\colon [0,1] \to S^n$ by

$$\gamma(t) = \frac{(1-t)x + ty}{|(1-t)x + ty|}.$$

The numerator defines a line segment in $\mathbb{R}^{n+1}$. Dividing by the norm forces $|\gamma(t)| = 1$ so that $\gamma(t) \in S^n$ for all t (see Figure 4). Then γ is a continuous function such that $\gamma(0) = x$ and $\gamma(1) = y$. Thus $x \sim y$. □

The following simple fact can be used to show that a manifold is disconnected.

Lemma 2.3. *Suppose $f : M \to N$ is a surjective continuous map between manifolds. If M is connected, then N is connected. (Equivalently, if N is disconnected, then M must be disconnected.)*

Proof. $f \circ \gamma$ is a continuous map from $[0,1]$ to N connecting $f(x)$ and $f(y)$ whenever $\gamma\colon [0,1] \to M$ is a continuous map connecting $x, y \in M$. □

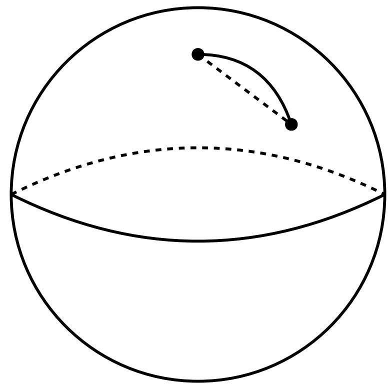


Figure 4: Visualization of γ used to prove S^n is connected

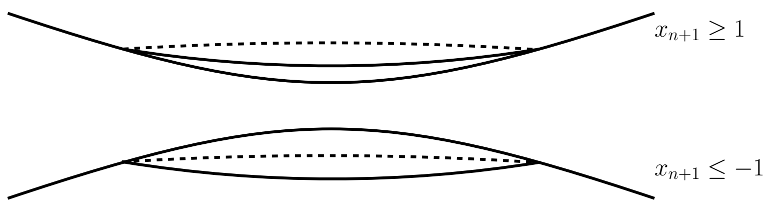


Figure 5: The n-dimensional hyperboloid H^n

Exercise 2.4. The n-dimensional hyperboloid $H^n = \{x \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_n^2 = x_{n+1}^2 - 1\}$ is an n-dimensional manifold by the implicit function theorem applied to the function $g(x) = x_1^2 + \cdots + x_n^2 - x_{n+1}^2 - 1$. Prove that H^n has two connected components. (See Figure 5.) *Hint:* The projection map $\mathbb{R}^{n+1} \to \mathbb{R}$ onto the last coordinate is a continuous map. Likewise, the projection map $\mathbb{R}^{n+1} \to \mathbb{R}^n$ onto the first n coordinates is continuous.

Definition 2.5. A subset N of a manifold $M \subset \mathbb{R}^n$ is an *open submanifold* if there is an open subset U of $\mathbb{R}^n$ such that $N = U \cap M$.

Proposition 2.6. *Let $M \subset \mathbb{R}^n$ be a manifold and $x \in M$. Then the connected component $[x]$ is an open submanifold.*

Proof. For any $y \in [x]$ pick an open set $U_y \subset \mathbb{R}^n$ as in the definition of a manifold. Then any $z \in M \cap U_y$ is connected to y, hence to x (see Figure 6). So $M \cap U_y \subset [x]$. Let $U = \bigcup_{y \in [x]} U_y$. Then U is open in $\mathbb{R}^n$ and $U \cap M = [x]$. $\square$

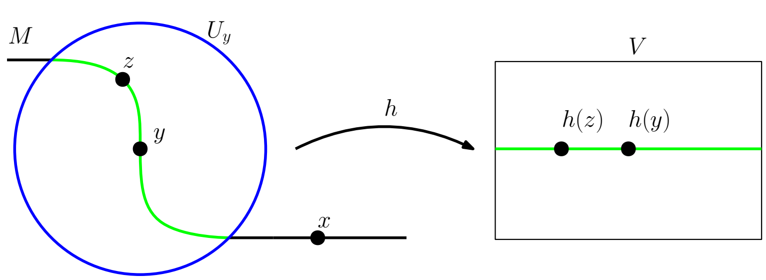


Figure 6: Visual justification that for any $z \in M \cap U_y$, $z \sim y$

2.2 Connected Lie Groups

Definition 2.7. A Lie group G is *connected* if it is connected as a manifold.

Notation 2.8. If G is a Lie group then G^0 denotes the connected component of G that contains the identity element. $G^0 = [e_G]$.

Remark 2.9. By Proposition 2.6 above, G^0 is an open submanifold of G.

Example 2.10. Any finite group G can be viewed as a 0-dimensional Lie group by placing its elements on the real line:

$$\bullet \quad \bullet \quad \overset{e}{\bullet} \quad \bullet \quad \bullet$$

In this case the connected component $G^0 = \{e\}$ only contains the identity element.

Example 2.11. The *orthogonal group* $O(n, \mathbb{R})$ is the group of orthogonal matrices:

$$O(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid A^T A = I = AA^T\}.$$

One can show, using the implicit function theorem, that $O(n, \mathbb{R})$ is a manifold of dimension $n(n-1)/2$. Since matrix multiplication and inverse are smooth, $O(n, \mathbb{R})$ is a Lie group. This group has two connected components consisting of orientation-preserving and orientation-reversing transformations respectively. In more detail, the determinant map $\det : O(n, \mathbb{R}) \to \{\pm 1\}$ sends an orthogonal matrix A to $\det(A)$ which is ± 1 since $1 = \det(I) = \det(AA^T) = \det(A)\det(A^T) = \det(A)^2$. Since det is surjective and continuous and $\{\pm 1\}$ is disconnected, $O(n, \mathbb{R})$ must have at least two connected components by Lemma 2.3. Moreover $O(n, \mathbb{R})^0 \subset \{A \in O(n, \mathbb{R}) \mid \det(A) = 1\} = SO(n, \mathbb{R})$ which is the *special orthogonal group*.

In fact, $SO(n, \mathbb{R})$ is connected, hence $O(n, \mathbb{R})^0 = SO(n, \mathbb{R})$. Furthermore, the matrix $g = \mathrm{diag}(-1, 1, 1, \ldots, 1)$ is orthogonal of determinant -1. Multiplication by g provides a diffeomorphism between the two connected components.

Theorem 2.12. *Let G be a Lie group. Then G^0 is a normal subgroup and is itself a Lie group. The quotient group G/G^0 is* discrete*, meaning each coset gG^0 is an open submanifold of G.*

Proof. $e \in G^0$ by definition. If $f\colon M \to N$ is continuous then $x \sim y$ in $M \Rightarrow f(x) \sim f(y)$ in N (Exercise). In particular, $f([x]) \subset [f(x)]$. Apply to $i\colon G \to G$, $i(g) = g^{-1}$ gives $i(G^0) \subset G^0$. Similarly, $m\colon G \times G \to G$, $m(g, h) = gh \Rightarrow m(G^0 \times G^0) \subset [m(e, e)] = G^0$. Lastly, fix $g \in G$, $c(h) = ghg^{-1}$ Then $c(G^0)) \subset [c(e)] = [e] = G^0$. Thus G^0 is a normal subgroup. Since $gG^0 = [g]$ each coset in G/G^0 is an open submanifold by Proposition 2.6, so G/G^0 is discrete. □

Example 2.13. Let $G = O(n, \mathbb{R})$. Then the determinant map $\det : G \to \{\pm 1\}$ is surjective with kernel equal to G^0. Thus $G/G^0 \cong \{\pm 1\}$.

3 Lecture 3

3.1 Simple Connectedness

Let M be a connected manifold and fix $x_0 \in M$, called a *base point*.

Definition 3.1. A *path* in M is a continuous map $\gamma\colon [0, 1] \to M$ such that $\gamma(0) = x_0$. γ is a *loop* if $\gamma(0) = \gamma(1)$. The *constant loop* γ_0 is given by $\gamma_0(t) = x_0$ $\forall t \in [0, 1]$.

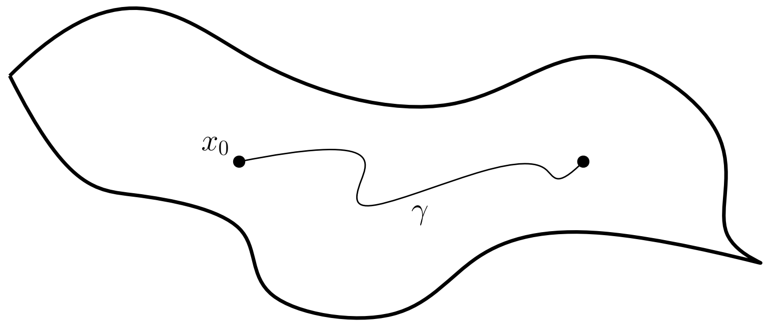


Figure 7: Example of a path γ with base point x_0

Two paths are homotopic if one can be continuously deformed into the other. The precise definition is as follows.

Definition 3.2. Let x_0 and x_1 be points in M. Two paths γ, δ in with $\gamma(0) = \delta(0) = x_0$ and $\gamma(1) = \delta(1) = x_1$ are *homotopic* if $\exists$ continuous map $h\colon [0,1]^2 \to M$ such that

$$h(0,s) = x_0 \text{ and } h(1,s) = x_1 \text{ for all } s \in [0,1],$$

and

$$h(t,0) = \gamma(t) \text{ and } h(t,1) = \delta(t) \text{ for all } t \in [0,1].$$

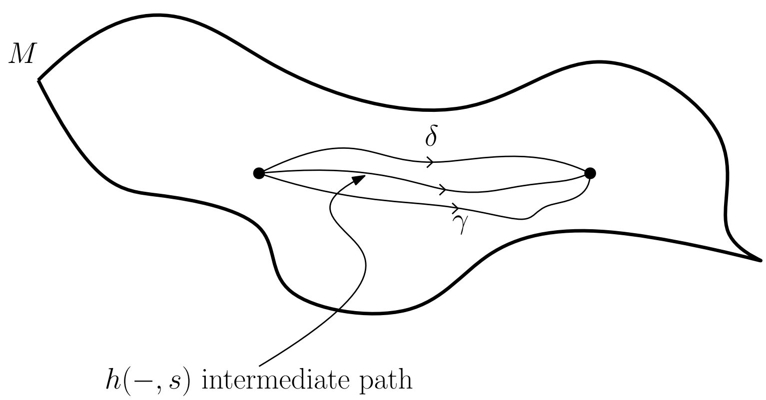


Figure 8: Example of homotopic paths

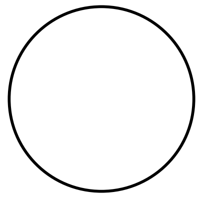
Definition 3.3. M is *simply connected* if every loop in M is homotopic to the constant loop. (Note: this is independent of the choice of x_0)

Example 3.4. In figure 9 we see that $\mathbb{R}^2$ is a simply connected manifold, while S^1 is not.

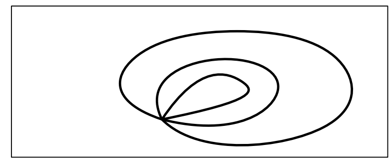
Example 3.5. The *projective plane* $\mathbb{RP}^2$ (in the Poincaré model) is D^1 but opposite points on S^1 identified: $D^1/\sim$ where $x \sim y$ iff $|x| = |y| = 1$ & $x + y = 0$. Then $\mathbb{P}^1$ is <u>not</u> simply connected. See figures 10a and 10b.

Definition 3.6. The *product* of two loops γ, δ in M is $\gamma * \delta\colon [0,1] \to M$

$$\gamma * \delta(t) = \begin{cases} \gamma(2t) & 0 \leqslant t \leqslant \frac{1}{2} \\ \delta(2(t - \frac{1}{2})) & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

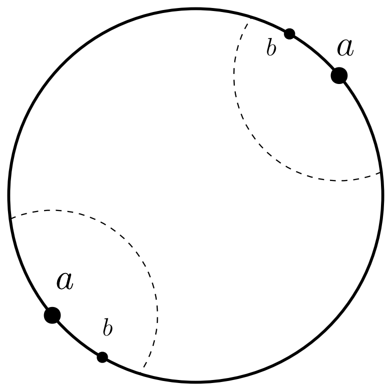


(a) S^1

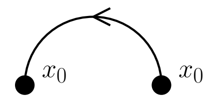


(b) $\mathbb{R}^2$

Figure 9: An example and non-example of simply connected manifolds



(a) Depiction of $\mathbb{RP}^2$ with identical points labeled



(b) A loop not homotopic to γ_0

Figure 10

Exercise 3.7. Homotopy defines an equivalence relation on the set of loops in M. The set of equivalence classes is denoted $\pi_1(M, x_0)$.

Exercise 3.8. $\pi_1(M, x_0)$ is a group with respect to the operation:

$$[\gamma][\delta] = [\gamma * \delta].$$

$\pi_1(M, x_0)$ is the (1st) *fundamental group* of M (a.k.a. *Poincaré group* of M).

Exercise 3.9. $\pi_1(M, x_0) \cong \pi_1(M, y_0)$ for any $x_0, y_0 \in M$ (Recall: we assume M connected).

Example 3.10. $\pi_1(\mathbb{P}^2) \cong \mathbb{Z}/2\mathbb{Z}$, notice $[\gamma]^2 = [\gamma_0]$ in figure 11

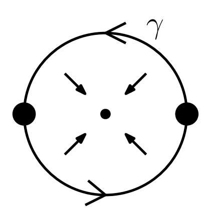


Figure 11

Example 3.11. $\pi_1(S^1) \cong \mathbb{Z}$. The correspondence is the winding number.

3.2 The Universal Cover

Some convenient terminology:

- A subset $U \subset M$ of a manifold M in $\mathbb{R}^n$ is *open in* M, or simply *open* (when no confusion can arise), if $U = M \cap A$ for some open set $A \subset \mathbb{R}^n$.

- A *neighborhood (abbreviated nbh) of* x is a set containing x.

Definition 3.12. Let M be a connected manifold. A *cover* $(\widetilde{M}, p)$ (covering space) for M is a connected manifold $\widetilde{M}$ together with a morphism $p\colon \widetilde{M} \to M$ such that: $\forall x \in M$, $\exists$ connected open neighborhood $U \subset M$ of x such that every connected component of $p^{-1}(U)$ diffeomorphically onto U. $(\widetilde{M}, p)$ is a *universal cover* if it is simply connected. Often we just write $\widetilde{M}$ for $(\widetilde{M}, p)$.

Example 3.13. $(\mathbb{R}, p)$ is a universal cover for S^1 where $p\colon x \mapsto e^{2\pi i x}$.

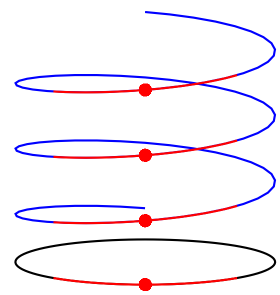


Figure 12: Visualization of $(\mathbb{R}, p)$ as covering space over S^1

Theorem 3.14. *Every connected manifold has a universal cover. Moreover, it is unique up to diffeomorphism.*

Proof (sketch). Pick a base point $x_0 \in M$. Define $\widetilde{M}$ to be the set of homotopy classes of paths in M starting at x_0 (see Figure 13). Define $p : \widetilde{M} \to M$, $p(\gamma) = \gamma(1)$. One can show that this is a universal cover. $\square$

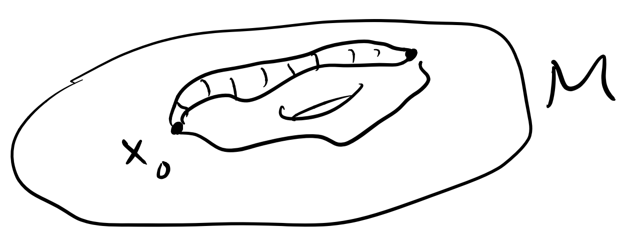


Figure 13: Visualization of Homotopy Classes

Theorem 3.15. *Any morphism of connected manifolds $f\colon M \to N$ can be lifted to a morphism of their respective universal covers $\widetilde{f}\colon \widetilde{M} \to \widetilde{N}$.*

Theorem 3.16. *If G is a connected Lie group, then its universal cover $\widetilde{G}$ has a canonical structure of a Lie group such that*

i) $p\colon \widetilde{G} \to G$ *is a morphism of Lie groups*

ii) $\ker p = \pi_1(G, e)$.

Moreover, $\ker p$ *is a discrete subgroup of* $\widetilde{G}$*, and* $\ker p \subset Z(\widetilde{G})$ *the center of* $\widetilde{G}$*.*

Example 3.17. $G = S^1 \times S^1 \times \mathbb{Z}$. The connected component at the identity element is $G^0 = S^1 \times S^1 \times \{0\}$. The universal cover of G^0 is $\widetilde{G^0} = \mathbb{R}^2$. See Figure 14.

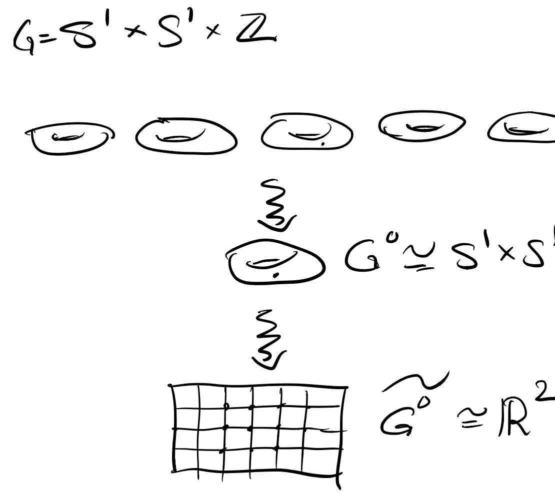


Figure 14: $G = S^1 \times S^1 \times \mathbb{Z}$

4 Lecture 4

4.1 Coordinate Systems

A coordinate system is a local parametrization of a manifold. Coordinate systems are used to construct tangent spaces.

Let M be a manifold and $x \in M$. Let (U_x, V_x, h_x) be a triple as in the definition of a manifold. Let W_x be the image of the projection of $V_x \cap \left(\mathbb{R}^k \times \{0\}\right)$ to the first k components.

Definition 4.1. The map $f_x : W_x \to M$ given by $f_x(a_1, a_2, \ldots, a_k) = h_x^{-1}(a_1, \ldots, a_k, 0, \ldots, 0)$ (with $n - k$ trailing zeros) is called a *coordinate system* around x.

Since h_x is a diffeomorphism, the derivative $h_x'(y)$ has rank n at all $y \in V_x$. Consequently f_x' has rank k at any point in W_x. Furthermore $f_x^{-1} : F_x(W_x) \to W_x$ is continuous. The following theorem shows that the data $\left\{(U_x, V_x, h_x)\right\}_{x \in M}$ can in fact be replaced by the data $\left\{(W_x, f_x)\right\}_{x \in M}$.

Theorem 4.2 (See Spivak, Theorem 5-2). *A subset M of $\mathbb{R}^n$ is a manifold iff $\forall x \in M$ there is an open set $U \subset \mathbb{R}^n$, $x \in U$, and an open set $W \subset \mathbb{R}^k$ and an injective differentiable function $f\colon W \to \mathbb{R}^n$ such that*

1) $f(W) = M \cap U$
2) $f'(y)$ *has rank* k $\forall y \in W$
3) $f^{-1}\colon f(W) \to W$ *is continuous.*

4.2 Tangent Space

Definition 4.3. Let M k-dimensional manifold, $x \in M$, $f\colon W \to \mathbb{R}^n$ be a coordinate system around x, and $a = f^{-1}(x)$. Since $f'(a)$ has rank k, the image of $f'(a)\colon \mathbb{R}^k \to \mathbb{R}^n$ given by the matrix

$$\begin{bmatrix} \partial_1 f_1(a) & \partial_2 f_1(a) & \cdots & \partial_k f_1(a) \\ \vdots & \vdots & & \vdots \\ \partial_1 f_n(a) & \partial_2 f_n(a) & \cdots & \partial_k f_n(a) \end{bmatrix}$$

is a k-linear subspace of $\mathbb{R}^n$ called the *tangent space of M at the point x*, denoted T_xM.

Often we draw/think of T_xM as the affine space $x + f'(a)\big(\mathbb{R}^n\big)$.

Note 4.4. By the chain rule, T_xM is independent of the choice of coordinate system f.

Example 4.5. Let us describe the tangent space of S^2 at $(0,0,1)$. Let $W = \{(s,t) \mid s^2 + t^2 < 1\}$ and $f\colon W \to \mathbb{R}^3$, $f(s,t) = (s, t, \sqrt{1 - s^2 - t^2})$. Then f is a coordinate system around $x = (0,0,1)$. Let $a = (0,0)$ (notice $f(a) = x$).

$$f'(a) = \begin{bmatrix} \partial_s f_1 & \partial_t f_1 \\ \partial_s f_2 & \partial_t f_2 \\ \partial_s f_3 & \partial_t f_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

so $T_xS^2 = f'(a)\big(\mathbb{R}^2\big) = \mathbb{R}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \oplus \mathbb{R}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Example 4.6. (See Figure 15.) A coordinate system for the 2-dimensional torus M embedded in $\mathbb{R}^3$ is given by (here $R > r > 0$ are the two radii):

$$f(s,t) = \big((R + r\cos s)\cos t,\ (R + r\cos s)\sin t,\ r\sin s\big), \qquad (s,t) \in (-\pi,\pi)^2$$

This is a coordinate system around any point except for the point $f(\pi,\pi) = (r - R, 0, 0)$. To get a coordinate system around that point one can use the same expression for f but change the domain to $(0, 2\pi)$. We can use this coordinate system to find the tangent space for a point $p = (x,y,z) = f(s,t)$. As an example, take $s = \pi/3$ and $t = \pi/4$. The derivative (matrix) of f is

$$f'(s,t) = \begin{bmatrix} (R - r\sin s)\cos t & -(R + r\cos s)\sin t \\ (R - r\sin s)\sin t & (R + r\cos s)\cos t \\ r\cos s & 0 \end{bmatrix}$$

Thus the tangent space T_pM at $p = f(\pi/3, \pi/4) = \big((R + r/2)/\sqrt{2},\ (R + r/2)/\sqrt{2},\ r\sqrt{3}/2\big)$ equals

$$T_pM = \mathbb{R}\begin{bmatrix} (R - r\sqrt{3}/2)/\sqrt{2} \\ (R - r\sqrt{3}/2)/\sqrt{2} \\ r/2 \end{bmatrix} + \mathbb{R}\begin{bmatrix} -(R + r/2)/\sqrt{2} \\ (R + r/2)/\sqrt{2} \\ 0 \end{bmatrix}$$

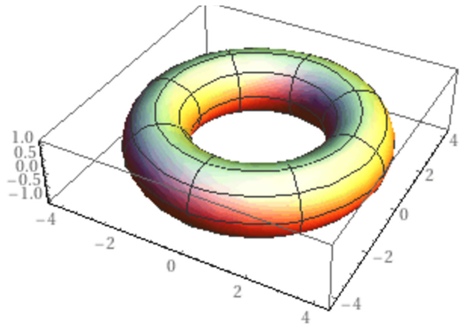


Figure 15: Torus with $R = 3$ and $r = 1$.

4.3 Vector Fields

Definition 4.7. A *vector field on* M is a map $F\colon M \to \mathbb{R}^n$ such that $F(x) \in T_x M$ for all $x \in M$. F is called *differentiable* if for every coordinate system $f\colon W \to \mathbb{R}^n$ at any $x \in M$, $a \mapsto f'(a)\big(F(f(a))\big)$[ii] is a differentiable vector field on W.

4.4 The Derivative (Differential) of a Morphism

If $f\colon M \to N$ is a morphism of manifolds there is an induced linear map $df = f_* = f'(p)\colon T_pM \to T_{f(p)}N$ given as follows

$$\begin{aligned} \alpha\colon U \to \mathbb{R}^n \quad & \alpha(a) = p \\ \beta\colon V \to \mathbb{R}^n \quad & \beta(b) = f(p) \\ T_pM &= \alpha'(a)\big(\mathbb{R}^k\big) \\ T_{f(p)}N &= \beta'(b)\big(\mathbb{R}^\ell\big) \end{aligned}$$

Define $f'(p)$ by the formula

$$(\beta^{-1} \circ f \circ \alpha)'(a) = (\beta'(b))^{-1} \circ f'(p) \circ \alpha'(a).$$

4.5 Vector Fields on Lie Groups

Fix $g \in G$. There are three maps $G \to G$:

$$\begin{aligned} \mathrm{L}_g &\colon h \mapsto gh \\ \mathrm{R}_g &\colon h \mapsto hg^{-1} \\ \mathrm{Ad}_g &\colon H \mapsto ghg^{-1} \end{aligned}$$

They are morphisms of manifolds $\Rightarrow$ have differentials.

Example 4.8. $d\mathrm{L}_g\colon T_1G \to T_gG$, $x \mapsto g.x$

Definition 4.9. A vector field F on a Lie group G is *left-invariant* if $g.F(x) = F(gx)$ for all $g, x \in G$.

[ii] this mapping is single-valued

Definition 4.10. The *Lie algebra* of a Lie group G is defined by

$$\mathfrak{g} = \operatorname{Lie} G = \{\text{all left-invariant v.fields on } G\}.$$

Theorem 4.11. *$v \mapsto v(1)$ is a linear isomorphism of $\mathfrak{g}$ with $T_1(G)$.*

Proof. Let $x \in T_1(G)$ Define $v(g) = g.x$. Then v is left invariant. Uniqueness is obvious. □

5 Lecture 5

5.1 Classical groups

The following is an assortment of Lie groups called *classical groups*. These are the most important but not an exhaustive list; the are many more. (Once we classify simple complex Lie algebras we can give a general definition of classical Lie group: A connected Lie group is *classical* if its complexified Lie algebra is a classical Lie algebra.)

- The *general linear group*:

  $$GL(n, \mathbb{R}) = \{\text{all invertible } n \times n \text{ matrices with real entries}\},$$

- The *special linear group*:

  $$SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid \det A = 1\}.$$

- The *orthogonal group*:

  $$\begin{aligned} O(n) &= \{A \in GL(n, \mathbb{R}) \mid A^T A = AA^T = I_n\} \\ &= \{A \in GL(n, \mathbb{R}) \mid \langle Ax, Ay \rangle = \langle x, y \rangle \ \forall x, y \in \mathbb{R}^n\}, \end{aligned}$$

  where $\langle \cdot, \cdot \rangle$ is the standard dot product: the unique (up to change of basis) positive definite, non-degenerate, symmetric, bilinear form on $\mathbb{R}^n$.

- The *special orthogonal group*:

  $$SO(n, \mathbb{R}) = SL(n, \mathbb{R}) \cap O(n).$$

- The *orthogonal group of signature* (p, q):

  $$O(p, q) = \{A \in GL(n, \mathbb{R}) \mid \langle Ax, Ay \rangle_{p,q} = \langle x, y \rangle_{p,q}\},$$

  where $\langle \cdot, \cdot \rangle_{p,q}$ is the (unique up to change of basis) nondegenerate, symmetric, bilinear form on $\mathbb{R}^n$ of signature (p, q): $\langle x, y \rangle_{p,q} = \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{p+q} x_i y_i$.

- The corresponding "special" version is as expected:

  $$SO(p, q) = O(p, q) \cap SL(n, \mathbb{R}).$$

- The *symplectic group*:

$$\begin{aligned} Sp(n) &= \{A \in GL(2n,\mathbb{R}) \mid \omega(Ax,Ay) = \omega(x,y)\ \forall x,y \in \mathbb{R}^{2n}\} \\ &= \{A \in GL(2n,\mathbb{R}) \mid A^T J A = J\}, \end{aligned}$$

  where ω is the (unique up to change of basis) non-degenerate, skew-symmetric, bilinear form on $\mathbb{R}^{2n}$ given by $\omega(x,y) = \sum_{i=1}^{n}(x_i y_{i+n} - y_i x_{i+n})$, and J is the Gram matrix of the form given by $2n \times 2n$-matrix $\begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix}$.

- The *unitary group*:

$$U(n) = \{A \in GL(n,\mathbb{C}) \mid A^*A = AA^* = I_n\},$$

  where $A^* = \overline{A^T}$ hermitian adjoint (conjugate-transpose).

- The *special unitary group*:

$$SU(n) = U(n) \cap SL(n,\mathbb{C}).$$

Exercise 5.1. Special symplectic group? Show that $Sp(n,\mathbb{R}) \subset SL(2n,\mathbb{R})$, so it is already "special".

5.2 The Exponential Map

How can one prove that the above matrix groups are actually Lie groups? There are three methods.

Method 1: Implicit function theorem (see Theorem 1.8). For example, $O(n,\Bbbk) = \{A = (a_{ij}) \mid AA^T = I_n\}$ is defined by $n(n+1)/2$ equations in $\Bbbk^{n^2}$ (since AA^T is symmetric). Compute Jacobian of this system and show it has full rank (i.e. rank $n(n+1)/2$). This can be computationally difficult.

Method 2: Observe that $O(n,\Bbbk)$ forms a closed subset of $GL(n,\Bbbk)$ and use a theorem about closed Lie subgroups (next time).

Method 3: Using the so called *the exponential map* as a coordinate system around the identity element of G.

To define this exponential map, first recall that $GL(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$. Hence the identity map $GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$ is a coordinate system around the identity map. Its derivative is the identity matrix. Therefore the Lie algebra of $GL(n,\mathbb{R})$, identified with the tangent space at the identity, is all of $\mathbb{R}^{n^2}$. In other words,

$$\mathfrak{gl}(n,\mathbb{R}) := \text{Lie algebra of } GL(n,\mathbb{R}) \cong T_1 GL(n,\mathbb{R}) = \{\text{all } n \times n \text{ matrices}\}.$$

The power series

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges absolutely in matrix norm[iii] for any $n \times n$ matrix x. So it defines a smooth (in fact, analytic) map of manifolds:

$$\exp\colon \mathfrak{gl}(n,\mathbb{R}) \to \mathfrak{gl}(n,\mathbb{R}).$$

[iii] The matrix norm is $\|x\| = \sup_{v \in \mathbb{R}^n, |v|=1} |xv|$

Furthermore,
$$\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k}$$
is smooth (in fact analytic) near $1 \in \mathfrak{gl}(n,\mathbb{R})$, where "near" means in the matrix norm.

Theorem 5.2.

1) $\log(\exp(x)) = x$ *and* $\exp(\log(x)) = x$ *whenever defined.*
2) $\exp(0) = 1, \exp'(0) = \mathrm{Id}$*, where* $\exp'(0)$ *denotes the derivative of* $\exp$ *at* 0*.*
3) *If* $xy = yx$ *then* $\exp(x+y) = \exp(x)\exp(y)$*. If* $xy = yx$ *then* $\log(xy) = \log(x) + \log(y)$ *near 1.*
4) *For fixed* $x \in \mathfrak{gl}(n,\mathbb{R})$ $t \mapsto \exp(tx)$ *is a morphism of Lie groups* $\mathbb{R} \to GL(n,\mathbb{R})$ *(a* one-parameter subgroup*).*
5) $\exp(AxA^{-1}) = A\exp(x)A^{-1}$ *and* $\exp(x^t) = (\exp(x))^t$

Theorem 5.3 (Thm 2.30 in Kirillov, Jr.)**.** *For each classical group* $G \subset GL(n,\mathbb{R})$ *there is a vector space* $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{R})$ *such that for some neighborhood* U *of* 1 *in* $GL(n,\mathbb{R})$ *and some neighborhood* u *of* 0 *in* $\mathfrak{gl}(n,\mathbb{R})$ *the logarithm and exponential maps restrict to diffeomorphisms*

$$\begin{tikzcd} U \cap G \arrow[r, bend left, "\log"] & u \cap \mathfrak{g} \arrow[l, bend left, "\exp"] \end{tikzcd}$$

Proof. This is done case by case. We will look at a few cases in the next lecture. □

Corollary 5.4 (Corollary 2.31 in Kirillov, Jr.)**.** *Each classical group* $G \subset GL(n,\mathbb{R})$ *is a Lie group with tangent space* $T_1G \cong \mathfrak{g}$ *and* $\dim G = \dim \mathfrak{g}$*.*

Proof. $\exp\colon u \cap \mathfrak{g} \to U \cap G$ is a coordinate system around $1 \in G$, by Theorem 5.3. Furthermore, $\exp'(0) = \mathrm{Id}$ which has full rank, hence has full rank when restricted to $u \cap \mathfrak{g}$. Let $g \in G$ be arbitrary. Then the composition
$$\mathrm{L}_g \circ \exp\colon u \cap \mathfrak{g} \to U \cap G$$
is a coordinate system around g. So every $g \in G$ has a coordinate system so by Theorem 4.2, G is a manifold. The multiplication map $G \times G \to G$ is the restriction of the multiplication in $GL(n,\mathbb{R})$ hence is smooth. Similarly the inverse map $G \to G$, $g \mapsto g^{-1}$, is smooth. Thus G is a Lie group.

For the second part,

$$\begin{array}{cccc} \exp_*\colon & T_0\,\mathfrak{g} & \to & T_1G \\ \shortparallel & \shortparallel & & \\ \exp'(0) & \mathfrak{g} & & \\ \shortparallel & & & \\ \mathrm{Id} & & & \end{array}$$

This gives an isomorphism between $\mathfrak{g}$ and T_1G. In particular $\dim G = \dim T_1G = \dim \mathfrak{g}$. □

Similarly one can show that the classical groups $G \subset GL(n,\mathbb{C})$ are Lie groups.

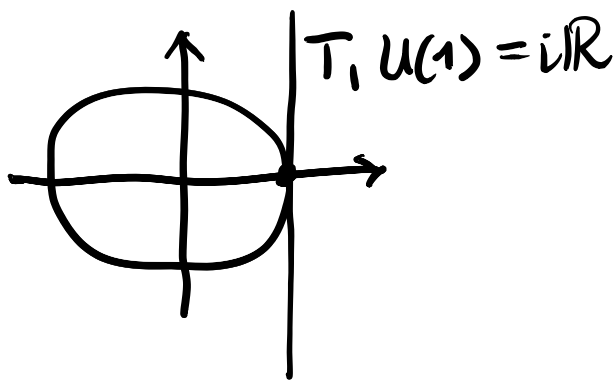
6 Lecture 6

6.1 Classical Groups (continued)

First we begin with two examples with tangent spaces.

Example 6.1. $T_a\mathbb{R}^k = \mathbb{R}^k$ because $\mathrm{Id}\colon \mathbb{R}^k \to \mathbb{R}^k$ is a coordinate system around a. $\mathrm{Id}(x_1, \ldots, x_k) = (x_1, \ldots, x_k)$ and $\mathrm{Id}'(a) = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = I_k$ and $I_k(\mathbb{R}^k) = \mathbb{R}^k$. Similarly $T_a\mathbb{C}^k = \mathbb{C}^k$.

Example 6.2. $T_1U(1) = i\mathbb{R} \cong \mathbb{R}$. $U(1) = \{z \in \mathbb{C} \mid zz^\star = 1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Now $\varphi\colon (-\varepsilon, \varepsilon) \to U(1)$ with $\varphi(t) = e^{2\pi it}$ is a coordinate system around $1 \in U(1)$, $\varphi(0) = 1$, $\varphi'(0) = 2\pi i$ (1×1 matrix). $\varphi'(0)\big(\mathbb{R}\big) = 2\pi i \cdot \mathbb{R} = i\mathbb{R}$



Definition 6.3. A *neighborhood* (nbg) of a point $x \in \mathbb{R}^n$ is an open subset $U \subset \mathbb{R}^n$ such that $x \in U$.

Recall from last time, Theorem 5.3.

Proof of thm 5.3.
$\underline{G = GL(n, \Bbbk)}$
By properties of exp and log.
$\underline{G = SL(n, \Bbbk)}$
For any $x \in \mathfrak{gl}(n, \Bbbk)$ we have the identity

$$\exp(\mathrm{Tr}(x)) = \det(\exp(x)) \tag{6.1}$$

For any $A \in GL(n, \Bbbk)$,

$$\exp(AxA^{-1}) = A\exp(x)A^{-1}$$

Proof of Equation 6.1. So 6.1 holds for x iff it holds for AxA^{-1}. Find $A \in GL(n, \mathbb{C})$ such that $AxA^{-1} = \mathfrak{s} + \mathfrak{n}$ where $\mathfrak{s}, \mathfrak{n} \in \mathfrak{gl}(n, \mathbb{C})$, $\mathfrak{s}$ diagonal, $\mathfrak{n}$ strictly upper triangular, $\mathfrak{s}\mathfrak{n} = \mathfrak{n}\mathfrak{s}$. Then use that

$$\exp(\mathfrak{s} + \mathfrak{n}) = \exp(\mathfrak{s})\exp(\mathfrak{n}).$$

Easy to check 6.1 for $\mathfrak{s}$ and $\mathfrak{n}$ $\Rightarrow$ holds for $\mathfrak{s} + \mathfrak{n}$ $\Rightarrow$ holds for x. □

Thus for $X \in GL(n, \Bbbk)$ near 1, $X = \exp x$ for some $x \in \mathfrak{gl}(n, \Bbbk)$.

$$\begin{aligned} \det(X) &= \det(\exp(x)) \\ &= \exp(\mathrm{Tr}(x)). \end{aligned}$$

So $\det(X) = 1 \Leftrightarrow \mathrm{Tr}(x) = 0$, so statement is true with $\mathfrak{g} = \{x \in \mathfrak{gl}(n, \Bbbk) \mid \mathrm{Tr}(X) = 0\}$.

Notation 6.4. $\mathfrak{sl}(n, \Bbbk)$.

$\underline{G = O(n, \Bbbk)} = \{X \in GL(n, \Bbbk) \mid XX^t = I\}$

For $X \in GL(n, \Bbbk)$ near 1 write $X = \exp(x), x \in \mathfrak{gl}(n, \Bbbk)$ by properties of exp. Then

$$\begin{aligned} XX^t = I &\Rightarrow I = X^{-1}(x^t)^{-1} = (X^t X)^{-1} \\ &\Rightarrow X^t X = XX^t = I \\ &\Rightarrow 0 = \log I = \log X + \log X^t = x + x^t \end{aligned}$$

Conversely if $x + x^t = 0$ then x and x^t commute $\Rightarrow$ $\exp(X)\exp(x^t) = \exp(x + x^t) = \exp(0) = I$. So statement true with $\mathfrak{g} = \mathfrak{o}(n, \Bbbk) = \{x \in \mathfrak{gl}(n, \Bbbk) \mid x + x^t = 0\}$ set of skew-symmetric matrices.

$\underline{G = SO(n, \Bbbk)}$

$\mathfrak{g} = \mathfrak{so}(n, \Bbbk) = \mathfrak{sl}(n, \Bbbk) \cap \mathfrak{o}(n, \Bbbk)$. However!

$$x + x^t = 0 \Rightarrow \operatorname{Tr} x = 0$$

So actually $\mathfrak{so}(n, \Bbbk) = \mathfrak{o}(n, \Bbbk)$. Which makes sense because $SO(n, \Bbbk) = O(n, \Bbbk)^0$ the connected component of I.

$\underline{G = U(n), SU(n)}$

$\mathfrak{u}(n) \neq \mathfrak{su}(n)$

$\underline{G = Sp(n, \Bbbk)}$

$\mathfrak{sp}(n, \Bbbk)$

$\underline{G = Sp(n)}$

$\mathfrak{sp}(n)$

Read about the previous few in the book. □

7 Lecture 7

7.1 Submanifolds

- Open
- Immersed
- Embedded

Recall:

A subset N of a manifold $M \subset \mathbb{R}^n$ is an *open submanifold* if $N = M \cap U$ for some open subset U of $\mathbb{R}^n$.

Example 7.1. $M = S^1 \subset \mathbb{R}^2$, $U = \{(x, y) \mid y > 0\}$, $N = M \cap U$ is an open submanifold of S^1.

Example 7.2. $GL(n, \Bbbk)$ is an open submanifold of $\Bbbk^{n^2}$.

Example 7.3. Every connected component of a manifold is an open submanifold. In particular G^0 is an open submanifold of G.

Definition 7.4. A morphism of manifolds $f\colon X \to Y$ is an *immersion* if $f_*\colon T_x X \to T_{f(x)} Y$ has full rank ($= \dim X$) for every $x \in X$.

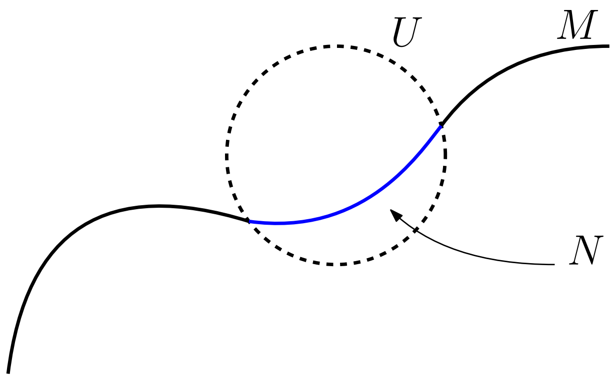


Figure 16: Visualization of an open submanifold

Example 7.5. $f\colon S^1 \to \mathbb{R}^2$, $f(\cos\theta, \sin\theta) = (\cos\theta, \sin 2\theta)$

$\varphi\colon \theta \mapsto \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} = x$, $\varphi'(a) = \begin{bmatrix}-\sin a\\ \cos a\end{bmatrix}$, $T_xS^1 = \mathbb{R}\cdot\begin{bmatrix}-\sin a\\ \cos a\end{bmatrix}$, $f_*\colon T_xS^1 \to T_{f(x)}\mathbb{R}^2 = \mathbb{R}^2$ is given by $f_*\left(t\cdot\begin{bmatrix}-\sin a\\ \cos a\end{bmatrix}\right) \overset{(*)}{=} t\cdot\begin{bmatrix}-\sin a\\ 2\cos 2a\end{bmatrix} \neq \begin{bmatrix}0\\ 0\end{bmatrix}$ for $t \neq 0$.

Proof of $(*)$.

$$(*):\quad \begin{array}{ccc} M & \xrightarrow{f} & N = \mathbb{R}^2 \\ \alpha \uparrow & & \uparrow \beta = \mathrm{Id} \\ W & \to & W' = \mathbb{R}^2 \end{array}$$

$(\beta^{-1}f\alpha)' = (\beta^{-1})'f'\alpha'$, so $\beta'(\beta^{-1}f\alpha)' = f'\alpha'$. Now since $\beta = \mathrm{Id}$, $(f\alpha)' = f'\alpha'$

$$\begin{bmatrix}-\sin a\\ 2\cos 2a\end{bmatrix} = f'\left(\begin{bmatrix}-\sin a\\ \cos a\end{bmatrix}\right)$$

□

Definition 7.6. The pair (X, f) is an *immersed submanifold* of Y. By abuse of terminology we sometimes say $f(X)$ is an immersed submaniofld.

Definition 7.7. If $f\colon X \to Y$ is an immersion such that

1) f is injective

2) $f\colon X \to f(X)$ is a homeomorphsim,

then f is an *embedding* and we say $f(X)$ is an *(embedded) submanifold* of Y.

Example 7.8. $f\colon \mathbb{R} \to S^1 \subset \mathbb{R}^2$ is an immersion

Example 7.9. $f\colon \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (t, \sin t)$ is an embedding. The graph of $y = \sin x$ is a submanifold of $\mathbb{R}^2$.

Example 7.10. $f\colon \mathbb{R} \to \mathbb{R}^2$

$$f(t) = \begin{cases} (0, t+2) & , -\infty < t \leqslant -1 \\ \gamma(t) & , -1 \leqslant t \leqslant 1 \\ (\frac{3}{t^2}, \sin\pi t) & , 1 \leqslant t < \infty \end{cases}$$

making f differentiable and injective. Then f is not an embedding since $f\colon \mathbb{R} \to f(\mathbb{R})$ is not a homeomorphism.

Example 7.11. $f\colon \mathbb{R} \to S^1 \times S^1$, $f(t) = (e^{ati}, e^{bti})$ where $a, b \in \mathbb{R}\backslash\{0\}$ such that a/b is irrational. Then f is an injective immersion, but $f(\mathbb{R})$ is dense in $S^1 \times S^1$ so $f\colon \mathbb{R} \to f(\mathbb{R})$ is not a homeomorphism $\Rightarrow f$ not an embedding.

Definition 7.12. A *closed Lie subgroup H* of a Lie group G is a subgroup which is also a submanifold.

Example 7.13. Any linear subspace U of a vector space V is a closed Lie subgroup.

Example 7.14. G^0 is a closed Lie subgroup of G using the indetity map embedding $G^0 \to G$ by $x \mapsto x$.

Example 7.15. If G_1 and G_2 are Lie groups then $G_1 \times \{1\}$ and $\{1\} \times G_2$ are closed Lie subgroups.

Theorem 7.16 (Thm 2.9 in Kirillov)**.**

1) *Let H be a closed Lie subgroup of Lie group G. Then $H = V \cap G$ for some closed subset $V \subset \mathbb{R}^n$. (i.e. H is* closed in *G)*

2) *Conversely, any subgroup H of a Lie group G such that H is closed in G, is a closed Lie subgroup.*

Proof. Skipped. □

Example 7.17. $Sp(n, \Bbbk)$ is a closed Lie subgroup of $GL(2n, \Bbbk)$.

Proof. Know $Sp(n, \Bbbk)$ subgroup. $\forall A \in GL(2n, \Bbbk)\colon A \in Sp(n, \Bbbk)$ iff $\omega(Ax, Ay) = \omega(x, y)$ $\forall x, y \in \Bbbk^{2n}$ $\Leftrightarrow (Ax)^t \cdot J \cdot Ay = x^t \cdot J \cdot y$ $\forall x, y \in \Bbbk^{2n} \Leftrightarrow A^tJA = J$, $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. So let $V = \{A \in M_{2n}(\Bbbk) \mid A^tJA = J\}$. Then V is a closed subset of $M_{2n}(\Bbbk) = \Bbbk^{(2n)^{2n}}$, and $Sp(2n, \Bbbk) = V \cap GL(2n, \Bbbk)$. Similar for other classical Lie groups. □

Further Reading

7.2 Quotient Groups and Homogeneous Spaces

[Corollary 2.10, read yourselves]

Definition 7.18. Let F be a manifold. A morphism $p\colon T \to B$ of manifolds is a *fiber bundle over B with fiber F* if

1) p is surjective,

2) p is *locally trivial:* Each $x \in B$ has a neighborhood U and a diffeomorphism $\mathcal{T}_U\colon p^{-1}(U) \to U \times F$ called a local trivialization, such that

$$\begin{tikzcd} U \times F \arrow[dr, "pr_1"'] & p^{-1}(U) \subset T \arrow[l, "\mathcal{T}_U"'] \arrow[dl, "p"] \\ & U \end{tikzcd}$$

commutes *i.e.* $p|_{p^{-1}(U)} = pr_1 \circ \mathcal{T}_U$.

3) At every $x \in M$, the map $\mathcal{T}_{U,x} : p^{-1}(\{x\}) \to F$ given by $\mathcal{T}_{U,x}(x, f) = f$ is a homeomorphism.

4) Whenever $U \cap V \neq \emptyset$, for any $x \in U \cap V$, the map $\mathcal{T}_{V,x}^{-1} \circ \mathcal{T}_{U,x} : F \to F$ is smooth.

B=base space, F=fiber, T=total space.

Example 7.19. Let F, B any manifolds $p\colon F \times B \to B$, $(x, y) \mapsto y$ is the *trivial fiber bundle* over B with fiber F.

Example 7.20. Tangent bundle on S^2

$$TS^2 = \{(x, v) \mid x \in S^2, v \in T_x S^2\}, \quad p : TS^2 \to S^2, \quad (x, v) \mapsto x$$

is a fiber bundle with fiber $\mathbb{R}^2$.

Example 7.21. M any manifold. *Tangent bundle* on M:

$$TM = \{(x, v) \mid x \in M, v \in T_x M\}, \quad p\colon TM \to M, \quad (x, v) \mapsto x$$

is a fiber bundle with fiber $\mathbb{R}^k$, $k = \dim M$.

Theorem 7.22 (Thm 2.11 in Kirillov)**.**

1) *Let G be a Lie group of dimension n, and $H \subset G$ a closed Lie subgroup of dimension k. Then the set of left cosets $G/H = \{gH \mid g \in G\}$ has a natural structure of a manifold of dimension $n-k$, such that the canonical map $p\colon G \to G/H$, $g \mapsto gH$ is a fiber bundle with fiber H. Also $T_{\bar{1}}(G/H) \cong T_1 G/T_1 H$ ($\bar{1} = p(1) = H$).*

2) *If H normal, closed Lie subgroup of a Lie group G, then G/H has a canonical structure of a Lie group, and $p\colon G \to G/H$ gives an isomorphism $T_{\bar{1}}(G/H) = T_1 G/T_1 H$.*

Proof. Beyond scope of class. □

2.3 Homomorphism Thm. Read yourselves.

7.3 Homogeneous Spaces

Let M be a manifold. Let $\mathrm{Diff}(M)$ be the group of diffeomorphisms $\varphi\colon M \to M$.

Definition 7.23. An *action* of a Lie group G on M is a group homomorphism $\rho\colon G \to \mathrm{Diff}(M)$ such that the map $G \times M \to M$, $(g, x) \mapsto \rho(g)(x)$ is a morphism of manifolds

Notation 7.24. $g.x := \rho(g)(x)$

Example 7.25. $GL(n, \mathbb{R})$ acts on $\mathbb{R}^n$

Example 7.26. $O(n, \mathbb{R})$ acts on S^{n-1}.

Example 7.27. G acts on G via $\mathrm{Ad}, \mathrm{L}, \mathrm{R}$ $g \mapsto \mathrm{Ad}\, g$.

Example 7.28. $GL(n, \mathbb{R})$ acts on the set of *flags* in $\mathbb{R}^n$:

$$\mathcal{F}_n(\mathbb{R}) = \{(V_0 \subset V_1 \subset \cdots \subset V_n) \mid V_d \text{ subspace of } \mathbb{R}^n, \dim V_d = d\}$$

Note 7.29. $\mathcal{F}_2(\mathbb{R}) \cong \mathbb{RP}^2$ and $g.(V_0 \subset V_1 \subset \cdots \subset V_n) := (gV_0 \subset gV_1 \subset \cdots \subset gV_n)$ $\forall g \in GL(n, \mathbb{R})$.

Theorem 7.30 (Thm 2.20 in Kirillov)**.** *Let M be a manifold with an action of a Lie group G. Then $\forall m \in M$, the stabilizer* $\mathrm{Stab}_G(m) = G_m = \{g \in G \mid g.m = m\}$ *is a closed Lie subgroup of G and the map*

$$G/G_m \to M \quad gG_m \mapsto g.m$$

is an injective immersion.

Proof. Future (?) □

Corollary 7.31 (Cor 2.21 in Kirillov)**.** *Each orbit $\mathcal{O}_m := \{g.m \mid g \in G\}$ is an immersed submanifold of M, and $T_m\mathcal{O}_m = T_1G/T_1G_m$. If $\mathcal{O}_m$ is a submanifold then $G/G_m \xrightarrow{\sim} \mathcal{O}_m$ is a diffeomorphism.*

Definition 7.32. A *G-homogeneous space* is a manifold with a transitive action of G.

Corollary 7.33. *Let M be a G-homogeneous space and fix $x \in M$. Then $G \to M$, $g \mapsto g.m$ is a fiber bundle with fiber G_m, and $M \cong G/G_m$ as G-homogeneous spaces.*

Example 7.34. $SO(n-1,\mathbb{R}) \to SO(n,\mathbb{R}) \to S^{n-1}$, where $SO(n-1,\mathbb{R})$ is the stabilizer of $m = (0,0,\ldots,1)$. So

$$S^{n-1} \cong \frac{SO(n,\mathbb{R})}{SO(n-1,\mathbb{R})}$$

Example 7.35. $GL(n,\mathbb{R})$ acts transitively on $\mathcal{F}_n(\mathbb{R})$. Pick standard flag

$$V^{\text{st}} = (\{0\} \subset \langle \mathbb{e}_1 \rangle \subset \cdots \subset \langle \mathbb{e}_1, \ldots, {}_{n-1} \rangle \subset \mathbb{R}^n).$$

Then the stabilizer $\mathrm{Stab}_{GL(n,\mathbb{R})}\left(V^{\text{st}}\right) = B(n,\mathbb{R})$ all invertible upper triangular matrices so

$$\mathcal{F}_n(\mathbb{R}) \cong \frac{GL(n,\mathbb{R})}{B(n,\mathbb{R})}$$

which equips $\mathcal{F}_n(\mathbb{R})$ with the structure of a manifold of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$

8 Lecture 8

8.1 The General Exponential Map

Let G be a Lie group (not necessarily classical) and $\mathfrak{g} = T_1G$ its Lie algebra.

Goal: Define a map $\exp\colon \mathfrak{g} \to G$ which generalizing matrix exponential map.

$$\mathfrak{gl}(n,\mathbb{R}) \to GL(n,\mathbb{R}) \quad x \mapsto e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

The problem is x^k has no meaning in general.

Definition 8.1. A *one-parameter subgroup of G* is a morphism of Lie groups $\gamma\colon \mathbb{R} \to G$.

Let $\gamma : \mathbb{R} \to G$ be a one-parameter subgroup of G. In particular $\gamma(0) = 1$, the identity element in the group G. So, the derivative at zero is a linear map $\dot\gamma(0)\colon T_0\mathbb{R} \to T_1G$. Since $T_0\mathbb{R}$ can be identified with $\mathbb{R}$ and T_1G can be identified with the Lie algebra of G, we obtain a linear map

$\dot{\gamma}(0)\colon \mathbb{R} \to \mathfrak{g}$. But a linear map $\mathbb{R} \to \mathfrak{g}$ is determined by the value at $1 \in \mathbb{R}$ and is really the same thing as an element of $\mathfrak{g}$. In this way we regard $\dot{\gamma}(0)$ as an element of $\mathfrak{g}$. Thus we have a map

$$\big\{\text{one-parameter subgroups of } G\big\} \to \mathfrak{g} \tag{8.1}$$

In fact, this map is a bijection. This gives a third way to think about the Lie algebra of G, the previous two being the space of all left-invariant vector field (the official definition), and the tangent space at the identity of G.

To prove it, we need the following theorem from differential equations, which we will not prove here.

Theorem 8.2 (Local Integrability of Vector Fields on Manifolds)**.** *Let v be a vector field on a manifold M and let $p \in M$. Then there exists an open interval $I \subset \mathbb{R}$ contatining 0 and a smooth map $\gamma : I \to M$ satisfying the differential equation:*

$$\begin{cases} \dot{\gamma}(t) = v\big(\gamma(t)\big) \\ \gamma(0) = p \end{cases} \tag{8.2}$$

Moreover, if $\widetilde{\gamma} : J \to M$ is another local solution to that differential equation, then $\gamma = \widetilde{\gamma}$ on $I \cap J$.

We can now prove that (8.1) is a bijection.

Proposition 8.3 (Prop 3.1 in Kirillov)**.** *Let G be a Lie group, $\mathfrak{g} = T_1 G$, and $x \in \mathfrak{g}$. Then there exists a unique one-parameter subgroup γ_x of G such that $\dot{\gamma}_x(0) = x$.*

Proof. Let v_x be the left invariant vector field on G with $v_x(1) = x$. Let $\gamma\colon I \to G$ be a solution to the system

$$\begin{cases} \dot{\gamma}(t) = v_x\big(\gamma(t)\big) \\ \gamma(0) = p \end{cases} \tag{8.3}$$

We claim that

$$\gamma(s+t) = \gamma(s)\gamma(t) \quad \text{when } s, t, s+t \in I. \tag{8.4}$$

To show this, fix s and let

$$\alpha(t) = \gamma(s)\gamma(t), \qquad \beta(t) = \gamma(s+t).$$

Then

$$\begin{aligned} \dot{\alpha}(t) &= \big(\mathrm{L}_{\gamma(s)} \circ \gamma\big)'(t) \\ &= \big(\mathrm{L}_{\gamma(s)}\big)_* \cdot \dot{\gamma}(t) \quad \text{by the chain rule} \\ &= \big(\mathrm{L}_{\gamma(s)}\big)_* \cdot v_x\big(\gamma(t)\big) \\ &= v\big(\gamma(s)\gamma(t)\big) \quad \text{by left invariance of } v_x \\ &= v\big(\alpha(t)\big). \end{aligned}$$

On the other hand

$$\dot{\beta}(t) = \dot{\gamma}(s+t) = v_x\big(\gamma(s+t)\big) = v_x\big(\beta(t)\big).$$

Thus α and β satisfy the same differential equation and $\alpha(0) = \beta(0) = \gamma(s)$. By the uniqueness part of Theorem 8.2, $\alpha(t) = \beta(t)$. This prove the claim (8.4). We leave it as an exercise to show that γ extends uniquely to a one-parameter group $\mathbb{R} \to G$. Taking $t = 0$ in (8.3) we have $\dot{\gamma}(0) = v_x(1) = x$. □

Definition 8.4. The *exponential map* $\exp\colon \mathfrak{g} \to G$ is defined by $\exp(x) = \gamma_x(1)$.

Example 8.5. $G = U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$ is real 1-dim Lie group. We have seen $\mathfrak{g} = i\mathbb{R}$. Given $x \in i\mathbb{R}$, the corresponding 1-parameter subgroup is

$$\gamma_x(t) = e^{tx}$$

because $\gamma_x(s+t) = e^{(s+t)x} = e^{sx}e^{tx} = \gamma_x(s)\gamma_x(t)$ and $\dot\gamma_x(t) = xe^{tx}$ so $\dot\gamma_x(0) = x$. So the exponential map in this in this case is $i\mathbb{R} \to U(1)$ by $x \mapsto \gamma_x(1) = e^x$ i.e. the usual one.

Example 8.6. $SO(3,\mathbb{R})$ it Lie algebra $\mathfrak{so}(3,\mathbb{R})$ has a basis $\{J_x, J_y, J_z\}$:

$$J_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad J_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad J_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

One can show that

$$\gamma_{J_x}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}.$$

Indeed it's a morphism of Lie groups and $\dot\gamma_{J_x}(0) = J_x$. So $\exp(tJ_x) = \gamma_{J_x}(t)$.

8.1.1 Properties of the Exponential Map

Exercise 8.7. Let G be a Lie group and $x \in T_1G$. Then for any real numbers s, t we have

$$\gamma_{sx}(t) = \gamma_x(st).$$

(Hint: Take $\frac{d}{dt}|_{t=0}$ on both sides and use uniqueness.)

Theorem 8.8 (thm 3.7 in Kirillov)**.** *The general exponential map satisfies the following:*

1) $\exp_*(0)\colon \mathfrak{g} \to T_1G = \mathfrak{g}$ *is the identity map*

2) $\exp$ *is a diffeomorphism between some neighborhood of* 0 *in* $\mathfrak{g}$ *and some neighborhood of* 1 *in* G*. The inverse is denoted by* $\log$.

3) $\exp((t+s)x) = \exp(tx)\exp(sx)\ \forall s, t \in \mathbb{R},\ x \in \mathfrak{g}$

4) *If* $\varphi\colon G_1 \to G_2$ *is a morphism of Lie groups, then*

$$\exp(\varphi_*(x)) = \varphi(\exp(x)) \quad \forall x \in \mathfrak{g}_1 = T_1G_1 \textit{(Recall: } \varphi_* = d\varphi(0)\textit{)}$$

5) *For* $X \in G, y \in \mathfrak{g}$ $X\exp(y)X^{-1} = \exp(\operatorname{Ad} X.y)$

Proof.
1) By Exercise 8.7, we have $\exp(tx) = \gamma_{tx}(1) = \gamma_x(t)$ for $x \in \mathfrak{g}$ and $t \in \mathbb{R}$. Now differentiate with respect to t and take $t = 0$ to get $\exp'(0) \cdot x = \dot\gamma_x(0) = x$.
2) Immediate by inverse function theorem and part 1).
3) Follows from the fact that $t \mapsto \exp(tx)$ is a one-parameter subgroup.
4) Follows from the uniqueness of the one-parameter subgroup: Let $x \in \mathfrak{g}_1$, consider $\gamma\colon \mathbb{R} \to G_1 \to G_2$ by $t \mapsto \gamma_x(t) \mapsto \varphi(\gamma_x(t))$. This is a one-parameter subgroup of G_2. Hence $\dot\gamma(0) = \varphi_*(\dot\gamma_x(0)) = \varphi_*(x)$. So by uniqueness of a one-parameter subgroup of G_2 with $\dot\gamma(0) = \varphi_*(x)$. So $\gamma = \gamma_{\varphi_*(x)}$ in G_2. This shows that $\varphi \circ \gamma_x = \gamma_{\varphi_*(x)}$. Now evaluate both sides at $t = 1$.
5) Follows from 4) by $G \to G$ by $Y \mapsto XYX^{-1} = (\operatorname{Ad} X)(Y)$. □

Further Reading

8.2 Classes of Manifolds

(1) *Complex manifold* $M \subset \mathbb{C}^n$: Coordinate systems $\varphi\colon U \to M$ where $U \subset \mathbb{C}^d$ are *holomorphic* functions.

(2) C^k*-manifolds* $M \subset \mathbb{R}^n$: Coordinate systems $\varphi\colon U \to M$ where $U \subset \mathbb{R}^d$, are differentiable of class C^k.

$$\begin{aligned} C^0 &= \text{ continuous functions} \\ C^1 &= \text{ 1st order differentiable functions} \\ C^k &= \text{ all partial derivatives up to order } k \text{ exist and are continuous} \\ C^\infty &= \text{ all partial derivatives exists } (\Rightarrow \text{ continuous}) \\ &= \textit{ smooth} \text{ functions} \\ C^\omega &= \text{ all real analytic functions (functions with Taylor series expansions)} \end{aligned}$$

One could imagine a theory of C^k-Lie groups for $k \in \mathbb{N} \cup \{\infty, \omega\}$, in the real case. However we have:

Theorem 8.9 (Deep Theorem (see Remark 2.2))**.** *Let G be a real Lie group of class C^0. Then there is a Lie group G' of class C^ω such that $G \cong G'$ as C^0-Lie groups. Moreover, G' is unique up to isomorphism of of C^ω-Lie groups.*

The conclusion to be drawn from this is "every real Lie group is a C^ω-Lie group". it suffices to consider

– Complex Lie groups

– Real Lie groups

We may WLOG assume all maps involved have Taylor series expansions.

9 Lecture 9: The Bracket (Commutator)

Let G be a Lie group and $\mathfrak{g} = T_1 G$. The vector space $\mathfrak{g}$ is equipped with a canonical bilinear operation, denoted $[x, y]$ for $x, y \in \mathfrak{g}$ and is called the *bracket* on $\mathfrak{g}$.

There are two equivalent ways to introduce it.

9.1 First Method to Introduce the Bracket: via the Logarithm

Recall exp is locally a diffeomorphism.

$$\exp\colon \mathfrak{g} \to G$$

Let $U \subset \mathfrak{g}$ be a neighborhood of 0 and $V \subset G$ neighborhood of 1 such that $\exp|_U\colon U \to V$ is a diffeomorphism with an inverse denoted by log. consider the map

$$\varphi\colon \mathfrak{g} \times \mathfrak{g} \to G \quad \text{by} \quad (x, y) \mapsto \exp(x) \cdot \exp(y).$$

Then for $(x,y) \in \varphi^{-1}(\mu(x,y))$ for some $\mu(x,y) \in \mathfrak{g}$. Explicitly,

$$\mu(x,y) = \log(\exp(x)\exp(y))$$

defined for $(x,y) \in \varphi^{-1}(V) \subset \mathfrak{g} \times \mathfrak{g}$. μ is a real analytic (or holomorphic) function and thus has a Taylor series at $(0,0)$.

Lemma 9.1. *$\mu(x,y) = x + y + \lambda(x,y) + \cdots$ where, $\lambda\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a skew-symmetric bilinear map.*

Proof. General Taylor series:

$$f(\underline{x}) = \frac{1}{0!}L_0 + \frac{1}{1!}L_1(\underline{x}) + \frac{1}{2!}L_2(\underline{x},\underline{x}) + \cdots$$

Where L_k is a multilinear function of k variables. In our case

$$\begin{aligned}\underline{x} &= (x,y) \in \mathfrak{g} \times \mathfrak{g} \cong \mathbb{R}^k \times \mathbb{R}^k \\ &= (x,0) + (0,y).\end{aligned}$$

So

$$\mu(x,y) = c_0 + \alpha_1(x) + \alpha_2(y) + \frac{1}{2}(Q_1(x,x) + \lambda(x,y) + Q_2(y,y)) + \cdots$$

for some linear functions α_1, α_2 and bilinear functions Q_1, λ, Q_2. Observe

$$\mu(x,0) = \log(\exp(x)\exp(0)) = x$$

So $c_0 = 0$, $\alpha_1(x) = x$, and $Q_1(x,x) = 0$. Similarly $\mu(0,y) = y$, which gives $\alpha_2(y) = y$ and $Q_2(y,y) = 0$. Lastly,

$$\mu(x,x) = \log(\exp(x)\exp(x)) = \log(\exp(2x)) = 2x.$$

So $\lambda(x,x) = 0$ $\forall x \in \mathfrak{g}$ which implies $\lambda(x+y, x+y) = 0$ $\forall x,y \in \mathfrak{g}$. Now since λ is bilinear we have, $\lambda(x,y) + \lambda(y,x) = 0$. ☐

Definition 9.2. The skew-symmetric bilinear function $\lambda\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ as introduced above is called the *commutator* (or *bracket*).

Notation 9.3. $[x,y] := \lambda(x,y)$.

Proposition 9.4.

1) *Let $\varphi\colon G_1 \to G_2$ be a morphism of Lie groups, and $\varphi_*\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ its differential. Then*

$$\varphi_*([x,y]) = [\varphi_*(x), \varphi_*(y)]$$

for any $x, y \in \mathfrak{g}_1$.

2) $\mathrm{Ad}_g([x,y]) = [\mathrm{Ad}_g(x), \mathrm{Ad}_g(y)]$ *for any $g \in G$, $x, y \in \mathfrak{g}$.*

3) *Let $X = \exp(x)$ and $Y = \exp(y)$. Then the* group commutator $[X,Y] = XYX^{-1}Y^{-1}$ *equals* $\exp([x,y] +$ *terms of higher order*$)$.

Proof. 1) Recall for $x \in \mathfrak{g}_1$

$$\exp(\varphi_*(x)) = \varphi(\exp(x))$$

hence

$$\begin{aligned}\exp(\mu(\varphi_*(x), \varphi_*(y))) &= \exp(\varphi_*(x)) \exp(\varphi_*(y)) \\ &= \varphi(\exp(x) \exp(y)) \\ &= \varphi(\exp(\mu(x, y))) \\ &= \exp(\varphi_*(\mu(x, y))).\end{aligned}$$

Now apply log to both sides.
2) $\mathrm{Ad}_g \colon G \to G$ is a morphism of Lie groups. Apply 1).
3) Explicit calc. □

Corollary 9.5. *If G is a commutative Lie group then $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.*

Proof. Use 3) from above. □

Example 9.6. Let $G \subset GL(n, \mathbb{R})$ and $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$. Then,

$$\exp(x) = (1 + x + \cdots) \quad \exp(y) = (1 + y + \cdots)$$

This implies that

$$\exp(x) \exp(y) \exp(-x) \exp(y) = (1+x+\cdots)(1+y+\cdots)(1-x+\cdots)(1-y+\cdots) = 1+(xy-yx)+\cdots$$

So $[x, y] = xy - yx$.

9.2 Second way to Introduce the Bracket: via Derivations

9.2.1 Derivations on an Algebra

If A is an (not necessarily associative) algebra, a *derivation* on A is a linear map $D : A \to A$ satisfying $D(ab) = D(a)b + aD(b)$ for all $a, b \in A$. Let $\mathrm{Der}(A)$ denote the set of all derivations of A. It is easy to see that $\mathrm{Der}(A)$ is a subspace of $\mathrm{End}_{\mathbb{R}}(A)$ (the space of all linear maps from A to itself). If D and E are two derivations on A we define their commutator as

$$[D, E] = D \circ E - E \circ D.$$

Lemma 9.7. $[D, E] \in \mathrm{Der}(A)$ *for all* $D, E \in \mathrm{Der}(A)$.

Proof. $(D \circ E)(fg) = D\big(E(f)g + fE(g)\big) = D\big(E(f)\big)g + E(f)D(g) + D(f)E(g) + fD\big(E(g)\big)$ so when we switch D and E and subtract the middle two terms cancel, giving the result. □

One checks the identities

$$[D, E] = -[E, D] \qquad \text{skew-symmetry}$$

$$[D, [E, F]] + [E, [F, D]] + [F, [D, E]] = 0 \qquad \text{Jacobi identity}$$

9.2.2 Vector Fields as Derivations

Let M be a manifold, $\mathcal{C}^\infty(M)$ the algebra of smooth real-valued functions on M with pointwise operations, $\mathrm{Vect}(M)$ be the vector space of smooth vector fields on M.

Lemma 9.8. *There is an isomorphism of vector spaces*

$$\mathrm{Vect}(M) \to \mathrm{Der}\big(\mathcal{C}^\infty(M)\big), \qquad X \mapsto D_X$$

where

$$(D_X f)(p) = \frac{d}{dt}|_{t=0} f\big(\gamma(t)\big) = f'(p) \cdot X_p,$$

and $\gamma : \mathbb{R} \to M$ is any smooth map with $\gamma(0) = p$ and $\dot\gamma(p) = X_p$. (Here $X_p \in T_pM$ denotes the tangent vector of the field X at p.)

Remark 9.9. As can be seen from the last expression, the result does not depend on the choice of γ satisfying the given conditions.

Proof. That $X \mapsto D_X$ is a linear map is easy to see. Suppose that $D_X = D_Y$ for some vector fields X, Y. Then

$$f'(p) \cdot X_p = f'(p) \cdot Y_p \tag{9.1}$$

for all smooth functions f on M and all $p \in M$. Here we can think of $f'(p)$ as the gradient of f at p, which is a row vector, while X_p and Y_p are column vectors upon which $f'(p)$ acts via matrix multiplication (dot product). By varying the smooth function f appropriately, we can get any row vector to appear as $f'(p)$. Then (9.1) implies that $u \cdot (X_p - Y_p) = 0$ for any row vector u, hence $X_p = Y_p$.

(We skip the proof of surjectivity) □

9.2.3 Bracket on Vector Fields

Combining these results we may define a bilinear operation on $\mathrm{Vect}(M)$ as follows: For two vector fields $X, Y \in \mathrm{Vect}(M)$, define $[X, Y]$ to be the unique vector field such that

$$D_{[X,Y]} = [D_X, D_Y]$$

Exercise 9.10. If X and Y are left invariant vector fields on a Lie group G, then the vector field $[X, Y]$ is also left invariant.

9.2.4 Explicit Form of Bracket on T_1G Using Exponential Map and Differentiation

Let G be a Lie group and in this section put $\mathfrak{g} = T_1G$, the tangent space at identity. We know that $X \mapsto X(1)$ gives a bijection between left invariant vector fields on G and T_1G. Thus we should be able to define a bracket on T_1G.

Recall that if $u \in T_1G$ is a tangent vector, the corresponding left invariant vector field X_u is defined by $X_u(g) = (\mathrm{L}_g)_* \cdot u$.

Thus the corresponding derivation $D_u = D_{X_u}$ on $\mathcal{C}^\infty(M)$ satisfies

$$(D_u f)(1) = \frac{d}{dt}\Big|_{t=0} f\big(\exp(tu)\big) = f'(1) \cdot u.$$

Therefore $[u, v]$ must be the unique tangent vector that satisfies

$$([D_u, D_v]f)(1) = f'(1) \cdot [u, v]$$

for all $f \in \mathcal{C}^\infty(M)$.

To find an explicit expression for $[u, v]$ we compute:

$$\begin{aligned}
(D_u D_v f)(1) &= \frac{d}{dt}\Big|_{t=0} (D_v f)\big(\exp(tu)\big) \\
&= \frac{d}{dt}\Big|_{t=0} \big(\exp(tu)^{-1} \cdot (D_v f)\big)(1) \quad \text{by } G\text{-action on } \mathcal{C}^\infty(M) \\
&= \frac{d}{dt}\Big|_{t=0} \big(D_v(\exp(-tu) f)\big)(1) \quad \text{by left invariance of } D_v \\
&= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} \big(\exp(-tu) f\big)\big(\exp(sv)\big) \\
&= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} f\big(\exp(tu)\exp(sv)\big) \\
&= \frac{d}{dt}\Big|_{t=0} f'\big(\exp(tu)\big) \cdot \frac{d}{ds}\Big|_{s=0} \exp(tu)\exp(sv) \\
&= f''(1) \frac{d}{dt}\Big|_{t=0} \exp(tu) \frac{d}{dt}\Big|_{s=0} \exp(sv) + f'(1) \cdot \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} \exp(tu)\exp(sv)
\end{aligned}$$

The second derivative term cancels when we switch u and v and subtract, giving:

$$([D_u, D_v] f)(1) = f'(1) \cdot \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} \big(\exp(tu)\exp(sv) - \exp(sv)\exp(tu)\big)$$

for all $f \in \mathcal{C}^\infty(M)$. We conclude that the Lie bracket on $T_1 G$ can be computed as follows:

$$[u, v] = \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} \big(\exp(tu)\exp(sv) - \exp(sv)\exp(tu)\big), \qquad \forall u, v \in T_1 G. \tag{9.2}$$

10 Further Reading

10.1 Computing Differentials Using Curves

Let $\varphi \colon M \to N$ be a morphism of manifolds. Let $p \in M$ and $x \in T_p M$. To find $d\varphi_p(x)$, also denoted $\varphi_*(x)$, let $\gamma \colon \Bbbk \to M$ be any morphism with $\gamma(0) = p$, and $d\gamma_0 = x$[iv]. Then $d(\varphi \circ \gamma)_0 = d\varphi_p \circ d\gamma_0 = d\varphi_p(x)$, where the first equality is by the chain rule. On the other hand

$$d(\varphi \circ \gamma)_0 = \frac{d}{dt}|_{t=0} \varphi(\gamma(t))$$

This is particularly useful for Lie groups: If $x \in \mathfrak{g}$ then $\gamma_x \colon \Bbbk \to G$, $\gamma_x(t) = \exp(tx)$ is a natural curve through $1 \in G$ with derivative x. So given a Lie group G, a manifold M and a morphism of manifolds $\varphi \colon G \to M$ its differential at 1 can be computed as follows:

$$\varphi_* = d\varphi_1 \colon \mathfrak{g} \to T_\varphi(1) M \quad \varphi_*(x) = \frac{d}{dt}|_{t=0} \varphi(\exp(tx)).$$

10.2 Differential of Ad

Recall $\operatorname{Ad} g \colon G \to G$ by $x \mapsto gxg^{-1}$. Its differential is

$$\operatorname{Ad} g \colon \mathfrak{g} \to \mathfrak{g}.$$

So $\operatorname{Ad} g \in GL(\mathfrak{g})$, and $\operatorname{Ad} \colon G \to GL(\mathfrak{g})$. So its differential is $\operatorname{ad} = \operatorname{Ad}_* \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$.

[iv]Here we identify $\operatorname{Hom}_\Bbbk(\Bbbk, T_p M)$ with $T_p M$

Lemma 10.1.

1) $\operatorname{ad} x.y = [x, y]$

2) $\operatorname{Ad}(\exp(X)) = \exp(\operatorname{ad} x)$

Proof. 1) For $g \in G$ consider $\operatorname{Ad} g\colon G \to G$. By formula for φ_* its differential is given by $\operatorname{Ad} g\colon \mathfrak{g} \to \mathfrak{g}$ by

$$(\operatorname{Ad} g)(y) = \frac{d}{dt}|_{t=0}(\operatorname{Ad} g)(\exp ty) = \frac{d}{dt}|_{t=0} g \exp(ty) g^{-1}$$

By the same formula again, $\operatorname{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is given by

$$\operatorname{ad} x.y = \frac{d}{ds}|_{s=0}\frac{d}{dt}|_{t=0} \exp(sx)\exp(ty)\exp(-sx) = \frac{d}{ds}|_{s=0}\frac{d}{dt}|_{t=0} \exp(ty + ts[x, y] + \cdots) = [x, y].$$

Where the second last equality is by Proposition 9.4 part 3).
2) Immediate by $X \exp(y) X^{-1} = \exp(\operatorname{Ad} X.y)$ which was proved earlier. □

Theorem 10.2 (Jacobi Identity)**.** *Let G be a Lie group and $\mathfrak{g} = T_1G$ Then the skew-symmetric bilinear map $[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ satisfies the* Jacobi Identity*:*

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

This identity can also be written (using skew-symmetry and bilinear-ity):

$$[x, [y, z]] = [[x, y], z] - [y, [x, z]]$$

$$\operatorname{ad} x.[y, z] = [\operatorname{ad} x.y, z] + [y, \operatorname{ad} x.z]$$

$$\operatorname{ad}[x, y] = \operatorname{ad} x \operatorname{ad} y - \operatorname{ad} y \operatorname{ad} x$$

Proof. Since $\operatorname{Ad}\colon G \to GL(\mathfrak{g})$ is a morphism of Lie groups, its differential preserves the commutator by Proposition 9.4 1). But the commutator on $\mathfrak{gl}(\mathfrak{g})$ is $[A, B] = AB - BA$ by example 9.6. So

$$\operatorname{ad}[x, y]_{\mathfrak{g}} = [\operatorname{ad} x, \operatorname{ad} y]_{\mathfrak{gl}(\mathfrak{g})} = \operatorname{ad} x \operatorname{ad} y - \operatorname{ad} y \operatorname{ad} x.$$

Applying both sides to $z \in \mathfrak{g}$ we get

$$[x, [y, z]] = [[x, y], z] - [y, [x, z]]$$

The other forms left as an exercise. □

11 Lecture 10: Lie Algebras

11.1 Lie Algebras and Homomorphisms

Definition 11.1. A *Lie algebra* $\mathfrak{g}$ over a field $\Bbbk$ is a vector space $\mathfrak{g}$ together with a bilinear map

$$[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$$

satisfying

i) $[x, y] = -[y, x]$ (*skew-symmetry*)[v]

[v] If $\operatorname{char}\Bbbk = 2$ this condition is replaced by $[x, x] = 0$.

ii) $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$ (*Jacobi identity*)

A *homomorphism of Lie algebras* $\psi\colon\ \mathfrak{g}_1 \to \mathfrak{g}_2$ is a linear map satisfying

$$\psi([x,y]) = [\psi(x),\psi(y)].$$

If ψ is moreover bijective, it is an *isomorphism*. Two Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$ are *isomorphic*, written $\mathfrak{g}_1 \cong \mathfrak{g}_2$, if there exists an isomorphism $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$.

Example 11.2. Let A be any associative algebra over $\Bbbk$ (i.e. a ring containing $\Bbbk$). Then define

$$[a,b] = ab - ba.$$

This operation turns A into a Lie algebra, denoted $\mathcal{L}(A)$.

Exercise 11.3. Show that $(A,[\cdot,\cdot])$ defined above is indeed a Lie algebra.

Definition 11.4. The *general linear Lie algebra over* $\Bbbk$ is

$$\mathfrak{gl}_n = \mathfrak{gl}(n,\Bbbk) = \mathcal{L}\big(M_n(\Bbbk)\big).$$

More generally, we may consider

$$\mathfrak{gl}(V) = \mathcal{L}\big(\operatorname{End}(V)\big) \quad V \text{ any vector space over } \Bbbk.$$

Clearly, if V is finite-dimensional, then

$$\mathfrak{gl}(V) \cong \mathfrak{g}(n,\Bbbk) \quad \text{where } n = \dim_{\Bbbk} V.$$

Example 11.5. Any vector space V can be regarded as a Lie algebra by defining $[x,y] = 0$ $\forall x,y \in V$.

Definition 11.6. A Lie algebra $\mathfrak{g}$ is *abelian* if $[x,y]=0$ for all $x,y \in \mathfrak{g}$.

Example 11.7. On $\mathbb{R}^3$ we may take $[u,v] = u \times v$. This is a Lie algebra. In fact, it is isomorphic to the Lie algebra $\mathfrak{so}(3)$ of all skew-symmetric 3×3-matrices. An isomorphism is given by

$$\mathbf{e}_1 \mapsto J_x, \quad \mathbf{e}_2 \mapsto J_y, \quad \mathbf{e}_3 \mapsto J_z$$

where $J_{x,y,z}$ are the natural basis elements of $\mathfrak{so}(3)$. This shows that the next natural Euclidean space where we can define the cross product isn't $\mathbb{R}^4$ but rather $\mathbb{R}^6$ because $\dim \mathfrak{so}(4) = \frac{4\cdot 3}{2} = 6$. (This is just one perspective; there are several distinct ways to generalize the cross product on $\mathbb{R}^3$.)

11.2 Subalgebras and Ideals

Definition 11.8. Let $\mathfrak{g}$ be a Lie algebra over $\Bbbk$. A linear subspace $\mathfrak{h} \subset \mathfrak{g}$ is a

- *(Lie) subalgebra* if $[x,y] \in \mathfrak{h}$ for all $x,y \in \mathfrak{h}$.
- *(Lie) ideal* if $[x,y] \in \mathfrak{h}$ for all $x \in \mathfrak{g}, y \in \mathfrak{h}$.

Exercise 11.9. If $\mathfrak{h} \subset \mathfrak{g}$ is an ideal then $\mathfrak{g}/\mathfrak{h}$ is a Lie algebra with operation

$$[x+\mathfrak{h}, y+\mathfrak{h}] = [x,y] + \mathfrak{h}$$

for all $x,y \in \mathfrak{g}$.

Exercise 11.10. (First Isomorphism Theorem) If $\varphi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, then $\ker(\varphi)$ is an ideal of $\mathfrak{g}$, $\operatorname{im}(\varphi)$ is a subgalgebra of $\mathfrak{h}$, and

$$\mathfrak{g}/\ker(\varphi) \cong \varphi(\mathfrak{g}).$$

Example 11.11. Let $\varphi : \mathfrak{gl}(n, \Bbbk) \to \Bbbk$ be given by $\varphi(A) = \operatorname{Tr} A$. This is a Lie algebra homomorphism. The kernel is

$$\ker(\varphi) = \mathfrak{sl}_n = \mathfrak{sl}(n, \Bbbk) = \{A \in \mathfrak{gl}(n, \Bbbk) \mid \operatorname{Tr} A = 0\}.$$

By the first isomorphism theorem $\mathfrak{gl}(n, \Bbbk)/\mathfrak{sl}(n, \Bbbk) \cong \Bbbk$.

Definition 11.12. Let $\mathfrak{g}$ be a Lie algebra. The *center* of $\mathfrak{g}$ is

$$\mathfrak{z}(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y] = 0 \quad \forall y \in \mathfrak{g}\}.$$

Exercise 11.13. $\mathfrak{z}(\mathfrak{g})$ is an ideal of $\mathfrak{g}$.

Exercise 11.14. The center of $\mathfrak{gl}(n, \Bbbk)$ is $\Bbbk I_n$ (the set of scalar multiples of the identity matrix).

11.3 Products of Lie Algebras

If $\mathfrak{g}_1, \mathfrak{g}_2$ are Lie algebras, the cartesian product $\mathfrak{g}_1 \times \mathfrak{g}_2$ is Lie algebra with bracket

$$[(x_1, x_2), (y_1, y_2)] = ([x_1, y_1], [x_2, y_2])$$

The projections $\pi_i : \mathfrak{g}_1 \times \mathfrak{g}_2 \to \mathfrak{g}_i$ are Lie algebra homomorphisms and the triple $(\mathfrak{g}_1 \times \mathfrak{g}_2, \pi_1, \pi_2)$ satisfies the universal property: Given any Lie algebra $\mathfrak{h}$ and homomorphisms $\psi_i : \mathfrak{h} \to \mathfrak{g}_i$ there is a unique homomorphism $\varphi : \mathfrak{g}_1 \times \mathfrak{g}_2 \to \mathfrak{h}$ such that $\psi_i = \pi_i \circ \varphi$.

Even though there are Lie algebra homomorphisms $\iota_i : \mathfrak{g}_i \to \mathfrak{g}_1 \times \mathfrak{g}_2$ sending x to $(x, 0)$ and $(0, x)$ respectively, the triple $(\mathfrak{g}_1 \times \mathfrak{g}_2, \iota_1, \iota_2)$ does NOT (in general) satisfy the dual universal property:

Exercise 11.15. Show that if $\mathfrak{g}$ is non-abelian, then $(\mathfrak{g} \times \mathfrak{g}, \iota_1, \iota_2)$ is not a coproduct in the category of Lie algebras.

> Conclusion: The category of Lie algebras has finite products but does not have finite coproducts. Nevertheless it is common to see the notation $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ instead of $\mathfrak{g}_1 \times \mathfrak{g}_2$ in the literature.

12 Further Reading

12.1 Lie Subgroups and Lie Subalgebras

Theorem 12.1 (Thm 3.22)**.** *Let G be a Lie group and $\mathfrak{g} = \mathrm{Lie}\,(G)$.*

(1) *If H is a Lie subgroup of G then $T_1 H$ is a Lie subalgebra in $\mathfrak{g}$.*

(2) *If H is a normal closed subgroup of G then $\mathfrak{h} = T_1 H$ is a Lie ideal of $\mathfrak{g}$ and $\mathrm{Lie}\,(G/H) \cong \mathfrak{g}/\mathfrak{h}$.*

Conversely, if H is a closed Lie subgroup in G, such that H and G are connected and $\mathfrak{h} = T_1 H$ is an ideal in $\mathfrak{g}$, then H is normal.

Proof. We only prove part (1) here. If $x \in T_1H$ then the one parameter subgroup $\gamma\colon \mathbb{R} \to H$ with $\dot\gamma(0) = x$ is also a one parameter subgroup of G and hence $\gamma(t) = \exp(tx)$ by uniqueness. So $\exp(tx) \in H$ $\forall t \in \mathbb{R}$. In particular for $x, y \in T_1H$:

$$\log(\exp(x)\exp(y)\exp(-x)\exp(-y))$$

belongs to $\mathfrak{h}$ hence by commutator formula, $[x, y] \in T_1H$. Similarly for part (2). (Read in book!) ☐

Theorem 12.2 (Thm 3.35 in Kirillov)**.** *Let G be a connected Lie group. Then its center*

$$Z(G) = \{g \in G \mid gh = hg \text{ for all } h \in G\}$$

is a closed Lie subgroup with Lie algebra $\mathfrak{z}(\mathfrak{g})$.

12.2 Connection Between Lie Groups and Lie Algebras

Let G be a Lie group. Then $\mathfrak{g} = \operatorname{Lie} G = T_1G$ with the bracket discussed previous lecture is a Lie algebra. Every morphism $\varphi\colon G_1 \to G_2$ of Lie groups gives a homomorphism of Lie algebras $\varphi_*\colon \mathfrak{g}_1 \to \mathfrak{g}_2$. Moreover, $(\varphi \circ \psi)_* = \varphi_* \circ \psi_*$, $\mathrm{Id}_* = \mathrm{Id}$. In other words, Lie is a functor from the category of Lie groups to the category of Lie algebras.

Theorem 12.3 (Thm 3.40)**.** *For any Lie group G there is a bijection between connected Lie subgroups $H \subset G$ and Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$ given by $H \mapsto \mathfrak{h} = \operatorname{Lie} H = T_1H$.*

Sketch of proof. An inclusion $\varphi : H \to G$ of a subgroup yields an inclusion $\varphi_* : \mathfrak{h} \to \mathfrak{g}$ of Lie algebras. For the converse, we have seen a special case: If $\mathfrak{h}$ is one-dimensional, say $\mathfrak{h} = \mathbb{R}x$ then we may take H to be the image of the exponential map $t \mapsto \exp(tx)$. This is a connected subgroup of G (being the image of the connected set $\mathbb{R}$ under a continuous map). The general case is not proved in the book and is more difficult. It relies on *Frobenius integrability criterion* which is a generalization of the "integrability of vector fields" theorem from differential equations. ☐

Theorem 12.4. *Let G_1 and G_2 be Lie groups and $\mathfrak{g}_i = \operatorname{Lie} G_i$.*

(1) *If G_1 is connected this functor is* faithful*, that is,* $\operatorname{Hom}(G_1, G_2) \to \operatorname{Hom}(\mathfrak{g}_1, \mathfrak{g}_2)$ *is injective.*

(2) *(Thm 3.41) If G_1 is connected and simply connected then the functor is* fully faithful*, that is,* $\operatorname{Hom}(G_1, G_2) \cong \operatorname{Hom}(\mathfrak{g}_1, \mathfrak{g}_2)$.

Proof. See Section 3.8 of the book. ☐

Theorem 12.5 (Thm 3.42, Lie's 3rd Thm)**.** *Any finite dimensional real Lie algebra is isomorphic to the Lie algebra of a Lie group.*

Idea of proof. Show that every Lie algebra is isomorphic to a subalgebra of $\mathfrak{gl}(n, \Bbbk)$. (Ado's Theorem[vi]). Then use theorem 12.3. ☐

Corollary 12.6 (Cor 3.43)**.** *) For any finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$, there is a unique up to isomorphism connected simply-connected Lie group G with* $\operatorname{Lie}(G) \cong \mathfrak{g}$. *Furthermore, any other connected Lie group G' with Lie algebra $\mathfrak{g}$ is of the form G/Z for some discrete central subgroup $Z \subset G$.*

Since the functor from connected simply connected Lie groups to real Lie algebras is fully faithful (Theorem 12.4 and essentially surjective on objects (Theorem 12.6), we obtain:

Corollary 12.7 (Cor 3.44)**.** *The category of finite dimensional Lie algebras over $\mathbb{R}$ is equivalent to the category of connected simply-connected Lie groups.*

[vi]If $\mathfrak{Z}(\mathfrak{g}) = 0$, then $x \mapsto \operatorname{ad} x$ is injective.

12.3 The Baker-Campbell-Hausdorff Formula

Recall: $\frac{1}{2}[x,y]$ $(x,y \in \mathfrak{g})$ is the quadratic term of the Taylor expansion of $\log(\exp(x)\exp(y))$ at 0.

Question 12.8. Do higher order terms give more information? Or does the bracket completely determine the multiplication in G?

Theorem 12.9 (Baker-Campbell-Hausdorff)**.** *For small enough $x, y \in \mathfrak{g}$ we have,*

$$\exp(x)\exp(y) = \exp\Big(\sum_0^\infty \mu_n(x,y)\Big)$$

where

$$\begin{aligned}
\mu_0(x,y) &= 0,\\
\mu_1(x,y) &= x+y,\\
\mu_2(x,y) &= \frac{1}{2}[x,y],\\
\mu_3(x,y) &= \frac{1}{12}\big([x[x,y]] + [y[y,x]]\big),\\
&\vdots
\end{aligned}$$

In general, for $n \geqslant 0$ μ_n is a universal (=independent of $\mathfrak{g}$) expression, in a linear combination of commutators of degree n.

Corollary 12.10. *The group operation in a connected Lie group can be recovered from its Lie algebra.*

12.4 Complex and Real Forms

Definition 12.11. The *complexification* of a real Lie algebra $\mathfrak{g}$ is

$$\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$$

with bracket determined by

$$[x \otimes \lambda, y \otimes \mu] = [x,y] \otimes \lambda\mu \qquad \forall x, y \in \mathfrak{g},\ \lambda, \mu \in \mathbb{C}.$$

$\mathfrak{g}$ is a *real form* of $\mathfrak{g}_{\mathbb{C}}$.

Under the isomorphism $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g} \oplus i\,\mathfrak{g}$, we can also write it

$$[x+iy, z+iw] = [x,z] - [y,w] + i\big([y,z] + [x,w]\big) \qquad \forall x, y, z, w \in \mathfrak{g}\,.$$

Example 12.12. $\mathfrak{g} = \mathfrak{u}(n) \to \mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$. With

$$X = \frac{1}{2}(X + X^*) + \frac{1}{2}(X - X^*)$$

with the first piece in $\mathfrak{u}(n)$ and the second piece in $i\mathfrak{u}(n)$. This is clear because X is hermitian $\Leftrightarrow$ iX skew-hermitian.

Example 12.13. $\mathfrak{su}(n)$ and $\mathfrak{sl}(n,\mathbb{R})$ are two (different) real forms of $\mathfrak{sl}(n,\mathbb{C})$, because

$$\mathfrak{sl}(n,\mathbb{C}) = \mathfrak{sl}(n,\mathbb{R}) \oplus i\mathfrak{sl}(n,\mathbb{R}) = \mathfrak{su}(n) \oplus i\mathfrak{su}(n).$$

Remark 12.14. Let G be a connected *complex* Lie group, $\mathfrak{g} = \mathrm{Lie}\,(G)$. Let $K \subset G$ be a closed real Lie subgroup in G such that $\mathfrak{k} = \mathrm{Lie}\,(K)$ is a real form of $\mathfrak{g}$ (i.e. $\mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}$). Then K is said to be a *real form* of G.

So, the above example shows that $SU(n)$ and $SL(n,\mathbb{R})$ are two real forms of $SL(n,\mathbb{C})$.

Since unitary matrices preserve the standard Hermitian inner product on $\mathbb{C}^n$, it is not hard to see that $SU(n)$ is a (closed) and bounded set, hence $SU(n)$ is a so called *compact real form* of $SL(n,\mathbb{C})$.

13 Lecture 11: Solvable Lie Algebras and Lie's Theorem

13.1 The Derived Subalgebra

The following lemma is easy to check.

Lemma 13.1. *$I, J \subset \mathfrak{g}$ ideals. Then*

$$\begin{aligned} &I + J = \{x + y \mid x \in I, y \in J\} \\ &I \cap J \\ &[I, J] = \mathrm{Span}\left\{[x,y] \mid x \in I, y \in J\right\} \end{aligned}$$

are ideals.

Definition 13.2. $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$ is the *derived subalgebra* (or *commutant*) of $\mathfrak{g}$.

Note that, somewhat confusingly, the derived subalgebra is actually an ideal of $\mathfrak{g}$.

Lemma 13.3.

(i) $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ *is abelian*

(ii) *If $I \subset \mathfrak{g}$ is an ideal such that $\mathfrak{g}/I$ is abelian, then $[\mathfrak{g},\mathfrak{g}] \subset I$.*

Example 13.4. Let $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{k})$, then $[\mathfrak{g},\mathfrak{g}] = \mathfrak{sl}(n,\mathbb{k})$

($\subset$): This is clear as $\mathrm{Tr}(xy - yx) = 0$ for every $x, y \in \mathfrak{gl}(n,\mathbb{k})$.

($\supset$): For $i \neq j$: $E_{ii} - E_{jj} = [E_{ij}, E_{ji}] \in \mathfrak{g}'$ and $E_{ij} = \frac{1}{2}[E_{ii} - E_{jj}, E_{ij}] \in \mathfrak{g}'$

13.2 The Derived Series

We construct the following sequence of subspaces of a Lie algebra $\mathfrak{g}$:

$$\begin{aligned} D^0\,\mathfrak{g} &= \mathfrak{g} \\ D^1\,\mathfrak{g} &= [\mathfrak{g},\mathfrak{g}] = \mathfrak{g}' \\ D^2\,\mathfrak{g} &= [[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]] = [\mathfrak{g}',\mathfrak{g}'] \\ \vdots\ &= \ \vdots \\ D^i\,\mathfrak{g} &= [D^{i-1}\,\mathfrak{g}, D^{i-1}\,\mathfrak{g}] \\ \vdots\ &= \ \vdots \end{aligned}$$

Notice that $\mathfrak{g} = D^0\,\mathfrak{g} \supset D^1\,\mathfrak{g} \supset D^2\,\mathfrak{g} \supset \cdots$. This sequence is known as the *derived series* of $\mathfrak{g}$.

Exercise 13.5. Show that $D^i\mathfrak{g}$ is an ideal of $\mathfrak{g}$ for every $i \geqslant 0$.

Definition 13.6. $\mathfrak{g}$ is *solvable* if there exists $n \geqslant 0$: $D^n\,\mathfrak{g} = 0$.

Proposition 13.7. *Let $\mathfrak{g}$ be a Lie algebra, then TFAE:*

(i) *$\mathfrak{g}$ is solvable;*

(ii) *There is a sequence of subspaces*

$$\mathfrak{g} = \mathfrak{a}^0 \supset \mathfrak{a}^1 \supset \mathfrak{a}^2 \supset \cdots \supset \mathfrak{a}^k = 0$$

such that $[\mathfrak{a}^i, \mathfrak{a}^{i+1}] \subset \mathfrak{a}^{i+1}$, $i = 0, \ldots, k-1$ and $\mathfrak{a}^i/\mathfrak{a}^{i+1}$ is one-dimensional;

(iii) *There is a sequence of subspaces*

$$\mathfrak{g} = \mathfrak{a}^0 \supset \mathfrak{a}^1 \supset \mathfrak{a}^2 \supset \cdots \supset \mathfrak{a}^k = 0$$

such that $[\mathfrak{a}^i, \mathfrak{a}^{i+1}] \subset \mathfrak{a}^{i+1}$, $i = 0, \ldots, k-1$ and $\mathfrak{a}^i/\mathfrak{a}^{i+1}$ is abelian.

Proof. Observe the condition $[\mathfrak{a}^i, \mathfrak{a}^{i+1}] \subset \mathfrak{a}^{i+1}$ for $i = 0, 1, \ldots, k-1$ implies that each $\mathfrak{a}^{i+1}$ is a subalgebra of $\mathfrak{g}$ and is an ideal of $\mathfrak{a}^i$.

(i) $\Rightarrow$ (ii): Let $\mathfrak{b}^1$ be any subspace in $\mathfrak{g}/\mathfrak{g}'$ of codimension one and let $\mathfrak{a}^1$ be the inverse image of $\mathfrak{b}^1$ in $\mathfrak{g}$ under the canonical map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{g}'$. Then $[\mathfrak{g}, \mathfrak{a}^1] \subset \mathfrak{g}' \subset \mathfrak{a}^1$. In particular $\mathfrak{a}^1$ is a (codimension one) subalgebra of $\mathfrak{g}$ hence is itself solvable. By induction on $\dim \mathfrak{g}$, the subalgebra $\mathfrak{a}^1$ has such a chain of subspaces.
(ii) $\Rightarrow$ (iii): Trivial.
(iii) $\Rightarrow$ (i): Since $\mathfrak{a}^i/\mathfrak{a}^{i+1}$ is abelian, Lemma 13.3(ii) implies that $[\mathfrak{a}^i, \mathfrak{a}^i] \subset \mathfrak{a}^{i+1}$ for all $i = 0, 1, \ldots, k-1$. We show by induction on i that $D^i\mathfrak{g} \subset \mathfrak{a}^i$ for all $i = 0, 1, \ldots, k$. It is trivial for $i = 0$. For $i > 0$ we have $D^i\mathfrak{g} = [D^{i-1}\mathfrak{g}, D^{i-1}\mathfrak{g}] \subset [\mathfrak{a}^{i-1}, \mathfrak{a}^{i-1}] \subset \mathfrak{a}^i$. Therefore $D^k\mathfrak{g} = 0$ and $\mathfrak{g}$ is solvable. □

13.3 Lie's Theorem

The following is the key result that is needed to prove Lie's Theorem.

Theorem 13.8. *Assume that $\Bbbk$ is algebraically closed and* $\operatorname{char} \Bbbk = 0$*. Let $V \neq 0$ be a finite-dimensional vector space over $\Bbbk$. Let $\mathfrak{g}$ be a solvable Lie subalgebra of $\mathfrak{gl}(V)$. Then there exists a nonzero vector $v \in V$ which is a common eigenvector for all elements of $\mathfrak{g}$.*

Proof. By Proposition 13.7(i)$\Rightarrow$(ii), there exists an ideal $\mathfrak{a} \subset \mathfrak{g}$ of codimension one. By induction on $\dim \mathfrak{g}$ there exists a nonzero vector $v' \in V$ and a linear functional $\lambda : \mathfrak{a} \to \Bbbk$ such that $av' = \lambda(a)v'$ for all $a \in \mathfrak{a}$. Consider the subspace

$$W = \{w \in V \mid aw = \lambda(a)w \ \forall a \in \mathfrak{a}\}.$$

Since $v' \in W$, $W \neq 0$. Pick $x \in \mathfrak{g}\backslash\mathfrak{a}$; then $\mathfrak{g} = \mathfrak{a} \oplus \Bbbk x$. We claim that $xW \subset W$. For any $a \in \mathfrak{a}$ and $w \in W$ we have

$$axw = xaw + [a, x]w = \lambda(a)xw + \lambda([a, x])w$$

(we can write this because $[a, x] \in \mathfrak{a}$). So the claim will follow if we show that $\lambda([a, x]) = 0$ for all $a \in \mathfrak{a}$.

Pick any nonzero $w \in W$ and let $W_i = \operatorname{Span}\{w, xw, x^2w, \ldots, x^{i-1}w\}$. Since V is finite-dimensional, the ascending sequence $W_1 \subset W_2 \subset \cdots$ must stabilize. Let n be minimal such that $W_n = W_{n+1}$. Then $\dim W_i = i$ for $1 \leqslant i \leqslant n$.

We show that any $a \in \mathfrak{a}$ preserves W_n and in the basis $w, xw, \ldots, x^{n-1}w$, a is an upper triangular matrix with all diagonal elements equal to $\lambda(a)$. That is, we show

$$ax^i w \in \lambda(a)x^i w + W_i \quad \forall a \in \mathfrak{a}, i \geqslant 0.$$

For $i = 0$ this holds since $aw = \lambda(a)w$. For $i > 0$ we have $ax^i w = xax^{i-1}w + [a,x]x^{i-1}w$ hence, by induction,

$$ax^i w \in x\big(\lambda(a)x^{i-1}w + W_{i-1}\big) + \lambda([a,x])x^{i-1}w + W_{i-1} \subset \lambda(a)x^i w + W_i$$

since $xW_{i-1} \subset W_i$. Thus the trace of a on W_n is $n\lambda(a)$, for all $a \in \mathfrak{a}$. Since both x and $a \in \mathfrak{a}$ preserve W_n, we have $n\lambda([a,x]) = Tr_{W_n}([a,x]) = 0$. Since $\operatorname{char} \Bbbk = 0$ this implies that $\lambda([a,x]) = 0$ for all $a \in \mathfrak{a}$. This proves the claim that $xW \subset W$. Since $\Bbbk$ is algebraically closed, there exists an eigenvector v for x in W. Since $\mathfrak{g} = \mathfrak{a} \oplus \Bbbk x$, v is a common eigenvector for all elements of $\mathfrak{g}$. $\square$

Lie's Theorem is frequently stated in terms of representations.

1. A *representation* of a Lie algebra $\mathfrak{g}$ is a vector space V together with a Lie algebra homomorphism $\rho_V : \mathfrak{g} \to \mathfrak{gl}(V)$. We say that the representation is *complex* (resp. *finite-dimensional*, etc.) if the vector space V is.
2. A *subrepresentation* of V is a subspace U such that $\rho_V(x)U \subset U\ \forall x \in \mathfrak{g}$. (Then U becomes a representation by $\rho_U(x) = \rho_V(x)\big|_U$.)
3. If U is a subrepresentation of a representation V then the quotient space V/U is a representation by $\rho_{V/U}(x)(v+U) = \big(\rho_V(x)v\big) + U\ \forall x \in \mathfrak{g}, v \in V$.

When no confusion can arise we frequently use *module notation*:

$$x.v = \rho_V(x)v, \quad \forall x \in \mathfrak{g},\ v \in V.$$

Note that with this notation we have

$$[x,y].v = x.y.v - y.x.v$$

for all $x, y \in \mathfrak{g}$ and $v \in V$.

Corollary 13.9 (Lie's Theorem)**.** *Any complex finite dimensional representation V of a solvable Lie algebra $\mathfrak{g}$ has a flag (sequence of subspaces)*

$$\mathcal{F} \;=\; \big(0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V\big)$$

which is

$$\begin{aligned} &\mathfrak{g}\text{-stable}: \ x.V_i \subset V_i \quad \forall x \in \mathfrak{g} \\ &\textit{and}\ \text{complete}: \ \dim V_i = i \end{aligned}$$

Remark 13.10. Consequently, choosing a basis for V_1 and extending to V_2, then extending to V_3, and so on, produces a basis for V in which the matrix $\rho_V(x)$ is upper-triangular for every $x \in \mathfrak{g}$. Thus, Lie's theorem is a generalization of the linear algebra result that commuting complex matrices can be simultaneously upper-triangularized.

Proof. Since $\mathfrak{g}$ is solvable, the image in $\mathfrak{gl}(V)$ under the representation map ρ_V is a solvable Lie subalgebra of $\mathfrak{gl}(V)$. By Theorem 13.8, there is a vector $v \in V$ such that $\Bbbk v$ is a subrepresentation of V. By induction on $\dim V$, the representation $W = V/\Bbbk v$ has a flag

$$0 = W_0 \subset W_1 \subset \cdots \subset W_n = W,$$

such that $x.W_i \subset W_i \,\forall x \in \mathfrak{g}$ and $\dim W_i = i \,\forall i$. Define $V_{i+1} = \{u \in V \mid u + \Bbbk v \in W_i\}$ be the inverse image of W_i in V and let $V_0 = 0$. Then

$$0 = V_0 \subset V_1 = \Bbbk v \subset V_2 \subset V_3 \subset \cdots \subset V_{n+1} = V$$

is a $\mathfrak{g}$-stable complete flag for V. □

14 Lecture 12: Nilpotent Lie Algebras and Engel's Theorem

14.1 Lower Central Series

$D_0\,\mathfrak{g} = \mathfrak{g}$, $D_i\,\mathfrak{g} = [\mathfrak{g}, D_{i-1}\,\mathfrak{g}]$ for $i > 0$.

Definition 14.1. $\mathfrak{g}$ is *nilpotent* if there exists $n > 0$: $D_n\,\mathfrak{g} = 0$.

Proposition 14.2. *Let $\mathfrak{g}$ be a Lie algebra, then TFAE:*

(i) $\mathfrak{g}$ *is nilpotent*

(ii) *There is a sequence of subspaces*

$$\mathfrak{g} = \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots \supset \mathfrak{a}_k = 0$$

such that $[\mathfrak{g}, \mathfrak{a}_i] \subset \mathfrak{a}_{i+1}$, $i = 0, \ldots, k-1$.

By induction it can be shown that $D^i\,\mathfrak{g} \subset D_i\,\mathfrak{g}$, so nilpotent implies solvable.

Example 14.3. $\mathfrak{g} = \mathfrak{gl}(n, \Bbbk)$. Consider,

$$\begin{aligned}\mathfrak{b} &= \{\text{upper triangular matrices}\} = \operatorname{span}_{\Bbbk}\{E_{ij} \mid i \leqslant j\}^{\text{vii}}\\ \mathfrak{n} &= \{\text{strictly upper triangular matrices}\} = \operatorname{span}_{\Bbbk}\{E_{ij} \mid i < j\}\end{aligned}$$

We claim that $\mathfrak{b}$ is solvable and $\mathfrak{n}$ is nilpotent.

Proof of claim. We will instead prove a more general statement. Let V be a vector space and let

$$\mathcal{F} = (0 \subset V_1 \subset V_2 \subset \cdots \subset V_k = V)$$

be a flag (not necessarily complete). The *standard flag* in $\Bbbk^n$ is given by $V_i := \operatorname{span}\{\mathbb{e}_1, \ldots, \mathbb{e}_i\}$.

Put

$$\begin{aligned}\mathfrak{b}(\mathcal{F}) &= \{x \in \mathfrak{gl}(V) \mid xV_i \subset V_i \;\forall i\}\\ \mathfrak{n}(\mathcal{F}) &= \{x \in \mathfrak{gl}(V) \mid xV_i \subset V_{i-1} \;\forall i\}\\ \mathfrak{a}_k(\mathcal{F}) &= \{x \in \mathfrak{gl}(V) \mid xV_i \subset V_{i-k} \;\forall i\}\end{aligned}$$

with $V_k = 0$ for $k < 0$. Note that $\mathfrak{b}(\mathcal{F}_{\text{std}}) = \mathfrak{b}$, $\mathfrak{n}(\mathcal{F}_{\text{std}}) = \mathfrak{n}$. Abbreviate $\mathfrak{a}_k = \mathfrak{a}_k(\mathcal{F})$. Obviously, $\mathfrak{a}_k \cdot \mathfrak{a}_\ell \subset \mathfrak{a}_{k+\ell}$ and $[\mathfrak{a}_k, \mathfrak{a}_\ell] \subset \mathfrak{a}_{k+\ell}$, hence $\mathfrak{n}(\mathcal{F})$ is nilpotent. Since diagonal entries of xy and yx coincide for $x, y \in \mathfrak{b}(\mathcal{F})$ (check!). We have $D^1\mathfrak{b}(\mathcal{F}) \subset \mathfrak{a}_1$, and by induction, $D^i\mathfrak{b}(\mathcal{F}) \subset \mathfrak{a}_{2i}$. This implies that $\mathfrak{b}(\mathcal{F})$ is solvable. □

[vii] This called the standard Borel subalgebra of $\mathfrak{gl}(n, \Bbbk)$

14.2 Engel's Theorem

The *adjoint representation* of a Lie algebra $\mathfrak{g}$ is $\mathfrak{g}$ itself with

$$\rho_{\mathfrak{g}} = \mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \quad (\mathrm{ad}\, x)(y) = [x, y].$$

That $[\mathrm{ad}\, x, \mathrm{ad}\, y] = \mathrm{ad}[x, y]$ is equivalent to the Jacobi identity (assuming $[x, y] = -[y, x]$). The map ad is also called the *adjoint action* of $\mathfrak{g}$ (on itself).

Theorem 14.4 (Key Result for Engel's Theorem)**.** *Let $\Bbbk$ be an arbitrary field. Let $V \neq 0$ be a finite dimensional vector space. Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie subalgebra consisting of nilpotent transformations. Then there exists a nonzero vector $v \in V$ such that $xv = 0$ for all $x \in \mathfrak{g}$.*

Proof. We use induction on $n = \dim \mathfrak{g}$. The conclusion is trivial for $n = 0, 1$ so suppose $n > 1$. Let $I \subsetneq \mathfrak{g}$ be a maximal subalgebra. Consider $\mathfrak{g}$ as a representation of I via restriction of the adjoint action. Then I is a subrepresentation, hence $\mathfrak{g}/I$ is a representation of I. Explicitly, we have a Lie algebra homomorphism $\rho : I \to \mathfrak{gl}(\mathfrak{g}/I)$ given by $\rho(a)(x + I) = [a, x] + I$ for all $a \in I, x \in \mathfrak{g}$. Since a is nilpotent, $x \mapsto [a, x] = a \circ x - x \circ a = (L_a - R_a)(x)$ is nilpotent (binomial theorem implies $(L_a - R_a)^{2k-1} = 0$ if $a^k = 0$). Thus, by the induction hypothesis applied to $\rho(I)$, there exists a nonzero $z + I \in \mathfrak{g}/I$ such that $\rho(I)(z + I) = 0$. That is, $[I, z] \subset I$. That implies that the vector space $I \oplus \Bbbk z$ is a subalgebra of $\mathfrak{g}$ strictly containing I. By maximality of I, $\mathfrak{g} = I \oplus \Bbbk z$. Let

$$W = \{v \in V \mid av = 0 \ \forall a \in I\}.$$

By the induction hypothesis applied to I, $W \neq 0$. Furthermore, W is z-invariant: if $v \in W$ and $a \in I$ then $azv \subset zav + [a, z]v = 0$ (because $[a, z] \in I$). Since z is nilpotent, $z|_W$ is nilpotent, so there exists $v \in W$ such that $zv = 0$. Then $\mathfrak{g}v = 0$. ☐

Corollary 14.5. *Let $\Bbbk$ be an arbitrary field. Let $V \neq 0$ be a finite dimensional vector space. Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie subalgebra consisting of nilpotent transformations. Then there exists a complete flag $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ such that $xV_i \subset V_{i-1}$ for all $x \in \mathfrak{g}$.*

Remark 14.6. The conclusion is equivalent to there being a basis for V in which all $x \in \mathfrak{g}$ are strictly upper-triangular.

Corollary 14.7. *If $\mathfrak{g} \subset \mathfrak{gl}(V)$ consists of nilpotent transformations, then $\mathfrak{g}$ is nilpotent.*

Proof. Pick a basis in which all $x \in \mathfrak{g}$ are strictly upper-triangular. Then $\mathfrak{g}$ is contained in a nilpotent Lie algebra hence is nilpotent. ☐

Theorem 14.8. *If $\mathfrak{g}$ is a finite dimensional Lie algebra, the $\mathfrak{g}$ is nilpotent iff $\forall x \in \mathfrak{g}$, the map $\mathrm{ad}\, x\colon \mathfrak{g} \to \mathfrak{g}$, $y \mapsto [x, y]$, is nilpotent.*

Proof. If $\mathfrak{g}$ is nilpotent, then $(\mathrm{ad}\, x)^n(y) \in D_n\mathfrak{g} = 0$ for $n \gg 0$. Conversely, if $\mathrm{ad}\, x$ is nilpotent for all $x \in \mathfrak{g}$, then $\mathrm{ad}\, \mathfrak{g} \cong \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is nilpotent by Corollary 14.7. That implies $\mathfrak{g}$ is nilpotent. ☐

15 Lecture 13: The Radical; semisimple Lie Algebras; Semi-Direct Products and Levi's Theorem

15.1 The Radical of a Lie Algebra

Definition 15.1. A Lie algebra $\mathfrak{g}$ is called *semisimple* if $\{0\}$ is the only solvable ideal in $\mathfrak{g}$.

Definition 15.2. A Lie algebra $\mathfrak{g}$ is called *simple* if it is non-abelian and $\{0\}$ and $\mathfrak{g}$ are the only ideals.

Lemma 15.3. $\mathfrak{g}$ *simple* $\Rightarrow$ $\mathfrak{g}$ *semisimple.*

Proof. Let $\{0\} \neq I \subset \mathfrak{g}$ be a solvable ideal. $\mathfrak{g}$ simple $\Rightarrow$ $I = \mathfrak{g}$ $\Rightarrow$ the ideal $[\mathfrak{g}, \mathfrak{g}]$ is proper $\Rightarrow$ $[\mathfrak{g}, \mathfrak{g}] = 0$ $\Rightarrow$ $\mathfrak{g}$ abelian, which is a contradiction. □

Example 15.4. $\mathfrak{sl}(2, \mathbb{C})$ is simple. Use the map $\operatorname{ad} h$.

Proposition 15.5. *Let* $\mathfrak{g}$ *be a finite-dimensional Lie algebra. Then the set of all solvable ideals of* $\mathfrak{g}$ *has a unique maximal element.*

Definition 15.6. The unique maximal solvable Lie algebra of $\mathfrak{g}$ is called the *radical* of $\mathfrak{g}$ and is denoted $\operatorname{rad}(\mathfrak{g})$.

Proof. Let $\operatorname{rad}(\mathfrak{g})$ be the sum of all solvable ideals of $\mathfrak{g}$. We need to show $\operatorname{rad}(\mathfrak{g})$ is itself solvable. If I and J are solvable ideals of $\mathfrak{g}$, then $\dfrac{I+J}{I} = \dfrac{J}{I \cap J}$. Since J is solvable, $\dfrac{J}{I \cap J}$ is solvable. Since I and $\dfrac{I+J}{I}$ are solvable $\Rightarrow$ $I + J$ solvable. By induction, any finite sum is solvable. $\mathfrak{g}$ finite dimensional implies there exists $N > 0$ such that $D^N I = 0$ for all solvable ideals of $\mathfrak{g}$ ($N = \dim \mathfrak{g}$). So $D^N \operatorname{rad}(\mathfrak{g}) = 0$. □

Exercise 15.7. $\operatorname{rad}(\mathfrak{g} / \operatorname{rad}(\mathfrak{g})) = 0$, i.e. $\mathfrak{g} / \operatorname{rad}(\mathfrak{g})$ is semisimple. (Note $\mathfrak{g}$ is semisimple iff $\operatorname{rad}(\mathfrak{g}) = 0$)

Definition 15.8. A finite-dimensional Lie algebra $\mathfrak{g}$ is *reductive* if $\operatorname{rad}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$.

16 Lecture 14: Jordan Decomposition and Cartan's First Criterion

Recall that any matrix $x \in M_n(\mathbb{k})$ is similar to a matrix in *Jordan canonical form*:

$$x \sim \begin{bmatrix} \lambda_1 & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \ddots & 1 & & & & \\ & & & \lambda_1 & & & & \\ & & & & \ddots & & & \\ & & & & & \lambda_k & 1 & & \\ & & & & & & \ddots & \ddots & \\ & & & & & & & \ddots & 1 \\ & & & & & & & & \lambda_k \end{bmatrix}.$$

Such a matrix can be written:

$$\begin{bmatrix} \lambda_1 & & & & & \text{\huge 0} \\ & \ddots & & & & \\ & & \lambda_1 & & & \\ & & & \ddots & & \\ & & & & \lambda_k & \\ & & & & & \ddots \\ \text{\huge 0} & & & & & \lambda_k \end{bmatrix} + \begin{bmatrix} 0 & 1 & & & & & \text{\huge 0} \\ & \ddots & \ddots & & & & \\ & & \ddots & 1 & & & \\ & & & \ddots & & & \\ & & & & \ddots & 1 & \\ & & & & & \ddots & 1 \\ \text{\huge 0} & & & & & & 0 \end{bmatrix}$$

where the first is diagonal and the second is nilpotent, and these matrices commute.

When $\Bbbk$ is algebraically closed, another word for diagonalizable is semisimple.

Definition 16.1. $A \in End(V)$, V vector space over $\Bbbk$ (algebraically closed) is *semisimple* if A is diagonalizable.

Theorem 16.2 (Jordan Decomposition)**.** *V finite dimensional vector space over an algebraically closed field $\Bbbk$. Let $A \in End(V)$. Then*

a) *There exist unique linear maps $A_s, A_n \in End(V)$ such that*

  i) $A = A_s + A_n$

  ii) $A_s A_n = A_n A_s$

  iii) *A_s is semisimple and A_n is nilpotent.*

b) *There exists $P(T) = P_A(T) \in \mathbb{C}[T]$ and $Q(T) = Q_A(T) \in \mathbb{C}[T]$ such that*

  i) *$A_s = P(A)$ and $A_n = Q(A)$*

  ii) *$\gcd(P(T), Q(T)) = T$ in particular $P(0) = Q(0) = 0$. Hence, if $B \in \mathrm{End}(V)$ commutes with A, then B commutes with A_s and A_n.*

c)

$$(\operatorname{ad} A)_S = \operatorname{ad}(A_s)$$
$$(\operatorname{ad} A)_n = \operatorname{ad}(A_n)$$

In particular, if A is semisimple (respectively nilpotent), then $\operatorname{ad} A$ *is semisimple (respectively nilpotent).*

Proof. a,b) $A \in \mathrm{End}(V)$ define a $\Bbbk$-algebra morphism

$$ev_A \colon \Bbbk[T] \to \mathrm{End}(V)$$
$$\sum c_n T^n \mapsto \sum c_n A^n$$

$\Rightarrow$ $c_A(T) = \prod (T - a_i)^{m_i}$ where $a_i \in \Bbbk$. The Sun-Tzu-Aryabhata's Remainder theorem gives:

$$\frac{\Bbbk[T]}{(c_A(T))} \cong \bigoplus_i \frac{\Bbbk[T]}{((T - a_i)^{m_i}) = I_i}$$

by

$$f(T) + (c_A(T)) \mapsto (f(T) + I_1, \dots, f(T) + I_k).$$

So the system $\begin{cases} P(T) \equiv a_i \mod I_i \quad \forall i \\ P(T) \equiv 0 \mod T \end{cases}$ has to be a solution $\mod (c_A(T))$. Put $Q(T) = T - P(T)$ and $A_s \colon P(A)$, $A_n = Q(A)$. correspondingly,

$$V = \bigoplus_{i=1}^{k} V_i$$

where $V_i = \{v \in V \mid (A - a_i)^{m_i} v = 0\}$. Then $A_s|_{V_i} = a_i \operatorname{Id}_{V_i}$ $\forall i$ i.e. $(A - a_i) \operatorname{Id}_{V_i} = 0$. This implies A_s is diagonal on each V_i, thus A_s is semisimple.

Then $A_n = A - A_s$ is nilpotent on each V_i

$$(A_n|_{V_i})^{m_i} = (A|_{V_i} - A_s|_{V_i})^{m_i} = (A - a_i)^{m_i} = 0.$$

Hence $A = A_s + A_n$ by construction $[A_s, A_n] = [P(A), Q(A)] = 0$. To show uniqueness, suppose $A = S + N$ also. Then $A = A_s + A_n = S + N \Rightarrow A_s - S = N - A_n$ is both semisimple and nilpotent $\Rightarrow$ they are 0.

c) $\operatorname{ad} A = \operatorname{ad}(A_s + A_n) = \operatorname{ad}(A_s) + \operatorname{ad}(A_n)$ as in part c) we get $\operatorname{ad}(A_s)$ is semisimple, $\operatorname{ad}(A_n)$ nilpotent, and they commute. So by uniqueness $(\operatorname{ad} A)_s = \operatorname{ad}(A_s)$ and $(\operatorname{ad} A)_n = \operatorname{ad}(A_n)$. □

Note 16.3. If $\psi \colon \operatorname{End}(V) \to \operatorname{End}(W)$ is linear and satisfies A is semisimple (nilpotent) implies $\psi(A)$ is semisimple (nilpotent), and $[A, B] = 0 \Rightarrow [\psi(A), \psi(B)] = 0$ then:

$$\psi(A)_s = \psi(A_s),$$
$$\psi(A)_n = \psi(A_n).$$

16.1 Cartan's First Criterion

Proposition 16.4. *Let $\mathbb{k}$ be an algebraically closed field of characteristic zero, and V be a finite-dimensional vector space over $\mathbb{k}$. Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$. Then*

(a) *If $\mathfrak{g}$ is solvable, then $\operatorname{Tr}(xy) = 0$ for all $x \in \mathfrak{g}$, $y \in \mathfrak{g}'$ (recall $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$);*

(b) *If $\operatorname{Tr}(xy) = 0$ for all $x, y \in \mathfrak{g}$, then $\mathfrak{g}$ is solvable.*

Proof. .... (to be typed up) .... □

Theorem 16.5 (Cartan's First Criterion - algebraically closed field case)**.** *Let $\mathbb{k}$ be an algebraically closed field of characteristic zero, and V be a finite-dimensional vector space over $\mathbb{k}$. Then $\mathfrak{g}$ is solvable if and only if $\operatorname{Tr}\big((\operatorname{ad} x) \circ (\operatorname{ad} y)\big) = 0$ for all $x \in \mathfrak{g}$ and $y \in \mathfrak{g}'$.*

Proof. If $\mathfrak{g}$ is solvable, then $\operatorname{ad}\mathfrak{g} \subset \mathfrak{gl}(\mathfrak{g})$ is solvable hence the conclusion follows from part (a) of Proposition 16.4.

Conversely, if $\operatorname{Tr}\big((\operatorname{ad} x)(\operatorname{ad} y)\big) = 0$ for all $x \in \mathfrak{g}$, $y \in \mathfrak{g}'$, then $\operatorname{ad}(\mathfrak{g}')$ is solvable by part (b) of Proposition 16.4. Since $\operatorname{ad}(\mathfrak{g}') \cong \mathfrak{g}'/\mathfrak{z}(\mathfrak{g}')$ and $\mathfrak{z}(\mathfrak{g}')$ is abelian hence solvable, it follows that $\mathfrak{g}'$ is solvable. Say $D^n(\mathfrak{g}') = 0$. But then $D^{n+1}\mathfrak{g} = D^n(\mathfrak{g}') = 0$ so $\mathfrak{g}$ is solvable. □

17 Lecture 15: Cartan's Second Criterion

This lecture will focus on a method for checking whether or not a given Lie algebra is semisimple or solvable based on the so called Killing[viii] form.

17.1 Invariant Symmetric Bilinear Forms

A *symmetric bilinear form* $b\colon \mathfrak{g} \times \mathfrak{g} \to \Bbbk$ is a function satisfying $b(\lambda x + \mu y, z) = \lambda b(x) + \mu b(y, z)$ and $b(x, y) = b(y, x)$ for all $x, y \in \mathfrak{g}$ and $\lambda, \mu \in \Bbbk$.

Definition 17.1. A symmetric bilinear form $b\colon \mathfrak{g} \times \mathfrak{g} \to \Bbbk$ is

(i) *invariant* if
$$b([y, x], z) = b(y, [x, z]) \qquad \forall x, y, z \in \mathfrak{g}, \tag{17.1}$$

(ii) *non-degenerate* if $\forall x \in \mathfrak{g}\backslash\{0\} \exists y \in \mathfrak{g} : b(x, y) \neq 0$.

Remark 17.2. The reason for the name "invariant" is as follows. First, a vector v in a representation of $\mathfrak{g}$ is called *invariant* if $x.v = 0$ for all $x \in \mathfrak{g}$ (recall we write $x.v$ for $\rho_V(x)v$). Now if V is a representation of $\mathfrak{g}$ then one can verify that the space of all bilinear forms on V is a representation of $\mathfrak{g}$ with action $(x.b)(v, w) = -b(x.v, w) - b(v, x.w)$. When $V = \mathfrak{g}$ is the adjoint representation, this is equivalent to $(x.b)(y, z) = -b\big([x, y], z\big) - b\big(y, [x, z]\big)$. Then (17.1) is equivalent to $x.b = 0$ (check!), i.e. b is an invariant vector of the representation $(V \otimes V)^*$.

Definition 17.3.

(i) Given a finite-dimensional representation V of $\mathfrak{g}$ the *trace form* on $\mathfrak{g}$ (with respect to V) is $b_V\colon \mathfrak{g} \times \mathfrak{g} \to \Bbbk$,
$$b_V(x, y) = \operatorname{Tr}\big(\rho_V(x)\rho_V(y)\big), \quad \forall x, y \in \mathfrak{g}.$$

(ii) The trace form on $\mathfrak{g}$ with respect to the adjoint representation is called the *Killing form* and is denoted by κ or $\kappa^{\mathfrak{g}}$. Explicitly:
$$\kappa(x, y) = \operatorname{Tr}\big(\operatorname{ad}(x)\operatorname{ad}(y)\big), \quad \forall x, y \in \mathfrak{g}.$$

Exercise 17.4. Show that the trace form (with respect to any finite-dimensional representation V) is an invariant symmetric bilinear form on $\mathfrak{g}$.

Lemma 17.5. *If $b\colon \mathfrak{g} \times \mathfrak{g} \to \Bbbk$ is an invariant symmetric bilinear form on $\mathfrak{g}$ and $I \subset \mathfrak{g}$ is an ideal, then*
$$I^\perp = \{x \in \mathfrak{g} \mid b(x, z) = 0 \ \forall z \in I\}$$
is an ideal in $\mathfrak{g}$.

Proof. $x \in I^\perp$, we show that $[x, y] \in I^\perp$ for all $y \in \mathfrak{g}$. Indeed, for all $z \in I$ we have
$$b([x, y], z) = b(x, [y, z]) = 0$$
hence, $[x, y] \in I^\perp$. □

Corollary 17.6. $\mathfrak{g}^\perp = \ker(b)$ *is an ideal in $\mathfrak{g}$.*

[viii] Wilhelm Killing (1847 – 1923)

Example 17.7. Let $\mathfrak{g} = \Bbbk a \oplus \Bbbk b$ be a two-dimensional vector space with bracket uniquely determined by the condition $[a, b] = a$. This is the unique non-abelian Lie algebra of dimension 2 up to isomorphism (exercise). In the ordered basis (a, b) the matrices for $\operatorname{ad} a$ and $\operatorname{ad} b$ are $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ respectively. Thus it is easy to check that $\kappa(a, a) = \kappa(a, b) = 0$ and $\kappa(b, b) = 1$.

Example 17.8. a) The space of column vectors $\Bbbk^n$ is a representation of $\mathfrak{g} = \mathfrak{gl}(n, \Bbbk)$ called the *vector representation* (or the *tautological representation*). The trace form is just $b(x, y) = \operatorname{Tr}(xy)$. When $n = 2$, put

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

with respect to the ordered basis (e, f, h, I), the Gram matrix of the trace form on $\mathfrak{gl}(2, \Bbbk)$ is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

For example, $b_{\Bbbk^2}(e, e) = \operatorname{Tr}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = 0$.

b) By a similar but much longer calculation one can see that the Gram matrix of the Killing form κ on $\mathfrak{gl}(2, \Bbbk)$ is

$$\begin{bmatrix} 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

17.2 Further reading: reductive Lie algebras

Theorem 17.9. *If there exists a representation V of $\mathfrak{g}$ such that the trace form b_V is non-degenerate, then $\mathfrak{g}$ is reductive i.e.* $\operatorname{rad}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$.

Proof. $\operatorname{rad}(\mathfrak{g}) \supset \mathfrak{z}(\mathfrak{g})$ is always true, so all that remains is to show the reverse containment. That is, we must show $[\mathfrak{g}, \operatorname{rad}(\mathfrak{g})] = 0$.

(1) $x \in [\mathfrak{g}, \operatorname{rad}(\mathfrak{g})]$ acts by 0 on any irreducible W (the proof of this is omitted. This implies that $x \in \ker(b_W)$.

(2) For

$$0 \to W' \to V \to W \to 0$$

we have that $b_V = b_{W'} + b_W$ by

$$\operatorname{Tr}\left(\begin{bmatrix} X_{W'} & * \\ 0 & X_W \end{bmatrix}\begin{bmatrix} Y_{W'} & * \\ 0 & Y_W \end{bmatrix}\right) = \operatorname{Tr}(X_{W'}Y_{W'}) + \operatorname{Tr}(X_W Y_W)$$

(3) By induction on $\dim V$ we show that $x \in \ker(b_V) = \{0\} \Rightarrow x = 0$. □

Theorem 17.10. *Each classical Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n, \Bbbk)$ is reductive.*

Proof. $b_{\Bbbk^n}$ is non-degenerate $\Rightarrow \mathfrak{g}$ reductive. □

17.3 Cartan's Criteria

Lemma 17.11. *Let I be an ideal of a finite-dimensional Lie algebra $\mathfrak{g}$ with Killing form $\kappa^{\mathfrak{g}}$. Let κ^I be the Killing form on I regarded as a Lie algebra in its own right. Then $\kappa^I(x,y) = \kappa^{\mathfrak{g}}(x,y)$ for all $x, y \in I$.*

Proof. Choose a linear complement V to I in $\mathfrak{g}$. Thus $\mathfrak{g} = I \oplus V$ as vector spaces. Consider a basis for $\mathfrak{g}$ which is the union of bases for I and V. Let $x, y \in I$. In the chosen basis,

$$\operatorname{ad} x = \begin{bmatrix} A_x & B_x \\ 0 & 0 \end{bmatrix}, \qquad \operatorname{ad} y = \begin{bmatrix} A_y & B_y \\ 0 & 0 \end{bmatrix},$$

for some matrices A_x, B_x, A_y, B_y of appropriate size. Note that A_x and A_y are the matrices for $\operatorname{ad}^I x : I \to I$ and $\operatorname{ad}^I y : I \to I$, where ad^I denotes the adjoint representation of I. We have

$$(\operatorname{ad} x)(\operatorname{ad} y) = \begin{bmatrix} A_x A_y & A_x B_y \\ 0 & 0 \end{bmatrix}$$

so that

$$\kappa^{\mathfrak{g}}(x,y) = \operatorname{Tr}\big((\operatorname{ad} x)(\operatorname{ad} y)\big) = \operatorname{Tr}(A_x A_y) = \operatorname{Tr}\big((\operatorname{ad}^I x)(\operatorname{ad}^I y)\big) = \kappa^I(x,y).$$

□

Lemma 17.12. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with nonzero radical. Then there exists a nonzero abelian ideal of $\mathfrak{g}$.*

Proof. Let I be the radical of $\mathfrak{g}$. Since I is an ideal of $\mathfrak{g}$, so is $[I, I]$ (by the Leibniz Rule form of Jacobi identity: $[x,[y,z]] = [[x,y],z] + [y,[x,z]]$). Repeating this argument we see that each term of the derived series for I is an ideal of $\mathfrak{g}$. In particular the last nonzero term of the series is an abelian ideal of $\mathfrak{g}$. □

Theorem 17.13 (Cartan's Criteria)**.** *Let $\Bbbk$ be any field of characteristic zero. Then:*

a) $\mathfrak{g}$ *solvable iff* $\kappa(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$*;*

b) $\mathfrak{g}$ *semisimple iff* κ *non-degenerate.*

Proof.
a) We proved this in the previous lecture in the case when $\Bbbk$ is algebraically closed. In general, let $\bar{\mathfrak{g}} = \bar{\Bbbk} \otimes_{\Bbbk} \mathfrak{g}$ be the Lie algebra over the algebraic closure $\bar{\Bbbk}$ of $\Bbbk$. (This process is called *extension of scalars*. The Lie bracket on $\bar{\mathfrak{g}}$ is $[\lambda \otimes x, \mu \otimes y] = (\lambda\mu) \otimes [x,y]$ for all $x, y \in \mathfrak{g}$ and $\lambda, \mu \in \bar{\Bbbk}$.) Then

$$\begin{aligned} \mathfrak{g} \text{ solvable} &\iff \bar{\mathfrak{g}} \text{ solvable} \\ &\iff \kappa\big(\bar{\mathfrak{g}}, [\bar{\mathfrak{g}}, \bar{\mathfrak{g}}]\big) = 0 \qquad \text{by last time} \\ &\iff \kappa\big(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]\big) = 0. \end{aligned}$$

b) We prove the contrapositive. Suppose $\mathfrak{g}$ is not semisimple. By Lemma 17.12, there exists a nonzero abelian ideal $\mathfrak{a}$ of $\mathfrak{g}$. Now for any $x \in \mathfrak{g}$ and $a \in \mathfrak{a}$, the composition $(\operatorname{ad} x)(\operatorname{ad} a)$ maps $\mathfrak{g}$ into $\mathfrak{a}$, since $\mathfrak{a}$ is an ideal. Since $\mathfrak{a}$ is abelian, $(\operatorname{ad} a)(\operatorname{ad} x)(\operatorname{ad} a)$ is identically zero. Thus $(\operatorname{ad} x)(\operatorname{ad} a)$ is nilpotent, hence has trace zero. Thus $\kappa(x, a) = 0$ for all $x \in \mathfrak{g}$ and $a \in \mathfrak{a}$. Since $\mathfrak{a} \neq 0$, this shows κ is degenerate.

For the reverse implication, suppose $\mathfrak{g}$ is semisimple. Consider the space $I = \{x \in \mathfrak{g} \mid \kappa(x,y) = 0 \forall y \in \mathfrak{g}\}$ (this is called the *radical* or *kernel* of the form κ). We claim that I is solvable hence equal to zero. First, I is an ideal by associativity of κ. By Lemma 17.11, $\kappa^I(I, [I, I]) = \kappa(I, [I, I]) \subset \kappa(I, \mathfrak{g}) = 0$. By part a), I is solvable. Since $\mathfrak{g}$ is semisimple, $I = 0$. Thus, κ is non-degenerate. □

Proposition 17.14. *If $\mathfrak{g}$ is semisimple, and $I \subset \mathfrak{g}$ is an ideal, then $I^\perp = \{x \in \mathfrak{g} \mid \kappa(x,y) = 0 \; \forall y \in I\}$ is an ideal of $\mathfrak{g}$, and $\mathfrak{g} = I \oplus I^\perp$.*

Proof. $I^\perp$ is an ideal by Lemma 17.5. We claim that $I \cap I^\perp = \{0\}$. Let $x, y, z \in I \cap I^\perp$. Then $\kappa\big(x, [y,z]\big) = 0$ simply because $x \in I^\perp$ and $[y,z] \in I$. By Lemma 17.11, $\kappa^{I \cap I^\perp}(x, [y,z]) = 0$ too. By Cartan's First Criterion, applied to $I \cap I^\perp$, we conclude that $I \cap I^\perp$ is solvable. Since $\mathfrak{g}$ is semisimple, $I \cap I^\perp = 0$.

Next we show that $I + I^\perp = \mathfrak{g}$. Let $\{x_i\}_{i=1}^m$ be a basis for I and extend it to a basis $\{x_i\}_{i=1}^n$ for $\mathfrak{g}$. By Cartan's Second Criterion, κ is non-degenerate. Therefore there exists a corresponding dual basis $\{x^i\}_{i=1}^n$ ($\mathfrak{g} \to \mathfrak{g}^*$, $x \mapsto \kappa(x,\cdot)$ is an isomorphism, so to the linear functional $\xi^j \in \mathfrak{g}^*$, $\xi^j(x_i) = \delta_{ij}$, there corresponds a $x^j \in \mathfrak{g}$ satisfying $\kappa(x_i, x^j) = \xi^j(x_i) = \delta_{ij}$). Then, clearly, $x^j \in I^\perp$ for $j = m+1, m+2, \dots, n$. This shows that $\dim I^\perp = n - m$. Since $I \cap I^\perp = 0$ we have $\dim(I + I^\perp) = \dim I + \dim I^\perp = m + (n-m) = n$. Thus $\mathfrak{g} = I + I^\perp$.

□

We can now prove the following theorem, which motivates the name "semisimple".

Theorem 17.15. *Let $\mathbb{k}$ be a field of characteristic zero and $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{k}$. Then $\mathfrak{g}$ is semisimple iff $\mathfrak{g} = I_1 \oplus I_2 \oplus \cdots \oplus I_k$, where I_j are simple (as Lie algebras) ideals. Moreover, in this case every ideal of $\mathfrak{g}$ is equal to $\bigoplus_{j \in S} I_j$ for some subset $S \subset \{1, 2, \dots, k\}$.*

Proof. Suppose $\mathfrak{g}$ is semisimple. Let I_1 be a minimal nonzero ideal of $\mathfrak{g}$. If $I_1 = \mathfrak{g}$ we are done. Otherwise, by Proposition 17.14, $\mathfrak{g} = I \oplus I^\perp$. Since any solvable ideal $I_1^\perp$ would be a solvable ideal of $\mathfrak{g}$, $I_1^\perp$ is also semisimple. By induction on $\dim \mathfrak{g}$ we are done.

The converse is an exercise (see below).

For the last claim, first suppose that I is a simple ideal of $\mathfrak{g}$. Then $[\mathfrak{g}, I] = \bigoplus_{j=1}^k [I_j, I]$. By simplicity of I, all but one summand is zero. Say $[I_j, I] \neq 0$. Then $[I_j, I] = I = I_j$ by simplicity of I and I_j. Now suppose I is any ideal of $\mathfrak{g}$. Then $\mathfrak{g} = I \oplus I^\perp$ and any ideal of I is an ideal of $\mathfrak{g}$. In particular, I is semisimple. Say $I = J_1 \oplus J_r$ for some simple ideals J_i of I. But then J_i are simple ideals of $\mathfrak{g}$ hence each J_i is equal to one of the I_i's. This proves the claim. □

Exercise 17.16. Let $\mathbb{k}$ be a field of characteristic zero and $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{k}$. Suppose that $\mathfrak{g}$ $\mathfrak{g} = I_1 \oplus I_2 \oplus \cdots \oplus I_k$, where I_j are simple (as Lie algebras) ideals. Prove that $\mathfrak{g}$ is semisimple.

18 Lecture 16: Semisimple Lie Algebras; the Casimir Operator

Before moving on we point out two important corollaries of previous lecture.

Corollary 18.1. $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ *if $\mathfrak{g}$ is semisimple.*

Proof.

$$
\begin{aligned}
[\mathfrak{g}, \mathfrak{g}] &= [\oplus_j I_j, \oplus_k I_k] \\
&= \oplus_{j,k} [I_j, I_k] \qquad \text{since for } j \neq k \ [I_j, I_k] \subset I_j \cap I_k = 0 \\
&= \oplus_j [I_j, I_j] \\
&= \oplus I_i \\
&= \mathfrak{g}\,.
\end{aligned}
$$

□

Corollary 18.2. *If $\mathfrak{g}$ is semisimple, then I and $\mathfrak{g}/I$ are semisimple for any ideal I of $\mathfrak{g}$.*

Proof. If I is any ideal of $\mathfrak{g}$ then $\mathfrak{g} = I \oplus I^\perp$ with respect to the Killing form. Then for any ideal J of I we have $[\mathfrak{g}, J] = [I, J] \oplus [I^\perp, J] \subset J$ since $[I^\perp, J] \subset I^\perp \cap I = 0$. Furthermore, $\mathfrak{g}/I \cong I^\perp$ which is an ideal of $\mathfrak{g}$ hence semisimple by the previous part. □

We also prove the following uniqueness theorem for the Killing form on a simple Lie algebra. Here we do need algebraically closed field to ensure the existence of at least one eigenvalue.

Proposition 18.3. *Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra over an algebraically closed field of characteristic zero. Then there exists a unique (up to nonzero scalar multiple) invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$.*

Proof. We know the Killing form κ is one such form. Suppose $\beta : \mathfrak{g} \times \mathfrak{g} \to \Bbbk$ is another. Then we have two vector space isomorphisms $\kappa_1 : \mathfrak{g} \to \mathfrak{g}^*$, $x \mapsto \kappa(x, \cdot)$ and similarly $\beta_1 : \mathfrak{g} \to \mathfrak{g}^*$. Let $x_0 \in \mathfrak{g}\backslash\{0\}$ be an eigenvector of the non-singular transformation $\beta_1^{-1}\kappa_1$ of $\mathfrak{g}$. Thus $\kappa_1(x_0) = \lambda\beta_1(x_0)$ for some nonzero $\lambda \in \Bbbk$. This means $\kappa(x_0, y) = \lambda\beta(x_0, y)$ for all $y \in \mathfrak{g}$. Let

$$I = \{x \in \mathfrak{g} \mid \kappa(x, y) = \lambda\beta(x, y) \ \forall y \in \mathfrak{g}\}.$$

We claim I is an ideal of $\mathfrak{g}$. For any $x \in I$ and $y, z \in \mathfrak{g}$ we have

$$\kappa\big([x, y], z\big) = \kappa\big(x, [y, z]\big) = \lambda\beta\big(x, [y, z]\big) = \lambda\beta([x, y], z).$$

Thus $[x, y] \in I$ for all $x \in I$, $y \in \mathfrak{g}$. So I is an ideal of $\mathfrak{g}$. Since $\mathfrak{g}$ is simple and I is nonzero (containing x_0), we have $I = \mathfrak{g}$. This shows that $\kappa(x, y) = \lambda\beta(x, y)$ for all $x, y \in \mathfrak{g}$. □

18.1 The Casimir Operator

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, and suppose $\langle \cdot, \cdot \rangle$ is a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$. For example, if $\mathfrak{g}$ is semi-simple we could use $\langle x, y \rangle = \kappa(x, y)$ (the Killing form) and if $\mathfrak{g} = \mathfrak{gl}_n$ we can use $\langle x, y \rangle = \mathrm{Tr}(xy)$. (One can show that the existence of such a form is equivalent to that $\mathrm{rad}\,(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$ i.e. $\mathfrak{g}$ is reductive.)

Let V be a representation of $\mathfrak{g}$. The *Casimir operator (on V with respect to $\langle \cdot, \cdot \rangle$)*, denoted $C_V \in \mathrm{End}_\Bbbk(V)$, is defined by

$$C_V = \sum_{i=1}^{n} \rho_V(x_i)\rho_V(x^i),$$

where $\{x_i\}_{i=1}^n$ is a basis for $\mathfrak{g}$ and $\{x^i\}_{i=1}^n$ is the corresponding dual basis with respect to $\langle \cdot, \cdot \rangle$.

Proposition 18.4 (Properties of the Casimir operator)**.** (a) *The Casimir operator C_V is independent of the choice of basis $\{x_i\}_{i=1}^n$ for $\mathfrak{g}$;*

(b) *$C_V \circ \rho_V(x) = \rho_V(x) \circ C_V$ for all $x \in \mathfrak{g}$. In other words, C_V is an* intertwining operator *from V to V.*

Before the proof, to put the second property in context, we give the following definition.

Definition 18.5. Let $\mathfrak{g}$ be a Lie algebra and V, W be representations of $\mathfrak{g}$. An *intertwining operator* $T : V \to W$ is a linear map such that

$$T \circ \rho_V(x) = \rho_W(x) \circ T, \qquad \forall x \in \mathfrak{g}.$$

Another term for intertwining operator is *morphism of representations*. The *category of representations of* $\mathfrak{g}$, denoted $\mathsf{Rep}_{\mathfrak{g}}$, is the category whose objects are representations V of $\mathfrak{g}$, and morphisms $V \to W$ are the intertwining operators. Thus the Casimir operators are canonical intertwining operators from any representation to itself. They play a critical role in the study of semisimple (and more generally, reductive) Lie algebras.

Proof of Proposition 18.4. (a) Let $\{y_i\}_{i=1}^n$ be another basis for $\mathfrak{g}$, and $\{y^i\}_{i=1}^n$ the corresponding dual basis with respect to β_V. Then

$$y_i = \sum_k a_{ik} x_k \qquad y^i = \sum_k b_{ik} x^k$$

and furthermore

$$\delta_{ij} = \langle y_i, y^j \rangle = \sum_{k,l} a_{ik} b_{jl} \langle x_k, x^l \rangle = \sum_k a_{ik} b_{jk},$$

which is to say $AB^T = I$ where $A = (a_{ij})$ and $B = (b_{ij})$. By matrix theory, this implies $B^T A = I$ as well, which can be written

$$\sum_i b_{ik} a_{il} = \delta_{kl}. \tag{18.1}$$

The Casimir operator with respect to the y-bases equals

$$\sum_i \rho_V(y_i)\rho_V(y^i) = \sum_{i,k,l} a_{ik} b_{il} \rho_V(x_k)\rho_V(x^l) \overset{(18.1)}{=} \sum_{kl} \delta_{kl} \rho_V(x_k)\rho_v(x^l) = C_V.$$

(b) Let $x \in \mathfrak{g}$. We will use that for any $x \in \mathfrak{g}$ we have

$$\begin{aligned} x &= \sum_i \langle x, x_i \rangle x^i, \\ x &= \sum_i \langle x, x^i \rangle x_i. \end{aligned} \tag{18.2}$$

The first equality follows from writing $x = \sum_j c_j x^j$ and then calculating that $\langle x, x_i \rangle = c_i$. The second one is proved similarly. Put $\rho = \rho_V$ for brevity. We have:

$$\begin{aligned} [C_V, \rho(x)] &= \sum_i [\rho(x_i), \rho(x)]\rho(x^i) + \rho(x_i)[\rho(x^i), \rho(x)] \qquad \text{by Leibniz' Rule} \\ &= \sum_{i,j} \rho\big(\langle [x_i, x], x^j \rangle x_j\big)\rho(x^i) + \rho(x_i)\rho\big(\langle [x^i, x], x_j \rangle x^j\big) \\ &= \sum_{i,j} \big(\langle [x_i, x], x^j \rangle + \langle [x^j, x], x_i \rangle\big)\rho(x_j)\rho(x^i) = 0 \end{aligned}$$

where, in the last equality, we used that $\langle [x^j, x], x_i \rangle = \langle x_i, [x^j, x] \rangle = -\langle x_i, [x, x^j] \rangle = -\langle [x_i, x], x^j \rangle$ by symmetry and invariance of the form, and by anti-commutativity of the bracket. $\square$

19 Lecture 17: Chevalley-Eilenberg Cohomology

Let $\mathfrak{g}$ be a Lie algebra and M be a representation of $\mathfrak{g}$. We use module notation: $x.v = \rho_V(x)v$ for $x \in \mathfrak{g}$ and $v \in M$.

19.1 Cochains

Definition 19.1. Let i be a non-negative integer. An *i-dimensional cochain for $\mathfrak{g}$ with values in M*, or *i-cochain* for short, is a linear map

$$f : \bigwedge^i \mathfrak{g} \to M.$$

In more concrete terms:

- a 0-cochain is an element $v \in M$ (because $\bigwedge^0 \mathfrak{g} = \Bbbk$ and a linear map $\Bbbk \to M$ is determined by the image of $1_{\Bbbk}$),
- a 1-cochain is a linear map $f : \mathfrak{g} \to M$,
- a 2-cochain is a bilinear map $f : \mathfrak{g} \times \mathfrak{g} \to M$ such that $f(x,y) = -f(y,x)$ for all $x, y \in \mathfrak{g}$.

In general, for $i > 1$ an i-cochain can be viewed as a multilinear map $f : \mathfrak{g}^i \to M$ which is alternating i.e. switching any two arguments results in the negative.

The vector space of all i-cochains is denoted by

$$C^i(\mathfrak{g}, M) = \mathrm{Hom}_{\Bbbk}\big(\bigwedge^i \mathfrak{g}, M\big).$$

19.2 The coboundary map

For each i-cochain f we define an $(i+1)$-cochain df (or $d^i f$ when i needs to be specified) as follows:

- If $v \in M$ is a 0-cochain, we define $dv \in C^1(\mathfrak{g}, M) = \mathrm{Hom}(\mathfrak{g}, M)$ by

$$(dv)(x) = x.v$$

- If $f : \mathfrak{g} \to M$ is a 1-cochain, we define $df \in C^2(\mathfrak{g}, M)$ by

$$(df)(x_1, x_2) = x_1.f(x_2) - x_2.f(x_1) - f([x_1, x_2])$$

- If $f : \mathfrak{g} \times \mathfrak{g} \to M$ is a 2-cochain, we define $df \in C^3(\mathfrak{g}, M)$ by

$$\begin{aligned}(df)(x_1, x_2, x_3) &= x_1.f(x_2, x_3) - x_2.f(x_1, x_3) + x_3.f(x_1, x_2) \\ &\quad - f\big([x_1, x_2], x_3\big) + f\big([x_1, x_3], x_2\big) - f\big([x_2, x_3], x_1\big).\end{aligned}$$

Here is the general definition:

Definition 19.2. Let i be a non-negative integer. The *(i:th) coboundary map* is the map

$$d = d^i : C^i(\mathfrak{g}, M) \to C^{i+1}(\mathfrak{g}, M)$$

given by

$$\begin{aligned}(df)(x_1, x_2, \ldots, x_{i+1}) &= \sum_{r=1}^{i+1} (-1)^{r+1} x_r.f(x_1, \ldots, \widehat{x}_r, \ldots, x_{i+1}) \\ &\quad + \sum_{1 \leqslant r < s \leqslant n} (-1)^{r+s} f\big([x_r, x_s], x_1, \ldots, \widehat{x}_r, \ldots, \widehat{x}_s, \ldots, x_{i+1}\big),\end{aligned}$$

for all $f \in C^i(\mathfrak{g}, M)$. A hat means the variable should be omitted from the list of arguments.

The fundamental property of the coboundary map is that applying it twice gives you zero:

Lemma 19.3. *For any non-negative integer i, we have*

$$d^{i+1} \circ d^i = 0.$$

Proof. We only prove this for $i = 0$ and $i = 1$ which are the only cases that we will need anyway.

For $i = 0$, let $v \in M$ be a 0-cochain. Then we have

$$(d^1 d^0 v)(x_1, x_2) = x_1.(x_2.v) - x_2.(x_1.v) - [x_1, x_2].v = 0$$

precisely because M is a representation of $\mathfrak{g}$: $x_1.x_2.v - x_2.x_1.v = [\rho_V(x_1), \rho_V(x_2)]v = \rho_V([x_1, x_2])v = [x_1, x_2].v$.

For $i = 1$, let $f : \mathfrak{g} \to M$ be a 1-cochain. Then

$$\begin{aligned}
(d^2 d^1 f)(x_1, x_2, x_3) &= x_1.\big(x_2.f(x_3) - x_3.f(x_2) - f([x_2, x_3])\big) \\
&\quad - x_2.\big(x_1.f(x_3) - x_3.f(x_1) - f([x_1, x_3])\big) \\
&\quad + x_3.\big(x_1.f(x_2) - x_2.f(x_1) - f([x_1, x_2])\big) \\
&\quad - \big([x_1, x_2].f(x_3) - x_3.f([x_1, x_2]) - f([[x_1, x_2], x_3])\big) \\
&\quad + \big([x_1, x_3].f(x_2) - x_2.f([x_1, x_3]) - f([[x_1, x_3], x_2])\big) \\
&\quad - \big([x_2, x_3].f(x_1) - x_1.f([x_2, x_3]) - f([[x_2, x_3], x_1])\big) \\
&= 0.
\end{aligned}$$

□

19.3 Cocycles, Coboundaries, and Cohomology

For convenience, if i is a negative integer we put $C^i(\mathfrak{g}, M) = 0$ and $d^i = 0$.

Definition 19.4. Let i be a non-negative integer.

- An i-cochain $f \in C^i(\mathfrak{g}, M)$ is called an *i-cocycle* if $df = 0$. The space of i-cocycles is denoted

$$Z^i(\mathfrak{g}, M) = \ker(d^i)$$

- An i-cochain $f \in C^i(\mathfrak{g}, M)$ is called an *i-coboundary* if $f = dg$ for some $g \in C^{i-1}(\mathfrak{g}, M)$. The space of i-coboundaries is denoted

$$B^i(\mathfrak{g}, M) = \operatorname{im}(d^{i-1})$$

Note that, since $d \circ d = 0$, every i-coboundary is an i-cocycle. Thus the following definition makes sense.

Definition 19.5. The *i:th cohomology group of $\mathfrak{g}$ with values in M* is defined as the quotient vector space

$$H^i(\mathfrak{g}, M) = \frac{Z^i(\mathfrak{g}, M)}{B^i(\mathfrak{g}, M)}.$$

The zeroth cohomology group is already somewhat interesting. Since $B^0(\mathfrak{g}, M) = \mathrm{im}(d^{-1}) = 0$, we have $H^0(\mathfrak{g}, M) \cong \ker(d^0) = \{v \in M \mid x.v = 0\ \forall x \in \mathfrak{g}\}$. This is the space of all $\mathfrak{g}$-*invariants* in M and is usually denoted by $M^{\mathfrak{g}}$.

Of particular interest to us will be the first and second cohomology groups:

$$H^1(\mathfrak{g}, M) = \frac{\ker(d^1)}{\mathrm{im}(d^0)}, \qquad H^2(\mathfrak{g}, M) = \frac{\ker(d^2)}{\mathrm{im}(d^1)}.$$

Exercise 19.6. Show that for the adjoint representation $M = \mathfrak{g}$, $Z^1(\mathfrak{g}, \mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$ and $B^1(\mathfrak{g}, \mathfrak{g}) = \mathrm{ad}\,\mathfrak{g}$ (the space of *inner derivations*). Conclude that $H^1(\mathfrak{g}, \mathfrak{g}) = \mathrm{Der}(\mathfrak{g})/\mathrm{ad}\,\mathfrak{g}$, sometimes called the space of *outer derivations*.

20 Lecture 18: Whitehead's First Lemma

In general, cohomology can be thought of as measuring the obstruction to carrying out certain constructions. Thus, vanishing of cohomology means there are no obstructions; a desirable property. Whitehead's First and Second Lemma are two fundamental results about the vanishing of cohomology for semisimple Lie algebras.

Theorem 20.1 (Whitehead's First Lemma)**.** *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic zero, and let M be a finite-dimensional representation of $\mathfrak{g}$. Then*

$$H^1(\mathfrak{g}, M) = 0.$$

In this section we will prove this theorem.

We need two lemmas. The first one is a variant of the Jordan decomposition over not necessarily algebraically closed fields.

Lemma 20.2 (Fitting's Lemma)**.** *Let T be a linear transformation of a finite-dimensional vector space V over a field of characteristic zero. Then there exists a decomposition*

$$V = V_0 \oplus V_1$$

into two subspaces V_i such that $T(V_i) \subset V_i$ for $i = 1, 2$; $T|_{V_0}$ is nilpotent and $T|_{V_1}$ is invertible.

Proof. Let $p(x)$ be the minimal polynomial of T. Write $p(x) = x^d q(x)$ where d is as large as possible. By the Remainder Theorem, the natural map $\Bbbk[x]/(p(x)) \to \Bbbk[x]/(x^d) \times \Bbbk[x]/(q(x))$ is a ring isomorphism. For $i = 0, 1$, let $e_i(x) \in \Bbbk[x]$ such that

$$\begin{cases} e_0(x) \equiv 1 \mod x^d \\ e_0(x) \equiv 0 \mod q(x) \end{cases} \qquad \begin{cases} e_1(x) \equiv 0 \mod x^d \\ e_1(x) \equiv 1 \mod q(x) \end{cases}$$

and define $V_i = e_i(T)V$. We have $e_i(T)^2 = e_i(T)$, $e_0(T) + e_1(T) = 1$ and $e_0(T)e_1(T) = 0$ (since when T is replaced by x the corresponding congruences hold modulo $p(x)$). Hence $V = V_0 \oplus V_1$. Since $e_i(T)$ is a polynomial in T hence commutes with T, it is clear that $T(V_i) \subset V_i$ for $i = 0, 1$. Since $e_0(x) = q(x)a(x)$ for some $a(x) \in \Bbbk[x]$ we have $T^d V_0 = p(T)a(T)V = 0$ so $T|_{V_0}$ is nilpotent. Since $q(x)$ is relatively prime to x we can write $b(x)q(x) + c(x)x = 1$ for some $b(x), c(x) \in \Bbbk[x]$ (Bezout's identity). Thus for $v = e_1(T)w \in V_1$ we have $v = b(T)q(T)e_1(T)w + c(T)Tv = c(T)Tv$ since x^d divides $e_1(x)$ and $q(x)x^d = p(x)$. Thus $T|_{V_1}$ is invertible. ∎

The second result we need is a property of the trace of the Casimir operator. A representation V is called *faithful* if ρ_V is injective.

Lemma 20.3. *Let $\mathfrak{g}$ be a semisimple Lie algebra, and V a faithful representation of $\mathfrak{g}$, and $\langle x, y\rangle = \operatorname{Tr}\big(\rho_V(x)\rho_V(y)\big)$ be the trace form (it is non-degenerate, invariant, symmetric, bilinear.) Let C_V be the corresponding Casimir operator. Then $\operatorname{Tr}(C_V) = \dim\mathfrak{g}$. In particular, if $\mathfrak{g} \neq 0$, then C_V is not nilpotent.*

Proof. $\operatorname{Tr}(C_V) = \sum_i \rho_M(x_i)\rho_M(x^i) = \sum_i \langle x_i, x^i\rangle = \dim\mathfrak{g}$. ∎

We are now ready to prove Theorem 20.1.

Proof of Whitehead's First Lemma. Let $f : \mathfrak{g} \to M$ be a 1-cocycle. Thus f is a linear map satisfying

$$f\big([x, y]\big) = x.f(y) - y.f(x) \quad \forall x, y \in \mathfrak{g}. \tag{20.1}$$

We must show that f is a 1-coboundary. That is, we must find $v \in M$ such that

$$f(x) = x.v \tag{20.2}$$

for all $x \in \mathfrak{g}$.

Let $I = \ker\rho_M$. Then, in module notation, $x.v = 0$ for all $x \in I$ and all $v \in M$. By Proposition 17.14 and Corollary 18.2, there exists a semisimple ideal $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{g} = I \oplus \mathfrak{h}$ as vector spaces. It suffices to find $v \in M$ such that (20.2) holds for all $x \in \mathfrak{h}$ because $I = [I, I]$ (by Corollary 18.1) hence, by (20.1), $f(I) = f([I, I]) = I.f(I) - I.f(I) = 0$. So (20.2) would then hold for all $x \in \mathfrak{g}$ by linearity.

First, suppose that $I = \mathfrak{g}$, i.e. $\mathfrak{h} = 0$. In this case we may simply take $v = 0$, since then (20.2) holds trivially for all $x \in \mathfrak{h}$.

So we may assume $I \neq \mathfrak{g}$. Note that M is a faithful representation of $\mathfrak{h}$ with representation map $\rho = \rho_M\big|_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{gl}(M)$. Let $\{x_i\}_{i=1}^m$ be a basis for $\mathfrak{h}$ and $\{x^i\}_{i=1}^m$ be the corresponding dual basis with respect to the trace form $\langle x, y\rangle = \operatorname{Tr}\big(\rho(x)\rho(y)\big)$, $x, y \in \mathfrak{h}$. By Lemma 20.3, the Casimir operator $C_M = \sum_i \rho(x_i)\rho(x^i)$ associated to $\mathfrak{h}$, is not nilpotent.

Thus, if $M = M_0 \oplus M_1$ is the Fitting decomposition of M relative to C_M (Lemma 20.2), we have $M_1 \neq 0$. Write $f(x) = f_0(x) + f_1(x)$ where $f_i(x) \in M_i$. It is easy to check that f_i both satisfy (20.1). If $\dim M_i < \dim M$ for both $i = 0, 1$ it therefore follows by induction on $\dim M$ that there exist $v_i \in M_i$ such that $f_i(x) = x.v_i$ for all $x \in \mathfrak{h}$. Taking $v = v_0 + v_1$ we have (20.2) for all $x \in \mathfrak{h}$.

So the case that remains is that $M_1 = M$. That is, C_M is invertible. Consider the following element of M:

$$w = \sum_i x_i.f(x^i).$$

For any $x \in \mathfrak{h}$ we have

$$\begin{aligned}
x.w &= \sum_i [x, x_i].f(x^i) + x_i.\big(x.f(x^i)\big) \quad \text{since } M \text{ is a representation of } \mathfrak{h} \\
&= \sum_{i,j} \langle [x, x_i], x^j \rangle x_j.f(x^i) + \sum_i x_i.\big(x.f(x^i)\big) \quad \text{by dual basis property (18.2)} \\
&= \sum_{i,j} x_j.f\big(-\langle x_i, [x, x^j]\rangle x^i\big) + \sum_i x_i.\big(x.f(x^i)\big) \quad \text{by linearity of } f \text{ and invariance of } \langle \cdot, \cdot \rangle \\
&= \sum_j x_j.f\big([x^j, x]\big) + x_j.\big(x.f(x^j)\big) \quad \text{by dual bases property (18.2) and bracket anti-commutativity} \\
&= \sum_j x_j.\big(x^j.f(x)\big) \quad \text{since } f \text{ is a 1-cocycle} \\
&= C_M\big(f(x)\big) \quad \text{by definition of } C_M.
\end{aligned}$$

Thus, define

$$v = C_M^{-1}(w).$$

Since C_M commutes with all $\rho(x)$, $x \in \mathfrak{h}$, the same is true of the inverse of C_M. Consequently, for all $x \in \mathfrak{h}$,

$$x.v = \rho(x)C_M^{-1}w = C_M^{-1}\rho(x)w = C_M^{-1}(x.w) = f(x).$$

□

21 Lecture 19: Abstract Jordan Decomposition

In this lecture we assume that $\Bbbk$ is algebraically closed.

Recall that in this case, the words "semisimple" and "diagonalizable" when talking about linear operators on a finite-dimensional vector space.

Lemma 21.1. *Assume $\Bbbk$ is algebraically closed. If $D \in \mathrm{Der}(\mathfrak{g})$ and $D = D_s + D_n$ is the Jordan decomposition of D, then D_s and D_n belong to $\mathrm{Der}(\mathfrak{g})$.*

Proof. Write $\mathfrak{g} = \bigoplus_{\lambda \in \Bbbk} \mathfrak{g}_\lambda$, where $\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid (D - \lambda)^N(x) = 0, N \gg 0\}$. Then $D_s x = \lambda x$ for all $x \in \mathfrak{g}_\lambda$. For $x, y \in \mathfrak{g}$ we have the identity

$$\big(D - (\lambda + \mu)\big)^N\big([x, y]\big) = \sum_{k=0}^{N} \binom{N}{k} \big[(D - \lambda)^{N-k}(x),\, (D - \mu)^k(y)\big]$$

which can be proved by induction on N. This proves that $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\mu+\lambda}$.

So if $x \in \mathfrak{g}_\lambda$ and $y \in \mathfrak{g}_\mu$ then $D_s\big([x, y]\big) = (\lambda+\mu)[x, y] = [\lambda x, y]+[x, \mu y] = [D_s(x), y]+[x, D_s(y)]$. By bilinearity, $D_s \in \mathrm{Der}(\mathfrak{g})$. Then $D_n = D - D_s \in \mathrm{Der}(\mathfrak{g})$ as well. □

Exercise 21.2. Prove the above identity.

The following proposition shows that in a semisimple Lie algebra over $\Bbbk = \bar{\Bbbk}$, any element can be decomposed in a way that is similar to the Jordan decomposition for a linear transformation.

Proposition 21.3. *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\Bbbk = \bar{\Bbbk}$ and let $x \in \mathfrak{g}$. Then there exists a unique pair $(x_s, x_n) \in \mathfrak{g}^2$ such that $x = x_s + x_n, [x_s, x_n] = 0$, $\mathrm{ad}\, x_s$ is semisimple and $\mathrm{ad}\, x_n$ is nilpotent. Moreover, for any $y \in \mathfrak{g}$ such that $[y, x] = 0$ we also have $[y, x_s] = 0$ and $[y, x_n] = 0$.*

Proof. Let $D = \operatorname{ad} x$ and let $D = D_s + D_n$ be the Jordan decomposition of D. Since $D \in \operatorname{Der}(\mathfrak{g})$ we have $D_s, D_n \in \operatorname{Der}(\mathfrak{g})$ by Lemma 21.1. By Whitehead's First Lemma (Theorem 20.1) and Exercise 19.6, $\operatorname{Der}(\mathfrak{g}) = \operatorname{ad} \mathfrak{g}$. Thus there exist $x_s \in \mathfrak{g}$ and $x_n \in \mathfrak{g}$ such that $\operatorname{ad}(x_s) = D_s$ and $\operatorname{ad}(x_n) = D_n$. Now $\operatorname{ad} x = D_s + D_n = \operatorname{ad}(x_s) + \operatorname{ad}(x_n) = \operatorname{ad}(x_s + x_n)$. Since ad is injective (its kernel equals $\mathfrak{z}(\mathfrak{g})$ which is a solvable ideal of $\mathfrak{g}$ hence zero since $\mathfrak{g}$ is semisimple), we obtain $x = x_s + x_n$. Similarly, $0 = [D_s, D_n] = [\operatorname{ad}(x_s), \operatorname{ad}(x_n)] = \operatorname{ad}[x_s, x_n]$ implies $[x_s, x_n] = 0$. And $\operatorname{ad} x_s = D_s$ is semisimple, $\operatorname{ad} x_n = D_n$ is nilpotent. This shows existence. If $(\widetilde{x}_s, \widetilde{x}_n)$ is another such pair, then $\operatorname{ad} \widetilde{x}_s + \operatorname{ad} \widetilde{x}_n$ is the Jordan decomposition of $\operatorname{ad} x$, hence $\operatorname{ad} \widetilde{x}_s = \operatorname{ad} x_s$ and $\operatorname{ad} \widetilde{x}_n = \operatorname{ad} x_n$ by uniqueness. Thus $\widetilde{x}_s = x_s$ and $\widetilde{x}_n = x_n$. Lastly, if $[y, x] = 0$ then $0 = \operatorname{ad}[y, x] = [\operatorname{ad} y, \operatorname{ad} x]$. So $\operatorname{ad} y$ commutes with $\operatorname{ad} x$. By the usual Jordan decomposition, $\operatorname{ad} y$ therefore also commutes with the semisimple and nilpotent parts, i.e. with $\operatorname{ad}(x_s)$ and $\operatorname{ad}(x_n)$. So $0 = [\operatorname{ad} y, \operatorname{ad}(x_s)] = \operatorname{ad}[y, x_s]$ which implies $[y, x_s] = 0$. Similarly $[y, x_n] = 0$. □

The abstract Jordan decomposition behaves well under representations:

Theorem 21.4. *Let $\mathfrak{g}$ be a semisimple Lie algebra and V a representation of $\mathfrak{g}$. Then*

$$\rho_V(x_s) = \rho_V(x)_s, \qquad \rho_V(x_n) = \rho_V(x)_n, \tag{21.1}$$

where all subscripts refer to the abstract Jordan decomposition.

Proof. Put $\rho = \rho_V$. Let ad denote the adjoint representation of the Lie algebra $\rho(\mathfrak{g})$. We have $\operatorname{ad} \rho(x) = \operatorname{ad} \rho(x_s) + \operatorname{ad} \rho(x_n)$ since $x = x_s + x_n$. Let $y_1, \ldots, y_n$ be a basis for $\mathfrak{g}$ such that $\operatorname{ad}(x_s) y_i = \lambda_i y_i$. That is $[x_s, y_i] = \lambda_i y_i y_i$. Applying ρ this gives $[\rho(x_s), \rho(y_i)] = \lambda_i \rho(y_i)$. Therefore $\operatorname{ad} \rho(x_s)$ is semisimple. Similarly $\operatorname{ad} \rho(x_n)$ is nilpotent, and they commute and sum to $\operatorname{ad} \rho(x)$. By uniqueness of the abstract Jordan decomposition, we have $\operatorname{ad} \rho(x_s) = \operatorname{ad} \rho(x)_s$ and $\operatorname{ad} \rho(x_n) = \operatorname{ad} \rho(x)_n$. Since the image of ρ is isomorphic to a quotient of a semisimple Lie algebra, it is semisimple by Corollary 18.2. Therefore ad is injective. Thus we conclude that (21.1) holds. □

Later on we will prove the following result. It actually requires some more work (Weyl's Theorem).

Theorem 21.5. *Suppose $\bar{\mathbb{k}} = \mathbb{k}$. If $\mathfrak{g} \subset \mathfrak{gl}(V)$ is a semisimple Lie subalgebra then the usual and abstract Jordan decompositions coincide.*

Once we establish this theorem, it means that representations map the abstract Jordan decomposition to the concrete (linear) Jordan decomposition.

22 Lecture 20: Weyl's Theorem on Complete Reducibility. Short Exact Sequences of Lie Algebras.

22.1 Complete Reducibility

Definition 22.1. A representation V is *irreducible* if $V \neq 0$ and the only subrepresentations are $\{0\}$ and V. A representation of is *completely reducible* if

$$V = \bigoplus_{i=1}^{n} V_i$$

where V_i are irreducible subrepresentations.

If V and W are representation of a Lie algebra $\mathfrak{g}$ then the space $\mathrm{Hom}(V, W)$ of all linear maps from V to W is also a representation of $\mathfrak{g}$ with action

$$(x.f)(v) = x.(f(v)) - f(x.v), \qquad \forall x \in \mathfrak{g}, f \in \mathrm{Hom}_{\Bbbk}(V, W), v \in V.$$

Exercise 22.2. Check that this makes $\mathrm{Hom}(V, W)$ into a representation of $\mathfrak{g}$. (That is, check that $[x, y].f = x.y.f - y.x.f$.)

Note that $x.f = 0$ for all $x \in \mathfrak{g}$ is equivalent to that f is an intertwining operator. The space of intertwining operators from V to W is denoted $\mathrm{Hom}_{\mathfrak{g}}(V, W)$. And the space of $\mathfrak{g}$-invariants of a representation V is $V^{\mathfrak{g}} = \{v \in V \mid x.v = 0 \ \forall x \in \mathfrak{g}\}$. Thus we have $\mathrm{Hom}(V, W)^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(V, W)$.

Theorem 22.3 (Weyl's Theorem on Complete Reducibility)**.** *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then every finite-dimensional representation of $\mathfrak{g}$ is completely reducible:*

$$V = \bigoplus_{i=1}^{n} V_i$$

where V_i are irreducible subrepresentations.

Proof. Let V be a finite-dimensional representation of $\mathfrak{g}$ and let $U \subset V$ be a subrepresentation. By induction on $\dim V$ it suffices to show that there exists a subrepresentation U' of V such that $V = U \oplus U'$.

Let

$$\begin{aligned} \mathscr{V} &= \{f \in \mathrm{Hom}(V, U) \mid f\big|_U = \Bbbk \,\mathrm{Id}_U\}, \\ \mathscr{U} &= \{f \in \mathrm{Hom}(V, U) \mid f\big|_U = 0\} \subset \mathscr{V}. \end{aligned}$$

Let $p : V \to U$ be any linear projection, i.e. $p|_U = \mathrm{Id}_U$. Then $p \in \mathscr{V}$ and in fact the image of p in the (one-dimensional) quotient space $\mathscr{V}/\mathscr{U}$ is a basis.

For $x \in \mathfrak{g}$ and $u \in U$ we have $(x.p)(u) = x.(p(u)) - p(x.u) = x.u - x.u = 0$. Thus $x.p \in \mathscr{U}$. Let $\varphi : \mathfrak{g} \to \mathscr{U}$ be given by $\varphi(x) = x.p$ for $x \in \mathfrak{g}$. We show that $\varphi \in Z^1(\mathfrak{g}, \mathscr{U})$:

$$\begin{aligned} \varphi([x, y]) &= [x, y].p = x.y.p - y.x.p \\ &= x.\varphi(y) - y.\varphi(x). \end{aligned}$$

By Whitehead's First Lemma (Theorem 20.1), there exists $T \in \mathscr{U}$ such that $\varphi = d^0 T$. That is, $x.p = x.T$ for all $x \in \mathfrak{g}$. Let $\pi = p - T$. Then $x.\pi = 0$ for all $x \in \mathfrak{g}$. By the comment preceeding the statement of the theorem, this means that π is an intertwining operator from V to U. Also, $\pi^2(v) = \pi(v)$ for all $v \in V$ which means π is a projection. Let $U' = \ker(\pi)$. Since π is an intertwining operator, U' is also a subrepresentation of V. By standard linear algebra arguments, $V = U \oplus U'$. (Any vector $v \in V$ can be written $v = \pi(v) + (v - \pi(v))$ and $\pi(v - \pi(v)) = 0$ so $v - \pi(v) \in U'$. If $v \in U \cap U'$ then $v = \pi(v) = 0$.) □

22.2 Short Exact Sequences

Let I, $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ be any three Lie algebras. A *short exact sequence (SES)*

$$0 \longrightarrow I \xrightarrow{\iota} \widetilde{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

is a sequence of Lie algebras and Lie algebra homomorphisms such that the kernel of each map is the image of the previous map. This means that $\ker(\iota) = \mathrm{im}(0) = 0$ so ι is injective; $\mathrm{im}(\pi) = \ker(0) = \mathfrak{g}$ so π is surjective; and $\ker(\pi) = \mathrm{im}(\iota)$ which means that $\iota(I)$ is an ideal of $\widetilde{\mathfrak{g}}$ and $\widetilde{\mathfrak{g}}/\iota(I) \cong \mathfrak{g}$.

Proposition 22.4. *Given a SES*

$$0 \longrightarrow I \xrightarrow{\iota} \tilde{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

the following are equivalent:

(i) $\tilde{\mathfrak{g}} \simeq \mathfrak{g} \ltimes_\alpha I$ *for some* α.

(ii) *There exists a Lie algebra map* $\sigma : \mathfrak{g} \to \tilde{\mathfrak{g}}$ *called a* section *such that* $\pi \circ \sigma = \mathrm{Id}_{\mathfrak{g}}$.

23 Lecture 21: Whitehead's Second Lemma and Levi's Theorem

23.1 Whitehead's Second Lemma

Theorem 23.1 (Whitehead's Second Lemma)**.** *Let* $\mathfrak{g}$ *be a semisimple Lie algebra over a field of characteristic zero, and let* M *be a finite-dimensional representation of* $\mathfrak{g}$*. Then*

$$H^2(\mathfrak{g}, M) = 0.$$

Proof. Let $I = \ker \rho_M$. By Proposition 17.14 and Corollary 18.2 we have $\mathfrak{g} = I \oplus \mathfrak{h}$ for some semisimple subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let $\{x_i\}_{i=1}^m$ be a basis for $\mathfrak{h}$ and $\{x^i\}_{i=1}^m$ be the dual basis for $\mathfrak{h}$ with respect to the non-degenerate invariant symmetric bilinear form $\langle x, y\rangle = \mathrm{Tr}\big(\rho(x)\rho(y)\big)$, $x, y \in \mathfrak{h}$, $\rho = \rho_M\big|_{\mathfrak{h}}$. Let $C_M = \sum_{i=1}^m \rho(x_i)\rho(x^i)$ be the corresponding Casimir operator.

Let $f \in Z^2(\mathfrak{g}, M)$. Thus $f : \mathfrak{g} \times \mathfrak{g} \to M$ is a bilinear map satisfying $f(x, y) = -f(y, x)$ and

$$\begin{aligned} &y_1.f(y_1, y_3) - y_2.f(y_1, y_3) + y_3.f(y_1, y_2) + \\ &- f\big([y_1, y_2], y_3\big) + f\big([y_1, y_2], y_2\big) - f\big([y_2, y_3], y_1\big) = 0. \end{aligned}$$

This may be written without minus signs in a cyclic permutation way reminiscent of the Jacobi identity:

$$\begin{aligned} &y_1.f(y_2, y_3) + f\big(y_1, [y_2, y_3]\big) \\ &+ y_2.f(y_3, y_1) + f\big(y_2, [y_3, y_1]\big) \\ &+ y_3.f(y_1, y_2) + f\big(y_3, [y_1, y_2]\big) = 0. \end{aligned}$$

Now choose $y_3 = x^i$ and act (via the representation) by x_i, then sum over i to get:

$$\begin{aligned} 0 = \sum_i \Big\{ &x_i.y_1.f(y_2, x^i) + x_i.f\big(y_1, [y_2 x^i]\big) \\ &+ x_i.y_2.f(x^i, y_1) + x_i.f\big(y_2, [x^i, x_i]\big) \\ &+ x_i.f\big(x^i, [y_1, y_2]\big) \Big\} + C_M\big(f(y_1, y_2)\big) \end{aligned}$$

Now use the module identity $x.y.v = y.v.x + [x, y].v$ in the first and second line (ignore the colors for now):

$$\begin{aligned} = \sum_i \Big\{ &y_1.x_i.f(y_2, x^i) + [x_i, y_1].f(y_2, x^i) + x_i.f\big(y_1, [y_2, x^i]\big) \\ &+ y_2.x_i.f(x^i, y_1) + [x_i, y_2].f(x^i, y_1) + x_i.f\big(y_2, [x^i, y_1]\big) \\ &+ x_i.f\big(x^i, [y_1, y_2]\big) \Big\} + C_M\big(f(y_1, y_2)\big). \end{aligned} \tag{23.1}$$

We have, using the dual basis properties (18.2),

$$\sum_i [x_i, y_1].f(y_2, x^i) = \sum_{i,j} \langle [x_i, y_1], x^j \rangle x_j.f(y_2, x^i) = \sum_j x_j.f(y_2, [y_1, x^j])$$

and

$$\sum_i [x_i, y_2].f(x^i, y_1) = \sum_{ij} \langle [x_i, y_2], x^j \rangle.f(x^i, y_1) = \sum_j x_j.f\big([y_2, x^j], y_1\big)$$

thus four terms (red and blue) in (23.1) cancel out pairwise, which yields

$$C_M\big(f(y_1, y_2)\big) + \sum_i \Big\{ y_1.x_i.f(y_2, x^i) + y_2.x_i.f(x^i, y_1) + x_i.f\big(x^i, [y_1, y_2]\big) \Big\} = 0. \tag{23.2}$$

First let us consider the case that $\mathfrak{h} \neq 0$. Let $M = M_0 \oplus M_1$ be the Fitting decomposition of M relative to C_M (Lemma 20.2). Since C_M is an intertwining operator, the M_i are actually $\mathfrak{g}$-subrepresentations of M (the action of I is zero anyway and C_M commutes with $\rho(x)$ for any $x \in \mathfrak{h}$). Since $\mathfrak{h} \neq 0$ and M is a faithful representation of $\mathfrak{h}$, the Casimir C_M is not nilpotent (Lemma 20.3). Thus, $M_1 \neq 0$. Suppose that $M_0 \neq 0$ as well. Write $f(x, y) = f_0(x, y) + f_1(x, y)$ for unique bilinear functions $f_i : \mathfrak{g} \times \mathfrak{g} \to M_i$. Since f is a 2-cocycle it is immediate that each f_i is a 2-cocycle with values in M_i. Thus, by induction on $\dim M$, there are $g_i \in C^1(\mathfrak{g}, M_i)$ such that $d^1 g_i = f_i$. Taking $g = g_0 + g_1$ we obtain that $d^1 g = f$ and thus $f \in B^2(\mathfrak{g}, M)$. So we may assume that $M_0 = 0$. That is, the linear operator $C_M : M \to M$ is invertible. In this case we define $g : \mathfrak{g} \to M$ by

$$g(y) = \sum_{i=1}^{m} C_M^{-1}\big(x_i.f(x^i, y)\big).$$

Then we have

$$\begin{aligned}(d^1 g)(y_1, y_2) &= y_1.g(y_2) - y_2.g(y_1) - g\big([y_1, y_2]\big) \\ &= C_M^{-1}\big(\sum_{i=1}^{m} y_1.x_i.f(x^i, y_2) - y_2.x_i.f(x^i, y) - x_i.f\big(x^i, [y_1, y_2]\big) = f(y_1, y_2)\end{aligned}$$

by (23.2).

It remains to deal with the possibility that $\mathfrak{h} = 0$. That is, $x.v = 0$ for all $x \in \mathfrak{g}$ and $v \in M$. In this case the identity for f can be written

$$f\big([y_1, y_2], y_3\big) + f\big([y_2, y_3], y_1\big) + f\big([y_3, y_1], y_2\big) = 0 \tag{23.3}$$

Put $\mathscr{M} = \operatorname{Hom}(\mathfrak{g}, M)$, regarded as a representatin of $\mathfrak{g}$, and define $F \in \operatorname{Hom}(\mathfrak{g}, \mathscr{M}) = C^1(\mathfrak{g}, \mathscr{M})$ by $F(x)(y) = f(x, y)$. Then (23.3) implies that $F \in Z^1(\mathfrak{g}, \mathscr{M})$. Indeed,

$$F\big([y_1, y_2]\big)(y_3) = f\big([y_1, y_2], y_3\big)$$

while

$$\begin{aligned}\big(y_1.F(y_2) - y_2.F(y_1)\big)(y_3) &= \cancel{y_1.f(y_2, y_3)} - f\big(y_2, [y_1, y_3]\big) \\ &\quad - \cancel{y_2.f(y_1, y_3)} + f\big(y_1, [y_2, y_3]\big) \\ &= -f\big([y_3, y_1], y_2\big) - f\big([y_2, y_3], y_1\big).\end{aligned}$$

Therefore, by Whitehead's first lemma (Theorem 20.1), there exists $g \in \mathscr{M}$ such that $d^0 g = F$. Written out, this is saying $(x.g)(y) = f(x, y)$. By the definition of $\mathfrak{g}$ action on $\mathscr{M} = \operatorname{Hom}(\mathfrak{g}, M)$ this means $x.\big(g(y)\big) - g\big([x, y]\big) = f(x, y)$. Since $\mathfrak{g}$ is acting trivially on M, this is equivalent to that $x.g(y) - y.g(x) - g\big([x, y]\big) = f(x, y)$. That is $d^1 g = f$ (now viewing $g \in \operatorname{Hom}(\mathfrak{g}, M) = C^1(\mathfrak{g}, M)$). Thus $f \in B^2(\mathfrak{g}, M)$. □

It is natural to wonder about the third cohomology group. Although we won't discuss it further here, in general it is nonzero, even for a semisimple Lie algebra.

23.2 Semi-Direct Products of Lie Algebras

Definition 23.2. If a Lie algebra $\mathfrak{g}$ is a vector space direct sum of a subalgebra $\mathfrak{a}$ and an ideal I, then $\mathfrak{g}$ is said to be the *(internal) semi-direct product* of $\mathfrak{a}$ and I and we write

$$\mathfrak{g} = \mathfrak{a} \ltimes I.$$

(or $I \rtimes \mathfrak{a}$.)

Note that if $\mathfrak{g} = \mathfrak{a} \ltimes I$ then the adjoint action on $\mathfrak{g}$ gives a map

$$\alpha : \mathfrak{a} \to \mathrm{Der}(I), \quad \alpha(a)(x) = [a, x] \in I$$

and the bracket in $\mathfrak{g}$ can be expressed as

$$\begin{aligned} [a + x, b + y] &= [a, b] + \big([a, y] + [x, b] + [x, y]\big) \\ &= [a, b] + \big(\alpha(a)(y) - \alpha(b)(x) + [x, y]\big) \end{aligned}$$

for any $a, b \in \mathfrak{a}$ and $x, y \in I$.

We can use this formula the basis for an *external* semi-direct product of any two Lie algebras (not a priori subalgebras of the same Lie algebra).

Definition 23.3. Let $\mathfrak{a}$ and I be any two Lie algebras, and $\alpha\colon \mathfrak{a} \to \mathrm{Der}(I)$ be a Lie algebra homomorphism. Define $\mathfrak{a} \ltimes_\alpha I$ to be the Lie algebra with underlying vector space $\mathfrak{a} \oplus I$ with bracket

$$[a + x, b + y] = [a, b] + \big(\alpha(a)(y) - \alpha(b)(x) + [x, y]\big)$$

for any $a, b \in \mathfrak{a}$ and $x, y \in I$.

Exercise 23.4. Let $\mathfrak{g} = \mathfrak{a} \ltimes_\alpha I$. Show that $\mathfrak{g}$ is indeed a Lie algebra and that $\widetilde{\mathfrak{a}} = \mathfrak{a} \times \{0\}$ is a subalgebra of $\mathfrak{g}$ and $\widetilde{I} = \{0\} \times I$ is an ideal of I. Conclude that $\mathfrak{g}$ is the internal semi-direct product of $\widetilde{\mathfrak{a}}$ and $\widetilde{I}$.

Thus we can go back and forth between internal and external semi-direct products.

23.3 Levi's Theorem

We define

$$\mathfrak{g}_{\mathrm{ss}} = \mathfrak{g} / \mathrm{rad}\,(\mathfrak{g})$$

This is the largest semisimple quotient of $\mathfrak{g}$.

Theorem 23.5 (Levi's Theorem)**.** *Suppose* $\mathrm{char}\,\Bbbk = 0$. *Any finite-dimensional Lie algebra* $\mathfrak{g}$ *is the semi-direct product of a semisimple and a solvable Lie algebra. More precisely, there exists a semisimple subalgebra* $\mathfrak{g}^{\mathrm{ss}}$ *of* $\mathfrak{g}$ *such that* $\mathfrak{g} = \mathfrak{g}^{\mathrm{ss}} \oplus \mathrm{rad}\,(\mathfrak{g})$ *as vector spaces. Equivalently, there is a Lie algebra homomorphism* $\alpha : \mathfrak{g}_{\mathrm{ss}} \to \mathrm{Der}(\mathrm{rad}\,(\mathfrak{g}))$ *such that*

$$\mathfrak{g} \cong \mathfrak{g}_{\mathrm{ss}} \ltimes_\alpha \mathrm{rad}\,(\mathfrak{g})\,.$$

Proof. $\square$

24 Lecture 22: Cartan Subalgebras and the Root Space Decomposition

24.1 Toral and Cartan Subalgebras

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{k}$ and assume $\mathbb{k}$ is algebraically closed.

Then we have the abstract Jordan decomposition available for any element $x \in \mathfrak{g}$.

Definition 24.1. Call $x \in \mathfrak{g}$ *semisimple* if $\operatorname{ad} x$ is semisimple, and *nilpotent* if $\operatorname{ad} x$ is nilpotent.

Definition 24.2. A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is *toral* if

(i) $\mathfrak{h}$ is abelian, and

(ii) $\mathfrak{h}$ consists of semisimple elements of $\mathfrak{g}$.

Note that, when $\mathfrak{g} \neq 0$, there is always at least one nonzero toral subalgebra: If every $x \in \mathfrak{g}$ is nilpotent then $\mathfrak{g}$ is nilpotent by Engels theorem, hence solvable hence $\mathfrak{g} = \operatorname{rad}(\mathfrak{g}) = 0$ which is a contradiction. Therefore there exists $x \in \mathfrak{g}$ with $x_s \neq 0$. Then $\mathbb{k}x_s$ is a toral subalgebra of $\mathfrak{g}$.

Definition 24.3. A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a *Cartan subalgebra* if $\mathfrak{h}$ is toral and not properly contained in another toral subalgebra.

In other words, a Cartan subalgebra is a maximal element of the family of all toral subalgebras of $\mathfrak{g}$.

24.2 The Root Space Decomposition

Suppose we fix a toral subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Eventually we will only be interested in the case of a Cartan subalgebra but the contents of this section does not depend on any assumption about maximality.

Let $\{h_1, h_2, \ldots, h_r\}$ be a basis for $\mathfrak{h}$. Then $\{\operatorname{ad} h_i\}_{i=1}^r$ is a family of commuting diagonalizable linear operators on $\mathfrak{g}$. So there exists a basis for $\mathfrak{g}$ consisting of vectors that are common eigenvectors for all the operators $\operatorname{ad} h_i$, $i = 1, 2, \ldots, r$.

Let x be such a common eigenvector. That means that there exist $\alpha_1, \alpha_2, \ldots, \alpha_r \in \mathbb{k}$ so that

$$[h_i, x] = \alpha_i x, \qquad \forall i = 1, 2, \ldots, r.$$

We wish to express this in a basis independent way. Notice that if $h \in \mathfrak{h}$ and we write $h = \sum_i c_i h_i$, $c_i \in \mathbb{k}$, then

$$[h, x] = [\sum_i c_i h_i, x] = \sum_i c_i [h_i, x] = \big(\sum_i c_i \alpha_i\big) x$$

Thus if we define a linear functional $\alpha \in \mathfrak{h}^* = \operatorname{Hom}(\mathfrak{h}, \mathbb{k})$ by

$$\alpha(h_i) = \alpha_i$$

then we may express the property that x has as follows:

$$[h, x] = \alpha(h) x \qquad \forall h \in \mathfrak{h}. \tag{24.1}$$

This is the basis independent form.

For any $\alpha \in \mathfrak{h}^*$ we put

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h) x \ \forall h \in \mathfrak{h}\}. \tag{24.2}$$

The case $\alpha = 0$ plays a special role, because $\mathfrak{g}_0$ coincides with the *centralizer of* $\mathfrak{h}$ *in* $\mathfrak{g}$:

$$\mathfrak{g}_0 = C_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \mid [h, x] = 0 \ \forall h \in \mathfrak{h}\}.$$

Definition 24.4. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\Bbbk = \bar{\Bbbk}$ and $\mathfrak{h} \subset \mathfrak{g}$ a toral subalgebra.

1. A nonzero linear functional $\alpha \in \mathfrak{h}^*$ is a *root* (of $\mathfrak{g}$ with respect to $\mathfrak{h}$) if $\mathfrak{g}_\alpha \neq 0$.

2. The set of roots of $\mathfrak{g}$ is denoted by $\Phi = \Phi(\mathfrak{g}) = \Phi(\mathfrak{g}, \mathfrak{h})$ and is called the *root system of* $\mathfrak{g}$.

3. $x \in \mathfrak{g}$ is called a *root vector* if $x \in \mathfrak{g}_\alpha$ for some $\alpha \in \Phi$.

4. When α is a root, the vector space $\mathfrak{g}_\alpha$ is called the *root space* associated to α.

Example 24.5. If $\mathfrak{h}$ is one-dimensional, say $\mathfrak{h} = \Bbbk h$, then a root of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is essentially just an eigenvalue of $\operatorname{ad} h$ acting on $\mathfrak{g}$. Similarly in this case a root vector is just an eigenvector of $\operatorname{ad} h$, and root spaces are eigenspaces of $\operatorname{ad} h$.

As discussed above, in this new terminology we may say that $\mathfrak{g}$ has a basis consisting of root vectors. That can also be expressed as follows:

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$$

Since $\mathfrak{g}_\alpha \neq 0$ only for $\alpha \in \{0\} \cup \Phi$ we obtain the so called *root space decomposition* of $\mathfrak{g}$ with respect to the toral subalgebra $\mathfrak{h}$:

$$\mathfrak{g} = C_{\mathfrak{g}}(\mathfrak{h}) \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \tag{24.3}$$

Example 24.6. Let $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{h}$ be the set of diagonal matrices in $\mathfrak{g}$. Then $\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \leqslant i, j \leqslant 3, i \neq j\}$ where $\varepsilon_i : \mathfrak{h} \to \Bbbk$ is defined by $\varepsilon_i(\operatorname{diag}(a_1, a_2, a_3)) = a_i$. To see this it suffices to note that $E_{ij} \in \mathfrak{g}_{\varepsilon_i - \varepsilon_j}$ which follows from the computation

$$[h, E_{ij}] = [\sum_i c_i E_{ii}, E_{ij}] = c_i E_{ij} - c_j E_{ij} = (\varepsilon_i - \varepsilon_j)(h) E_{ij}$$

for any $\mathfrak{h} = \sum_i c_i E_{ii} \in \mathfrak{h}$.

This decomposition will play a key role in the classification of semisimple Lie algebras. Our first steps towards that goal is the following proposition.

Proposition 24.7. *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\Bbbk = \bar{\Bbbk}$ and $\mathfrak{h} \subset \mathfrak{g}$ a toral subalgebra.*

(a) *For all $\alpha, \beta \in \mathfrak{h}^*$ we have*

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}. \tag{24.4}$$

(b) *If $\alpha \in \mathfrak{h}^*$ and $\alpha \neq 0$ then every $x \in \mathfrak{g}_\alpha$ is nilpotent.*

(c) *If $\alpha, \beta \in \mathfrak{h}^*$ and $\alpha + \beta \neq 0$ then the spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal with respect to the Killing form:*

$$\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0. \tag{24.5}$$

Proof. (a) Let $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$ and $h \in \mathfrak{h}$ then by the Lebniz rule form of the Jacobi identity:

$$[h, [x, y]] = [[h, x], y] + [x, [h, y]] = \alpha(h)[x, y] + \beta(h)[x, y] = (\alpha + \beta)(h)[x, y].$$

(b) Since $\mathfrak{g}$ is finite-dimensional, Φ is a finite set. Let $\alpha \in \mathfrak{h}^*$ be nonzero. Choose n to be a large enough positive integer such that $\forall \beta \in \Phi : \beta + n\alpha \notin \Phi$. Then $(\operatorname{ad} \mathfrak{g}_\alpha)^n = 0$.

(c) Let $h \in \mathfrak{h}$ be such that $(\alpha + \beta)(h) \neq 0$. Then for every $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$:

$$\alpha(h)\kappa(x, y) = \kappa\big([h, x], y\big) = -\kappa\big([x, h], y\big) = -\kappa\big(h, [h, y]\big) = -\beta(h)\kappa(x, y)$$

This implies that $\kappa(x, y) = 0$. □

Corollary 24.8. *The restriction of the Killing form κ to $C_{\mathfrak{g}}(\mathfrak{h}) \times C_{\mathfrak{g}}(\mathfrak{h})$ is non-degenerate.*

25 Lecture 23: Properties of the Root Space Decomposition

Example 25.1. Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, then $\mathfrak{h} = \{\text{diagonal matrices of trace } 0\}$ is a Cartan subalgebra. Indeed, $\mathfrak{h}$ is commutative, and if $h \in \mathfrak{h}$,

$$\begin{aligned} \operatorname{ad} h \colon\ & \mathfrak{g} \to \mathfrak{g} \\ & x \mapsto [h, x] \quad \forall x \in \mathfrak{g} \end{aligned}$$

is diagonalizable $h = \begin{bmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{bmatrix} \in \mathfrak{h} = \sum_i h_i E_{ii}$. Then

$$\begin{aligned} (\operatorname{ad} \mathfrak{h})(E_{ij}) &= [\mathfrak{h}, E_{ij}] \\ &= \sum h_k [E_{kk}, E_{ij}] \\ &= (h_i - h_j) E_{ij}. \end{aligned}$$

So $\mathfrak{h}$ is toral. Also, if $x \in C_{\mathfrak{g}}(\mathfrak{h})$, then $[h, x] = 0\ \forall h \in \mathfrak{h}$. Pick h with distinct eigenvalues. This implies any eigenvector for h is an eigenvector for x. Which implies x is diagonal and thus in $x \in \mathfrak{h}$.

Lemma 25.2. *$C_{\mathfrak{g}}(\mathfrak{h})$ is a reductive Lie algebra.*

Proof. If I is any ideal in $C_{\mathfrak{g}}(\mathfrak{h})$ then so is $I^{\perp}$ (with respect to the Killing form on $\mathfrak{g}$) and $C_{\mathfrak{g}}(\mathfrak{h}) = I \oplus I^{\perp}$. Repeating this argument (the Killing form on $\mathfrak{g}$ will be non-degenerate on both I and $I^{\perp}$) we can write $C_{\mathfrak{g}}(\mathfrak{h})$ as a sum of ideals which are either simple or one dimensional. The sum of the one dimensional ideals make up the center. Thus $C_{\mathfrak{g}}(\mathfrak{h})$ is the sum of a semisimple Lie algebra and a central ideal. ☐

Theorem 25.3. *If $\mathfrak{h}$ is a Cartan subalgebra then $C_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.*

Proof. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra.

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha, \qquad \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h) x\ \forall h \in \mathfrak{h}\}.$$

Note, $C_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{g}_0$. We claim that $\mathfrak{g}_0$ is toral. (Then, since $\mathfrak{h} \subset \mathfrak{g}_0$ and $\mathfrak{h}$ maximal among toral subalgebras, $\mathfrak{h} = \mathfrak{g}_0$.) Let $x \in \mathfrak{g}_0$. Then $(\operatorname{ad} x)|_{\mathfrak{g}_0}$ is nilpotent. Otherwise, it has a nonzero eigenvalue and $(\operatorname{ad} x_s)|_{\mathfrak{g}_0} \neq 0$. This implies $x_s \notin \mathfrak{h}$, and $\mathfrak{h} \oplus \Bbbk x_s$ is a toral subalgebra that strictly contains $\mathfrak{h}$ which is a contradiction. By Engel's Theorem (Theorem 14.8), $\mathfrak{g}_0$ is nilpotent. By Lemma 25.2, therefore $\mathfrak{g}_0$ is abelian.

It remains to show that $\mathfrak{g}_0$ consists of semisimple elements. Let $x \in \mathfrak{g}_0$. We want to show that $x_n = 0$. $\operatorname{ad} x_n$ is nilpotent and $\mathfrak{g}_0$ is commutative; therefore, $(\operatorname{ad} x_n)(\operatorname{ad} y)$ is nilpotent for every $y \in \mathfrak{g}_0$. Thus $\operatorname{Tr}\big((\operatorname{ad} x_n)(\operatorname{ad} y)\big) = 0 \qquad \forall y \in \mathfrak{g}_0$. Hence, $x_n = 0$ since $\kappa|_{\mathfrak{g}_0 \times \mathfrak{g}_0}$ is non-degenerate. ☐

Fact: If $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are two Cartan subalgebras of $\mathfrak{g}$, then there exists a Lie algebra automorphism φ of $\mathfrak{g}$ such that $\varphi(\mathfrak{h}_1) = \mathfrak{h}_1$.

Definition 25.4. The *rank* of a semisimple Lie algebra $\mathfrak{g}$ is $\operatorname{rank} \mathfrak{g} = \dim \mathfrak{h}$ where $\mathfrak{h}$ is any Cartan subalgebra.

Example 25.5. $\operatorname{rank} \mathfrak{sl}(n, \mathbb{C}) = n - 1$

25.1 Representations of $\mathfrak{sl}(2,\Bbbk)$

Recall $\mathfrak{sl}(2,\mathbb{C})$ has a basis $\{e,f,h\}$

$$e=\begin{bmatrix}0&1\\0&0\end{bmatrix},\qquad f=\begin{bmatrix}0&0\\1&0\end{bmatrix}, h=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

with $[e,f]=h$, $[h,e]=2e$, $[h,f]=-2f$.

Theorem 25.6. *For each non-negative integer λ there exists an $(\lambda+1)$-dimensional irreducible representation $V(\lambda)$ of $\mathfrak{sl}(2,\Bbbk)$. Furthemore, any finite-dimensional irreducible representation of $\mathfrak{sl}(2,\Bbbk)$ is isomorphic to $V(\lambda)$ for a unique non-negative integer λ.*

Proof. Let $\lambda\in\mathbb{Z}_{\geqslant 0}$, let $V_\lambda=\Bbbk[x,y]_\lambda=\Bbbk x^\lambda\oplus\Bbbk x^{\lambda-1}y\oplus\cdots\oplus\Bbbk y^\lambda$ and define

$$\rho_\lambda\colon\mathfrak{sl}(2,\Bbbk)\to\mathfrak{gl}(V(\lambda))$$

by

$$\rho_\lambda(e)=x\partial_y\qquad\rho_\lambda(f)=y\partial_x\qquad\rho_\lambda(h)=x\partial_x-y\partial_y$$

Note that $\dim V_\lambda=\lambda+1$.

Exercise 25.7. $V(\lambda)$ is an irreducible representation of $\mathfrak{sl}(2,\Bbbk)$.

Conversely, let $(V,\rho=\rho_V)$ be any finite dimensional irreducible representation of $\mathfrak{sl}(2,\Bbbk)$. Put $(E,F,H)=(\rho(e),\rho(f),\rho(h))$ and for $\mu\in\Bbbk$ put $V[\mu]=\{v\in V\mid Hv=\mu v\}$.

Step 1:

$$\begin{aligned}EV[\mu]&\subset V[\mu+2]\\FV[\mu]&\subset V[\mu-2]\\HV[\mu]&\subset V[\mu]\end{aligned}$$

hence, $V':=\bigoplus\limits_{\mu\in\Bbbk}V[\mu]$ is a subrepresentation of V. Since V is finite dimensional and $\Bbbk$ is algebraically closed, $\exists\mu\in\Bbbk$ such that $V[\mu]\neq 0\Rightarrow V'\neq 0\Rightarrow V=V'$ since V is irreducible.

Step 2: $\exists\lambda\in\Bbbk$ and $v_\lambda\in V[\lambda]\backslash\{0\}$ with $Ev_\lambda=0$. Indeed, pick any nonzero $w_\mu\in V[\mu]$ some $\mu\in\Bbbk$. Then $E^n w_\mu\in V[\mu+2n]$. Since V is finite dimensional, eigenvectors are linearly independent, $\Rightarrow\ \exists n\geqslant 0\ E^n w_\mu\neq 0$, $E^{n+1}w_\mu=0$. Put $\lambda=\mu+2n$, $V_\lambda=E^n w_\mu$.

Step 3: $W_\lambda=\operatorname{span}\{F^n v_\lambda\}_{n\geqslant 0}$ is a submodule of V, hence $V=W_\lambda$. Indeed

$$\begin{aligned}HF^n v_\lambda&=(\lambda-2n)F^n v_\lambda\\FF^n v_\lambda&=F^{n+1}v_\lambda\\EF^0 v_\lambda=Ev_\lambda&=0\end{aligned}$$

and for $n>0$:

$$\begin{aligned}EF^n v_\lambda&=\big([E,F]+FE\big)F^{n-1}v_\lambda\\&=HF^{n-1}v_\lambda+FEF^{n-1}V_\lambda.\end{aligned}$$

Now $HF^{n-1}v_\lambda\in W_\lambda$ and by induction $EF^{n-1}v_\lambda\in W_\lambda$. Thus $HF^{n-1}v_\lambda+FEF^{n-1}V_\lambda\in W_\lambda$.

Step 4: $\lambda \in \mathbb{Z}_{n\geqslant 0}$, $V = V[\lambda] \oplus V[\lambda-2] \oplus \cdots \oplus V[-\lambda]$ and $V[\lambda - 2n] = \Bbbk F^n v_\lambda$. To see this, let n be the smallest positive integer such that $F^n v_\lambda = 0$. Then

$$\begin{aligned} 0 = EF^n v_\lambda &= [E, F^n] v_\lambda \\ &= (HF^{n-1} + FHF^{n-2} + \cdots + F^{n-1}H) v_\lambda \\ &= (\lambda - 2(n-1) + \lambda - 2(n-2) + \cdots + \lambda) F^{n-1} v_\lambda \end{aligned}$$

This implies $\lambda = n-1 \Rightarrow n = \lambda + 1$ and $\{v_\lambda, Fv_\lambda, \ldots, F^\lambda v_\lambda\}$ is a basis.

Step 5: Define

$$\Phi \colon V \to V(\lambda) = \mathbb{C}[x,y]_\lambda, \qquad F^n v_\lambda \mapsto \frac{1}{(\lambda-n)!} x^{\lambda-n} y^n.$$

Then

$$\big(\rho(f) \circ \Phi\big)(F^n v_\lambda) = \Phi(F^{n+1} v_\lambda) = \frac{1}{(\lambda-n-1)!} x^{\lambda-n-1} y^{n+1}$$

while

$$(\rho_\lambda(f) \circ \Phi)(F^n v_\lambda) = y\partial_x\Big(\frac{1}{(\lambda-n)!} x^{\lambda-n} y^n\Big) = \frac{1}{(\lambda-n-1)!} x^{\lambda-n-1} y^{n+1}.$$

Similarly $\Phi \circ \rho(e) = \rho_\lambda(e) \circ \Phi$ and $\Phi \circ \rho(h) = \rho_\lambda(h) \circ \Phi$. This implies a bijective intertwining operator. □

25.2 From Semisimple to Simple

$\mathfrak{g}$ is finite dimensional semisimple Lie algebra over $\Bbbk$ and $\mathfrak{h} \subset \mathfrak{g}$ a fixed choice of Cartan subalgebra.

The following is easy to check:

Theorem 25.8 (Theorem 6.39 in Kirillov)**.** *Let $\mathfrak{g} = \prod_i \mathfrak{g}_i$. Then*

i) *Every Cartan subalgebra of $\mathfrak{g}$ has the form $\mathfrak{h} = \prod_i \mathfrak{h}_i$ where $\mathfrak{h}_i$ is a Cartan subalgebra of $\mathfrak{g}_i$.*

ii) *Then $\Phi = \bigsqcup_i \Phi_i$ a disjoint union where $\Phi_i \subset (\mathfrak{h}_i)^* \hookrightarrow \oplus(\mathfrak{h}_i)^* = \mathfrak{h}^*$ is the root system of $\mathfrak{g}_i$.*

25.3 Coroots

Definition 25.9. For each $\alpha \in R$, there is a corresponding *coroot* $\alpha^\vee = h_\alpha \in \mathfrak{h}$.

Let $(\ ,\ )$ be an invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$. We know that the restriction to $\mathfrak{h}$ is non-degenerate. This implies that $\mathfrak{h} \cong \mathfrak{h}^*$ by $h \mapsto (-, h)$. Let H_α be the inverse image of α under this map. Then

$$(H_\alpha, h) = \alpha(h) \qquad \forall h \in \mathfrak{h}\,.$$

Also convenient to define $\mathfrak{h}^*$ by $(\alpha, \beta) := (H_\alpha, H_\beta) = \alpha(H_\beta)$.

Lemma 25.10. $(\alpha, \alpha) = (H_\alpha, H_\alpha) \neq 0$.

Then we can define $h_\alpha = \dfrac{2H_\alpha}{(\alpha,\alpha)}$.

Note 25.11.

i) The 2 is to get an integer later.

ii) If $\mathfrak{g}$ is simple we know that $(\cdot,\cdot)$ is unique up to scalar i.e. $(\cdot,\cdot)' = \xi(\cdot,\cdot), \xi \in \Bbbk^\times$. Then

$$h'_\alpha = \frac{2H'_\alpha}{(\alpha,\alpha)'} = \frac{2\xi^{-1}H_\alpha}{\xi^{-2}\cdot\xi(\alpha,\alpha)} = h_\alpha$$

Hence, h_α is independent of $(\cdot,\cdot)$.

26 Lecture 24: The Root System of a semisimple Lie Algebra

Today we discuss the connections between finite dimensional semisimple Lie algebras over $\Bbbk$ and Root systems.

finite dimensional semisimple Lie algebra over $\Bbbk \to$ Root systems $\big(\to$ Dynkin diagrams$\big)$[ix]

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\Bbbk$, $(\cdot,\cdot)$ is the Killing form, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, and R root system of $\mathfrak{g}$.

Lemma 26.1 ($\mathfrak{sl}(2,\Bbbk)$ triples)**.** *Let $\alpha \in R$. Pick $e_\alpha \in \mathfrak{g}_\alpha$ where $e_\alpha \neq 0$. Choose $f_\alpha \in \mathfrak{g}_{-\alpha}$ by*

$$(e_\alpha, f_\alpha) = \frac{2}{(\alpha,\alpha)}.$$

(Recall by lemma 25.10 that $(\alpha,\alpha) \neq 0$.) Define $\mathfrak{sl}(2,\mathbb{C})_\alpha := \mathbb{C}e_\alpha \oplus \mathbb{C}f_\alpha \oplus \mathbb{C}h_\alpha$, which is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2,\mathbb{C})$.

Proof. Claim: $[e_\alpha, f_\alpha] = (e_\alpha, f_\alpha)H_\alpha$. Indeed for all $h \in \mathfrak{h}$

$$\begin{aligned}(h,[e_\alpha,f_\alpha]) &= -([e_\alpha,h],f_\alpha)\\ &= ([h,e_\alpha],f_\alpha)\\ &= (\alpha(h)e_\alpha,f_\alpha)\\ &= (e_\alpha,f_\alpha)\cdot(H_\alpha,h)\\ &= ((e_\alpha,f_\alpha)H_\alpha,h)\\ &= (h,(e_\alpha,f_\alpha)H_\alpha).\end{aligned}$$

Since the form is non-degenerate, the claim holds. Now $h_\alpha = \dfrac{2H_\alpha}{(\alpha,\alpha)}$, so

$$[e_\alpha,f_\alpha] = (e_\alpha,f_\alpha)\cdot\frac{(\alpha,\alpha)}{2}\cdot h_\alpha = h_\alpha.$$

Also,

$$[h_\alpha,e_\alpha] = \alpha(h_\alpha)e_\alpha = (H_\alpha,h_\alpha)e_\alpha = \frac{2(H_\alpha,H_\alpha)}{(\alpha,\alpha)} = 2e_\alpha.$$

Similarly $[h_\alpha,f_\alpha] = -2f_\alpha$. ∎

Lemma 26.2 (Lemma 6.43 in Kirillov)**.** *Let $\alpha \in R$. Then*

$$V = \Bbbk h_\alpha \oplus \bigoplus_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \mathfrak{g}_{k\alpha} \subset \mathfrak{g}$$

is an irreducible subrepresentation of $\mathfrak{g}$ with respect to the adjoint action of $\mathfrak{sl}(2,\mathbb{C})_\alpha$ on $\mathfrak{g}$.

[ix] It turns out these maps are isomorphisms, but we save this for a later time.

Proof. We check V is a subrepresentation of $\mathfrak{g}$. We have $e_\alpha.\,\mathfrak{g}_{k\alpha} = [e_\alpha, \mathfrak{g}_{k\alpha}] \subset \mathfrak{g}_{(k+1)\alpha}$. Note that $V[0] = \Bbbk h_\alpha$ and $V[2k+1] = 0$ since $h_\alpha.x = 2kx$ for all $x \in \mathfrak{g}_{k\alpha}$. By Exercise 4.11 in the text, V is irreducible. $\square$

Theorem 26.3 (Structure of semisimple Lie Algebras over $\Bbbk = \bar{\Bbbk}$)**.**

(1) *R spans $\mathfrak{h}^*$ as a $\Bbbk$-vector space and $\{h_\alpha\}_{\alpha\in R}$ span $\mathfrak{h}$ as a $\Bbbk$-vector space.*

(2) $\dim \mathfrak{g}_\alpha = 1$ *for all* $\alpha \in R$.

(3) $\forall \alpha, \beta \in R$ *then*

$$\beta(h_\alpha) = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}.$$

(4) *For any $\alpha \in R$, the reflection*

$$\begin{aligned} s_\alpha \colon \mathfrak{h}^* &\to \mathfrak{h}^* \\ \lambda &\mapsto \lambda - \lambda(h_\alpha)\alpha \\ &\quad\ \lambda - 2\frac{(\lambda,\alpha)}{(\alpha,\alpha)}\alpha \end{aligned}$$

preserves R, i.e. $s_\alpha(\beta) \in R$ $\forall \alpha, \beta \in R$. In particular, $-\alpha = s_\alpha(\alpha) \in R$ $\forall \alpha \in R$.

(5) $\forall \alpha \in R$ $(\Bbbk\alpha) \cap R = \{\alpha, -\alpha\}$

(6) *$\forall \alpha \in R$, $\forall \beta \in R$ such that $\beta \neq \pm\alpha$ then $V = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{\beta+k+\alpha}$ is an irreducible $\mathfrak{sl}(2,\Bbbk)_\alpha$ representation with respect to the adjoint action.*

(7) *If $\alpha, \beta \in R$ such that $\alpha+\beta \in R$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.*

Proof. (1) Suppose $h \in \mathfrak{h}$ with $\alpha(h) = 0$ for all $\alpha \in R$. Then $\operatorname{ad} h \colon \mathfrak{g} \to \mathfrak{g}$ is the zero map. This implies $h \in \mathfrak{z}(\mathfrak{g}) = 0 \Rightarrow h = 0$. Thus R spans $\mathfrak{h}^*$. The final part comes from $\mathfrak{h} \leftrightarrow \mathfrak{h}^*$ by $\mathfrak{h}_\alpha \mapsto \alpha$.

(2) By Lemma 26.2 and representation theory for $\mathfrak{sl}(2,\Bbbk)$, $V = \Bbbk h_\alpha \oplus \bigoplus \mathfrak{g}_{k\alpha}$ $\mathfrak{g}_{k\alpha}$ has $\dim 1$ for all $k, \alpha \Rightarrow k = 1$ for all $\alpha \in R$.

(3) $\beta(h_\alpha)$ is the weight of $x \in \mathfrak{g}_\beta$ with $\mathfrak{sl}(2,\Bbbk)$-action $h_\alpha(x) = \beta(h_\alpha)x$. By representation theory of $\mathfrak{sl}(2,\Bbbk)$, they are all integers.

(4) $x \in \mathfrak{g}_\beta$ has weight $n := \beta(h_\alpha)$ suppose $n > 0$, Then $f_\alpha^n \colon \mathfrak{g}_\beta \xrightarrow{\cong} \mathfrak{g}_{\beta-\alpha n}$, so if $0 \neq v \in \mathfrak{g}_\beta$ then $\mathfrak{g}_{\beta-n\alpha} \neq 0$. Which implies $\beta - n\alpha \in R$ with $s_\alpha(\beta) = \beta - n\alpha$. For $n < 0$ we use e_α^n instead. (5)-(7) Read yourselves. $\square$

Example 26.4. $\mathfrak{g} = \mathfrak{sl}(3,\Bbbk)$, $\mathfrak{h} = \Bbbk h_1 + \Bbbk h_2$ with

$$h_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad h_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Note that

$$[h_1, E_{12}] = 2E_{12} \qquad \text{and} \qquad [h_1, E_{12}] = -E_{12}$$

so $E_{12} \in \mathfrak{g}_\alpha$ with $\alpha(h_1) = 2$ and $\alpha(h_2) = -1$. Similarly $E_{23} \in \mathfrak{g}_\beta$ with $\beta(h_1) = -1$ and $\beta(h_2) = 2$. Now $E_{13} = [E_{12}, E_{23}] \in \mathfrak{g}_{\alpha+\beta}$. So $R = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$ and $\begin{bmatrix} & \alpha & \alpha+\beta \\ -\alpha & & \beta \\ -\alpha-\beta & -\beta & \end{bmatrix}$ with $(\alpha,\beta) = (H_\alpha, H_\beta) = 2\cos(120°) = -1$.

27 Lecture 25: (Abstract) Root Systems

Definition 27.1. An *(abstract) root system* is a finite subset of a Euclidean space $R \subset E\backslash\{0\}$ where E is a Euclidean space, such that

(R1) $\mathrm{span}_{\mathbb{R}} R = E$;

(R2) $\forall \alpha, \beta \in R$: $n_{\alpha\beta} = \dfrac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z}$;

(R3) Let $s_\alpha \colon E \to E$ given by

$$s_\alpha(\lambda) = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}\alpha.$$

Then $s_\alpha(\beta) \in R$ $\forall \alpha, \beta \in R$.

(R4) $\forall \alpha \in R$ $(\mathbb{R}\alpha) \cap R = \{\alpha, -\alpha\}$.

Note 27.2. s_α is the orthogonal reflection in the *root hyperplane* L_α:

$$L_\alpha = \alpha^\perp = \{\lambda \in E \mid (\lambda, \alpha) = 0\}.$$

Example 27.3. Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over $\mathbb{k}$ and $\mathfrak{h} \subset \mathfrak{g}$ Cartan subalgebra. Then the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is a root system is a root system in $\mathfrak{h}_{\mathbb{R}}^*$ where

$$\mathfrak{h}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{h_\alpha \mid \alpha \in R\}$$

a real form of $\mathfrak{h}$ i.e. $\mathfrak{h}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{k} = \mathfrak{h}$. $\mathfrak{h}_{\mathbb{R}}^* = \{\lambda \in \mathfrak{h}^* \mid \lambda(\mathfrak{h}_\alpha) \in \mathbb{R}\ \forall \alpha \in R\}$ (By a lemma $(\cdot, \cdot)$ Killing form restricted to $\mathfrak{h}_{\mathbb{R}}^*$ is a positive definite inner product.)

Notation 27.4. If $v \in V$, $\lambda \in V^*$ we define $\langle v, \lambda \rangle = \langle \lambda, v \rangle := \lambda(v)$.

Definition 27.5. Let $R \subset E$ be an (abstract) root system. The *coroot* $\alpha^\vee$ of $\alpha \in R$ is defined by $\alpha^\vee \in E^*$ where

$$\langle \alpha^\vee, \lambda \rangle = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}.$$

Note 27.6.

(1) This is consistent with the Lie algebra definition of coroot:

$$\alpha^\vee = h_\alpha = \frac{2H_\alpha}{(\alpha, \alpha)}$$

where $(H_\alpha, h) = \alpha(h)$ $\forall h \in \mathfrak{h}$.

(2) Integrality says $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ $\forall \alpha, \beta \in R$ and $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$.

Example 27.7 (Root system of type A_{n-1})**.** Let $\{\varepsilon_i\}_{i=1}^n$ be the the orthonormal basis for $\mathbb{R}^n$. Let

$$E = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R} \mid \sum_{i=1}^n \lambda_i = 0\}$$
$$R = \{\varepsilon_i - \varepsilon_j \mid 1 \leqslant i, j \leqslant n,\ i \neq j\}.$$

Then R spans E over $\mathbb{R}$. $(\varepsilon_i - \varepsilon_j, \varepsilon_i - \varepsilon_j) = 2 \Rightarrow (\alpha, \beta) \in \mathbb{Z} \quad \forall \alpha, \beta \in R \Rightarrow \dfrac{2(\alpha,\beta)}{(\beta,\beta)} = (\alpha, \beta) \in \mathbb{Z}$. Now,

$$\begin{aligned} s_{\varepsilon_i - \varepsilon_j}(\lambda_1, \ldots, \lambda_i, \ldots, \lambda_j, \ldots, \lambda_n) &= (\lambda, \varepsilon_i - \varepsilon_j) \cdot (\varepsilon_i - \varepsilon_j) \\ &= \lambda - (\lambda_i - \lambda)(\varepsilon_i - \varepsilon_j) \\ &= \lambda - \lambda_i \varepsilon_i - \lambda \varepsilon_j + \lambda_i \varepsilon_j + \lambda_j \varepsilon_i \\ &= \lambda_1, \ldots, \lambda_j, \ldots, \lambda_i, \lambda_n \\ &\Rightarrow s_{\varepsilon_i - \varepsilon_j} \leftrightarrow (ij) \in S_n. \end{aligned}$$

Clearly $s_\alpha(\beta) \in R$. Lastly, $\mathbb{R}_\alpha \cap R = \{\pm\alpha\}$ is clear. Fact: A_{n-1} "is" the root system of $\mathfrak{sl}(n, \Bbbk)$.

Definition 27.8. An *isomorphism* from $R_1 \subset E_1$ to $R_2 \subset E_2$ is a $\mathbb{R}$-linear isomorphism $\varphi \colon E_1 \to E_2$ such that

i) $\varphi(R_1) = R_2$;

ii) $n_{\varphi(\alpha)\varphi(\beta)} = n_{\alpha\beta}$ for all $\alpha, \beta \in R_1$.

Example 27.9. $R \subset E$ is isomorphic to $c.R$ a root system by $\varphi \colon E \to E$ by $\lambda \mapsto c \cdot \lambda$ for all $c \in \mathbb{R} \backslash \{0\}$.

Definition 27.10. The *Weyl group* $W = W(R)$ of a root system $R \subset E$ is the subgroup of $GL(E)$ generated by $\{s_\alpha \mid \alpha \in R\}$.

Lemma 27.11. *Let W be the Weyl group of a root system $R \subset E$.*

1) *W is finite;*

2) *$W \leqslant O(E)$ the orthogonal group*

3) *If $w \in W$, $\alpha \in R$ then $w s_\alpha w^{-1} = s_{w(\alpha)}$.*

Proof. 1) We know $W(R) \subset R$ since $s_\alpha(R) \subset R$, so we get a map $\varphi \colon W \to S_R$. We claim that φ is injective. Suppose $w \in \ker \varphi$, then $w(\alpha) = \alpha$ for all $\alpha \in R$. By R spanning E we get that $w = \mathrm{Id}_E$.

2) Each s_α is an orthogonal transformation (i.e, $(s_\alpha(\lambda), s_\alpha(\mu)) = (\lambda, \mu)$ for all $\lambda, \mu) \Rightarrow W \leqslant O(E)$

3) Consider the following calculation

$$\begin{aligned} (w s_\alpha w^{-1})(\lambda) &= w(s_\alpha(w^{-1}(\lambda))) \\ &= w\left(w^{-1}(\lambda) - \frac{(w^{-1}(\lambda), \alpha)}{(\alpha, \alpha)} \alpha \right) \\ &= \lambda - 2 \frac{(\lambda, w(\alpha))}{(w(\alpha), w(\alpha))} w(\alpha) \\ &= s_{w(\alpha)}(\lambda). \end{aligned}$$

□

28 Lecture 26: More on Root Systems

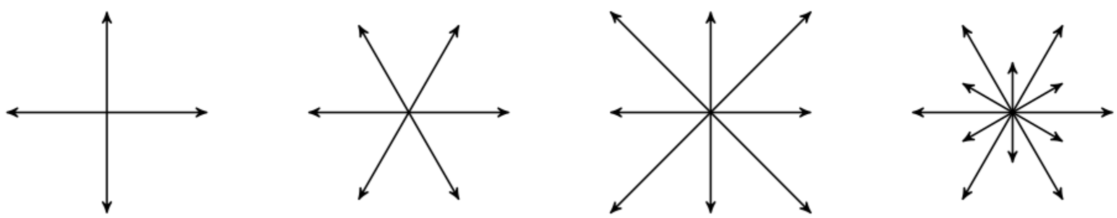
Theorem 28.1. *Let α, β be non-parallel roots of a root system R. Let $\theta \in (0, 2\pi)$ be the angle between them. Then $\theta \in \mathbb{Z}\frac{2\pi}{12} \cup \mathbb{Z}\frac{2\pi}{8}$ and $\pm\{n_{\alpha\beta}, n_{\beta\alpha}\} \in \big\{\{0,0\}, \{1,1\}, \{1,2\}, \{1,3\}\big\}$.*

Proof.

$$\begin{aligned} \mathbb{Z} \ni n_{\alpha\beta} n_{\beta\alpha} &= \frac{2(\alpha,\beta)}{(\beta,\beta)} \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \\ &= \frac{4|\alpha|^2|\beta|^2 \cos^1\theta}{|\alpha|^2|\beta|^2} \\ &= 4\cos^2\theta \end{aligned}$$

□

28.1 Root Systems of Rank 2



$\mathrm{A}_1 \times \mathrm{A}_1$ $\quad$ A_2 $\quad$ $\mathrm{B}_2 \cong \mathrm{C}_2$ $\quad$ G_2

$$\begin{aligned} A_1 \times A_1 \text{ (or } A_1 \sqcup A_1) &\leftrightarrow \mathfrak{sl}(2, \Bbbk) \times \mathfrak{sl}(2, \Bbbk) \\ A_2 &\leftrightarrow \mathfrak{sl}(3, \Bbbk) \\ B_2 = C_2 &\leftrightarrow \mathfrak{so}(5, \Bbbk) \cong \mathfrak{sp}(4, \Bbbk) \\ G_2 &\leftrightarrow \mathrm{Der}(\mathbb{O}) \end{aligned}$$

Where $\mathrm{Der}(\mathbb{O})$ is the Lie algebra of derivations of the octonions, which is one of the five exceptional simple Lie algebras.

28.2 E_{reg}, Weyl chambers, and Positive Roots

Definition 28.2. Let $R \subset E$ be a root system. The set of *regular vectors* in E is

$$E_{\mathrm{reg}} = \{\tau \in E \mid \forall \alpha \in R\colon (\tau, \alpha) \neq 0\} = E \setminus \bigcup_{\alpha \in R} L_\alpha \text{ where } L_\alpha = \alpha^\perp$$

The connected components of E_{reg} are called the *Weyl chambers* of R.

Given a Weyl chamber C, we associate a *polarization* of R:

$$R = R_+ \sqcup R_-, \text{ where } R_\pm = \{\alpha \in R \mid \pm(\alpha, \tau) > 0\},\ \tau \in E_{\mathrm{reg}}$$

Call $\alpha \in R_+$ (respectively R_-) a *positive* (respectively *negative*) *root.*

Remark 28.3. This doesn't depend on the choice of $\tau \in C$. C, being defined as a set of linear inequalities of the form $(\alpha, \tau) > 0$ (or $(\alpha, \tau) < 0$), is intersection of (open) half-spaces. Therefore, C is convex. So if $\tau, \tau' \in C$ then the straight line segment $[\tau, \tau'] \subset C$. By the intermediate value theorem, $R_+^\tau = R_+^{\tau'}$.

Remark 28.4. If $\tau \in E_{\text{reg}}$ and $\alpha \in R$, then $\forall \beta \in R$:

$$(s_\alpha(\tau), \beta) = (\tau, s_\alpha(\beta)) \neq 0$$

Now $s_\alpha(\beta) \in R$, hence $s_\alpha(\tau) \in E_{\text{reg}}$. Moreover, if τ, τ' belong to the same connected component C of E_{reg}. Let $p = [0,1] \to E_{\text{reg}}$ be a path from τ to τ'. Then $s_\alpha \circ p$ is a path from $s_\alpha(\tau)$ to $s_\alpha(\tau')$. So W acts on $\pi_0(E_{\text{reg}}) = \{C \mid \text{ connected components}\}$. Lastly, note that, the Weyl group acts on R by root system automorphisms. (More generally we have $O(E) \to \text{Aut}(R)$).

Theorem 28.5. *The action of W on $\pi_0(E_{\text{reg}})$ is transitive. $\forall C, C' \in \pi_0(E_{\text{reg}})\ \exists w \in W:\ w(C) = C'$. Thus every root system has a unique polarization, up to a Weyl group automorphism of R.*

Proof. Let C, C' be any two Weyl chambers. Pick $\tau \in C$, $\tau' \in C'$ such that $\tau + \tau' \neq 0$. Then the line segment $[\tau, \tau']$ is contained in $E \backslash \{0\}$ and crosses some of the hyperplanes $L_{\beta_1}, L_{\beta_2}, \ldots, L_{\beta_k}$. Claim: If C_1 and C_2 are adjacent, i.e. $L_\beta = \text{span}_{\mathbb{R}}(\overline{C_1} \cap \overline{C_2})$, then $s_\beta(C_1) = C_2$. Pick $\tau \in C_1$, the line segment $[\tau, s_\beta(\tau)]$ only intersects L_β. Hence, $(s_\beta(\tau), \alpha)$ and (τ, α) have the same sign for all $\alpha \in R \backslash \{\beta\}$. Thus $C' = s_{\beta_k} \cdots s_{\beta_1}(C)$. □

29 Lecture 27: Simple Roots

Last time discussed how any root system R has a unique (up to Weyl group automorphism) polarization $R = R_+ \sqcup R_-$.

Definition 29.1. $\alpha \in R_+$ is *simple* if it is not a sum of two positive roots.

Lemma 29.2. *Every positive root is a sum of simple roots.*

Proof. Let $\alpha \in R$. If α is simple we are done. If not, $\alpha = \beta + \gamma$, for $\beta, \gamma \in R_+$. Then pick $\tau \in C_+ = \{\lambda \in E \mid (\lambda, \alpha') > 0\ \forall \alpha' \in R_+\}$. Then $(\alpha, \tau) = (\beta, \tau) + (\gamma, \tau)$ each of which is strictly greater of zero $\Rightarrow (\beta, \tau), (\gamma, \tau) < (\alpha, \tau)$. The set $\{(\alpha', \tau) \mid \alpha' \in R_+\}$ is finite; therefore, totally ordered. So we proceed by induction (or by contradiction). □

Proposition 29.3. *Every root is a unique combination of simple roots with integer coefficients:*

$$\alpha = \sum_{i=1}^{r} n_i \alpha_i, \quad n_i \in \mathbb{Z}$$

Where $\Pi = \{\alpha_i, \ldots, \alpha_r\}$ is the set of simple roots. Moreover, $\alpha \in R_\pm \Leftrightarrow \pm n_i \geqslant 0\ \forall i$

Proof. With out loss of generality $\alpha \in R_+$.

Step 1: Linear algebra fact: If $\{v_i\}_i \subset E$ is a set of vectors, all lying on the same side of some hyperplane, such that $(v_i, v_j) \leqslant 0\ \forall i \neq j$, then $\{v_i\}_i$ is linearly independent. (Exercise)

Step 2: By Step 1, it suffices to show that $(\alpha_i, \alpha_j) \leqslant 0\ \forall i \neq j$. Fix $i \neq j$ and let $R' = (\mathbb{Z}\alpha_i + \mathbb{Z}\alpha_j) \cap R$. By Exercise 29.4 below, R' is a root system of rank two. Let $R'_\pm = R' \cap R_\pm$. Then $R' = R'_+ \sqcup R'_-$ is a polarization of R' and α_i, α_j are the two simple roots with respect to this polarization. By the description of the root systems of rank two, it now follows that the angle between α_i and α_j is obtuse, hence $(\alpha_i, \alpha_j) \leqslant 0$.

□

Exercise 29.4. If $\alpha, \beta \in R$, $\alpha \neq \pm\beta$, then $(\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap R$ is a root system of rank 2.

Corollary 29.5. *Π is a basis for E over $\mathbb{R}$, so $|\Pi| = \dim E = \operatorname{rank} R$.*

Proof. Π linear independent and spans R over $\mathbb{Z}$, and R spans E over $\mathbb{R} \Rightarrow \Pi$ spans E over $\mathbb{R}$. ☐

29.1 Simple Reflections

Definition 29.6. $R = R_+ \sqcup R_-$, $\Pi = \{\alpha_1, \dots, \alpha_r\}$. Then $s_i = s_{\alpha_i}$ are called *simple reflections.*

Proposition 29.7. *Any simple reflection s_i permutes the positive roots other than α_i, i.e. $s_i(R_+ \backslash \{\alpha_i\}) = R_+ \backslash \{\alpha_i\}$.*

Proof. Let $\beta \in R_+$ with $\beta = \sum_{j=1}^{\infty} n_j \alpha_j$ $n_j \geqslant 0$. Then $s_i(\beta) = \beta - \langle < \beta, \alpha_i \rangle \alpha_i$. So if $s_i(\beta) \in R_-$, then $n_j \leqslant 0$ for all $j \neq i$. This implies that $n_j = 0$ for all $j \neq i$. Thus $\beta = n_i \alpha_i \Rightarrow \beta = \alpha_i$. ☐

Corollary 29.8. *The* Weyl vector, *$\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$ satisfies:*

$$\langle \rho, \alpha_i^\vee \rangle = 1 \qquad \forall i = 1, \dots, r$$

equivalently $s_i(\rho) = \rho - \alpha_i$.

Theorem 29.9.

i) *The simple reflections generate the Weyl group.*

ii) *$\forall \alpha \in R$ $\exists w \in W$, $\alpha_i \in \Pi$: $\alpha = w(\alpha_i)$.*

iii) *W acts simple transitively on $\pi_0(E_{\mathrm{reg}})$ i.e. if $w(C) = C$, then $w = 1$.*

30 Example: $\mathfrak{sp}(4, \mathbb{k})$

For $\mathbb{C}^4$ we have *symplectic form* $w\colon \mathbb{C}^4 \times \mathbb{C}^4 \to \mathbb{C}$, i.e. w is

- bilinear
- non-degenerate
- skew-symmetric

$\mathfrak{sp}(4, \mathbb{C}) = \{x \in \mathfrak{gl}(4, \mathbb{C}) \mid w(x.v, u) + w(v, x.u) = 0\}$.

Theorem 30.1. *Over $\mathbb{C}$ all symplectic forms are equivalent (i.e coincide after a change of basis).*

We have matrix for w:

$$J = (w(\mathbb{e}_i, \mathbb{e}_j))_{i,j}$$

i.e., $w(a, b) = a^T J b$.

Example 30.2. A good choice $J = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix} \quad \left(\text{or } \begin{bmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{bmatrix} \right)$.

As such

$$\begin{aligned}
\mathfrak{sp}(4,\mathbb{C}) &= \{x \mid (x.v)^T Ju + v^T J(x.u) = 0 \ \forall u,v \in \mathbb{C}^4\} \\
&= \{x \mid v^T x^T Ju + v^T Jxu = 0 \ \forall u,v \in \mathbb{C}^4\} \\
&= \{x \mid x^T J + Jx = 0\} \\
&= \left\{ x = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \middle| \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix} \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\} \\
&= \left\{ x = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \middle| -C^T + C = 0,\ A^T + D = 0,\ B^T - B = 0 \right\} \\
&= \left\{ \left[\begin{array}{c|c} A & B = B^T \\ \hline C = C^T & -A^T \end{array} \right] \right\}
\end{aligned}$$

Now we claim we can generate a Cartan subalgebra $\mathfrak{h}$ by the following two matrices (here and everywhere else we adopt the convention that empty entries are supposed to be zero):

$$h_1 = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & -1 & \\ & & & 0 \end{bmatrix} \qquad h_2 = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & -1 \end{bmatrix}.$$

Now we construct matrices for generating the rest of $\mathfrak{sp}(4,\mathbb{C})$:

$$F_{12} = \begin{bmatrix} 0 & 1 & & \\ 0 & 0 & & \\ & & 0 & 0 \\ & & -1 & 0 \end{bmatrix} \quad F_{13} = \begin{bmatrix} & & 1 & 0 \\ & & 0 & 0 \\ & & & \\ & & & \end{bmatrix} \quad F_{14} = \begin{bmatrix} & & 0 & 1 \\ & & 1 & 0 \\ & & & \\ & & & \end{bmatrix} \quad F_{24} = \begin{bmatrix} & & 0 & 0 \\ & & 0 & 1 \\ & & & \\ & & & \end{bmatrix}$$

$$F_{21} = F_{12}^T \qquad F_{31} = F_{13}^T \qquad F_{41} = F_{14}^T \qquad F_{42} = F_{24}^T.$$

Recall: $[E_{ij}, E_{k\ell}] = \delta_{jk} E_{i\ell} - \delta_{i\ell} E_{kj}$. Where E_{ij} are matrix units.

Note 30.3.

$$\begin{aligned}
[h_i, x^T] &= [x, h_i^T]^T \\
&= [x, h_i]^T \\
&= -[h_i, x]^T \\
&= -\alpha(h_i) x^T
\end{aligned}$$

$\Rightarrow$ $x^T \in \mathfrak{g}_{-\alpha}$ and $(\mathfrak{g}_\alpha)^T = \mathfrak{g}_{-\alpha}$, when $\mathfrak{h} \subset$ {diagonal matrices} (or {symmetric}).

Now we calculate the root decomposition.

$$\begin{aligned}
[h_1, F_{12}] &= [E_{11} - E_{33}, E_{12} - E_{43}] \\
&= E_{12} - E_{43} \\
&= 1 \cdot F_{12}; \\
[h_2, F_{12}] &= [E_{22} - E_{44}, E_{12} - E_{43}] \\
&= -E_{12} + E_{43} = (-1) \cdot F_{12}.
\end{aligned}$$

So $F_{12} \in \mathfrak{g}_\alpha$ and $F_{21} \in \mathfrak{g}_{-\alpha}$, where $\alpha(h_1) = 1$ and $\alpha(h_2) = -1$. Similarly,

$$\begin{aligned}[h_1, F_{13}] &= [E_{11} - E_{33}, E_{13}] \\ &= 2 \cdot F_{13}; \\ [h_2, F_{12}] &= [E_{22} - E_{44}, E_{13}] \\ &= 0.\end{aligned}$$

So $F_{13} \in \mathfrak{g}_\beta$ and $F_{31} \in \mathfrak{g}_{-\beta}$, where $\beta(h_1) = 2$ and $\beta(h_2) = 0$. By similar calculations we get

$$\begin{aligned}&[h_1, F_{14}] = [E_{11} - E_{23}, E_{14} + E_{23}] = \cdots = F_{14}; \\ &[h_2, E_{14}] = F_{14};\end{aligned}$$

with $F_{14} \in \mathfrak{g}_\gamma$ and $F_{41} \in \mathfrak{g}_{-\gamma}$, where $\gamma(h_1) = \gamma(h_2) = 1$, and

$$\begin{aligned}[h_1, F_{24}] &= [E_{11} - E_{33}, E_{24}] \\ &= 0; \\ [h_2, F_{24}] &= [E_{22} - E_{44}, E_{24}] \\ &= 2 \cdot F_{24};\end{aligned}$$

with $F_{24} \in \mathfrak{g}_\delta$ and $F_{42} \in \mathfrak{g}_{-\delta}$, where $\delta(h_1) = 0$ and $\delta(h_2) = 2$.

Note 30.4. The following relations hold for our roots: $\alpha + \delta = \gamma$, $\alpha + \gamma = 2\alpha + \delta = \beta$.

Thus $R = \pm\{\alpha, \delta, \gamma = \alpha + \delta, \beta = \delta + 2\alpha\}$. Using the trace form on $\mathfrak{h}_\mathbb{R} = \mathbb{R}h_1 \oplus \mathbb{R}h_2$, we have $(h_i, h_j) = \mathrm{Tr}(h_i h_j) = 2\delta_{ij}$. By rescaling we define $(h_i, h_j) := \delta_{ij}$

Now we work to find the coroots. $H_\alpha = a_1 h_1 + a_2 h_2$, by the definition of coroots we know that

$$\begin{aligned}1 &= \alpha(h_1) = (H_\alpha, h_1) = a_1 \\ -1 &= \alpha(h_2) = (H_\alpha, h_2) = a_2.\end{aligned}$$

This implies that $H_\alpha = h_1 - h_2 \Rightarrow \alpha^\vee = h_\alpha = \dfrac{2H_\alpha}{(H_\alpha, H_\alpha)} = H_\alpha = h_1 - h_2$. Similarly, $H_\delta = 2h_2$ implies $\delta^\vee = \dfrac{2H_\delta}{(H_\delta, H_\delta)} = \dfrac{1}{2} H_\delta = h_2$.

Now we compute $n_{\delta\alpha}$ and $n_{\alpha\delta}$:

$$\begin{aligned}n_{\delta\alpha} &= \frac{2(\delta, \alpha)}{(\alpha, \alpha)} = \delta(\alpha^\vee) = \delta(h_1 - h_2) = -2 \\ n_{\alpha\delta} &= \alpha(\delta^\vee) = \alpha(h_2) = -1.\end{aligned}$$

Note 30.5. $(\delta, \delta) = 2(\alpha, \alpha)$.

So

$$\begin{aligned}4\cos^2\theta &= n_{\alpha\delta} n_{\delta\alpha} \\ &= (-1)(-2) \\ \cos^2\theta &= \frac{1}{2}.\end{aligned}$$

Now $(\alpha, \delta) < 0$ implies that $\cos\theta < 0$. Hence, $\cos\theta = \frac{-1}{\sqrt{2}} \Rightarrow \theta = 3 \cdot \frac{2\pi}{8}$. A suitable choice for R_+

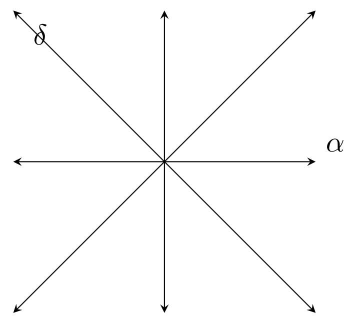


Figure 17: Root System of $\mathfrak{sp}(4, \mathbb{C})$

would be $R_+ = \{\alpha, \delta, \delta + \alpha, \delta + 2\alpha\}$ and $\Pi = \{\alpha, \delta\}$.

The Weyl group $W = \langle s_1 = s_\alpha, s_2 = s_\delta \rangle$ such that $s_1 s_2 = \rho_{\frac{2\pi}{4}}$, so $W \cong D_4$ the dihedral group of order 8.

The Cartan matrix (which will be discussed more next time) is:

$$\begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}.$$

31 Lecture 28: Cartan Matrices and Dynkin Diagrams

Let R root system $\subset E$; R_+ choice of positive roots; $\Pi \subset R_+$ simple roots $\Pi = \{\alpha_1, \cdots, \alpha_r\}$.

Theorem 31.1. *R can be recovered from Π and the numbers $n_{\alpha\beta} \in \Pi$.*

Proof. Recall $R = W(\Pi)$ and W is generated by the simple reflections (by Theorem 29.9). So if $\alpha \in R$, then $\exists i_1, \ldots, i_k, j \in \{1, \ldots, r\}$ such that $\alpha = s_{i_k} \cdots s_{i_2} s_{i_1}(\alpha_j)$.

Note 31.2. $s_{i_1}(\alpha_j) = \alpha_j - \langle \alpha_j, \alpha_{i_1}^\vee \rangle \alpha_{i_1} = \alpha_j - (n_{\alpha_j \alpha_{i_1}})\alpha_{i_1}$.

By linearity, $s_{i_2} s_{i_1}(\alpha_j)$ can be computed from $n_{\alpha_j \alpha_{i_2}}, n_{\alpha_j \alpha_{i_1}}, n_{\alpha_{i_1} \alpha_{i_2}}$. By induction α can be computed using only knowledge of Π and $n_{\alpha\beta}$ for $\alpha, \beta \in \Pi$. $\square$

Definition 31.3. The *cartan matrix* of R is defined by: $A = (a_{ij})_{i,j=1}^r$ where

$$a_{ij} = n_{\alpha_j \alpha_i} = \langle \alpha_j, \alpha_i^\vee \rangle = \frac{2(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)}.$$

Corollary 31.4. *R is uniquely determined by A.*

Example 31.5. We look at the cartan matrices for rank 2 root systems.

$$A_1 \sqcup A_1: \quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$A_2: \quad \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$B_2 = C_2: \quad \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$$

$$G_2: \quad \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

Note 31.6.

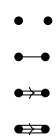
i) $a_{ii} = 2$ for all i

ii) $(a_{ij}, a_{ji}) \in \{(0,0), (-1,-1)\} \cup (\{-2,-3\} \times \{-1\}) \cup (\{-1\} \times \{-2,-3\})\}$

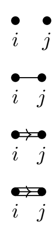
iii) $|a_{ij}| < |a_{ji}| \Leftrightarrow |\alpha_i| > |\alpha_j|$.

31.1 Rank 2 Dynkin diagrams

$A_1 \sqcup A_1$
A_2
B_2
G_2



Definition 31.7. The *Dynkin diagram* D of R is a graph with vertex set Π (often identified with $\{1, \dots, r\}$) and edges of the following four kinds:



no-edge $a_{ij} = a_{ji} = 0$

$a_{ij} = a_{ji} = -1$

$(a_{ij}, a_{ji}) = (-1, -2)$ note: $|\alpha_i| > |\alpha_j|$

$(a_{ij}, a_{ji}) = (-1, -3)$ note: $|\alpha_i| > |\alpha_j|$

Main point: A, hence R, can be recovered from D.

Definition 31.8. A set $S \subset E$ is the *orthogonal union* of two subsets $S_1, S_2 \subset S$ if

1) $S = S_1 \cup S_2$

2) $u \perp v \ \forall u \in S_1, v \in S_2$.

Notation: $S = S_1 \biguplus S_2$

Definition 31.9. $S \subset E$ *decomposable* if $S = S_1 \biguplus S_2$, $S_i \neq 0$.

32 Lecture 29: Classification of Root Systems

Lemma 32.1. *Let R be a root system, Π a set of simple roots, A the cartan matrix, and D the Dynkin diagram. TFAE:*

1) *R is* irreducible, *i.e.* $R = R_1 \biguplus R_2 \Rightarrow R_1 = \varnothing$ *or* $R_2 = \varnothing$.

2) *Π is irreducible.*

3) *D is connected.*

4) *A is* indecomposable, *i.e. it cannot be written as a block diagonal matrix* $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ *even after reordering* Π.

Proof. Bonus Homework Question. ☐

Lemma 32.2. *If R is a reducible subset of E: $R = R_1 \biguplus R_2$ with $R_i \neq 0$, then R_i are root systems in $E_i = \operatorname{span} R_i$.*

Proof. Bonus Homework Question. ☐

Corollary 32.3. *Without loss of generality, we may assume R is irreducible.*

Thus our goal should be to describe all connected Dynkin diagrams!

32.1 Coxeter Graphs

Definition 32.4. A *coxeter graph* $\Gamma = (\Gamma_0, \Gamma_1)$ is an undirected loopless graph such that each edge $e \in \Gamma_1$ has multiplicity $m_e \in \mathbb{Z}_{>0}$. By thinking of no-edge as multiplicity 0, we may think of

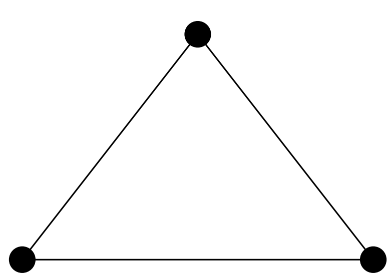
$$\Gamma_1 \subset (\mathbb{Z}_{\geqslant 0})^{\binom{\Gamma_0}{2}}$$

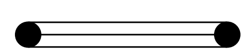
and the adjacency mattrix for Γ

$$A = \begin{bmatrix} 0 & & a_{ij} \\ & \ddots & \\ a_{ji} & & 0 \end{bmatrix}$$

is symmetric with $a_{ij} \in \mathbb{Z}_{\geqslant 0}$.

Example 32.5. Below are two examples of coxeter graphs.





Example 32.6. Forgetting the direction of a Dynkin diagram gives a coxeter-graph.

32.2 Admissible Sets

Definition 32.7. An *admissible set* $U = \{e_1, \ldots, e_r\} \subset E$ is a set of linearly independent unit vectors such that $\forall i \neq j$:

1) $\cos \varphi_{ij} = (e_i, e_j) \leqslant 0$;

2) $4(e_i, e_j)^2 \in \{0, 1, 2, 3\}$.

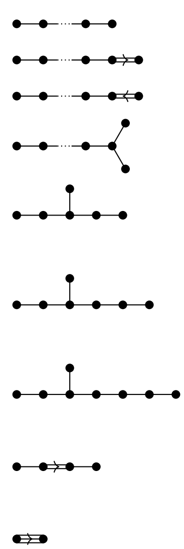
Example 32.8. The following will be our main example to consider $U = \left\{ \frac{\alpha}{|\alpha|} \mid \alpha \in \Pi \right\}$.

To any admissible set U, we can attach a coxeter graph $\Gamma = (\Gamma_0, \Gamma_1)$ where $\Gamma_0 = U$ and e_j, e_i are connected by an edge of multiplicity $4(e_i, e_j)^2$. So we have the following commuting diagram:

$$\begin{array}{ccc} \Pi & \rightsquigarrow & U \\ \wr & \circlearrowleft & \wr \\ D & \rightsquigarrow & \Gamma \end{array}$$

Theorem 32.9 (Classification of Root systems)**.** *Let R be any root system. Then its Dynkin diagram D is a union of diagram of the following type:*

A_r $(r \geqslant 1)$
B_r $(r \geqslant 2)$
C_r $(r \geqslant 3)$
D_r $(r \geqslant 4)$
E_6
E_7
E_8
F_4
G_2



Moreover, any two from this list correspond to non-isomorphic root systems.

Remark 32.10. Types A through D are called *classical types*, and types E through G are the five *exceptional types*.

Proof. We show that any connected admissible graph is one the above types (with orientation removed). Let $U = \{e_1, \ldots, e_r\}$ be any admissible set, and Γ be its coexeter graph. Then:

1) Any subset of an admissible set is admissible.

Proof of 1). Clear. $\square$

2) $c := |\{\{e_i, e_j\} \mid i \neq j \text{ and } e_i, e_j \text{ connected}\}|$. Then $c < r$.

Proof of 2). Let $e = \sum e_i$. Then

$$0 < (e, e) = r + \sum_{i<j} 2(e_i, e_j) \leqslant r + (-c).$$

Hence, $c < r$. $\square$

3) Γ has no cycles.

Proof of 3). If $\Gamma' \subset \Gamma$ is a cycle, by 1) Γ' is an admissible graph, but this contradicts 2) because in cycles $c = r$. $\square$

4) The degree of any vertex (with multiplicity) is $\leqslant 3$.

Proof of 4). Suppose $e \in U$ has edges to $\eta_1, \eta_2, \dots, \eta_k$, then $4(e, \eta_i)^2 \in \{1, 2, 3\}$. By 3) $(\eta_i, \eta_j) = 0$ $\forall i \neq j$ (else Γ would contain a cycle). $W = \operatorname{span}\{\eta_i\}$ $e' := \operatorname{proj}_W e = \sum_{i=1}^k (e, \eta_i)\eta_i$. Now $e' \neq e$ since $\{e, \eta_1, \dots, \eta_k\}$ are linearly independent. Hence,

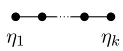
$$4 = 4 \cdot (e, e) > 4(e', e') = \sum_{i=1}^{k} 4(e, \eta_i)^2 = \deg e.$$

$\square$

5) By 4) the only possible diagrams containing a triple edge is

 .

6) If $\{\eta_1, \dots, \eta_k\} \subset U$ with their induced subgraph is



then they can be replaced by a single point, i.e. $(U \backslash \{\eta_1, \dots, \eta_k\}) \cup \{\eta\}$, where $\eta = \sum_{i=1}^k \eta_i$ is, is admissible. $\square$

33 Lecture 30: Classification Proof Continued

Proof of Theorem 32.9 Con't. 6) Linear roots can be deformed to a single point. $U' = (U \backslash \{\eta_1, \dots, \eta_k\}) \cup \{\eta\}$ with $\eta = \sum_{i=1}^k \eta_i$.

Proof of 6). Clearly, U' is linearly independent. Now

$$(\eta, \eta) = (\sum \eta_i, \sum \eta_i) = k + (k-1) \cdot (-1) = k - k + 1 = 1$$

We want to show that $4(\varepsilon, \eta)^2 \in \{0, 1, 2, 3\}$ for all $\varepsilon \in U' \backslash \{\eta\}$. Now any $\epsilon \in U' \backslash \{\eta\}$ is connected to at most one of the $\{\eta_1, \dots, \eta_k\}$ (else there would be a cycle), so $(\varepsilon, \eta) = 0$ or $(\varepsilon, \eta) = (\varepsilon, \eta_i)$ for exactly one i. Hence, $4(\varepsilon, \eta)^2 = 4(\epsilon, \eta_i)^2 \in \{0, 1, 2, 3\}$. $\square$

7) Γ contain no subgraphs of the form shown in Figure 19.

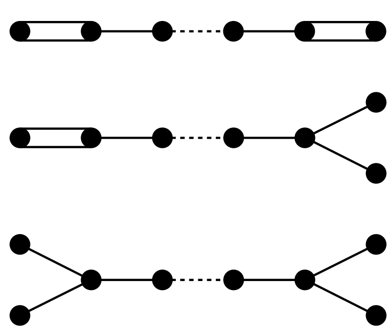


Figure 19: None of these can occur as subgraphs in an admissible graph.

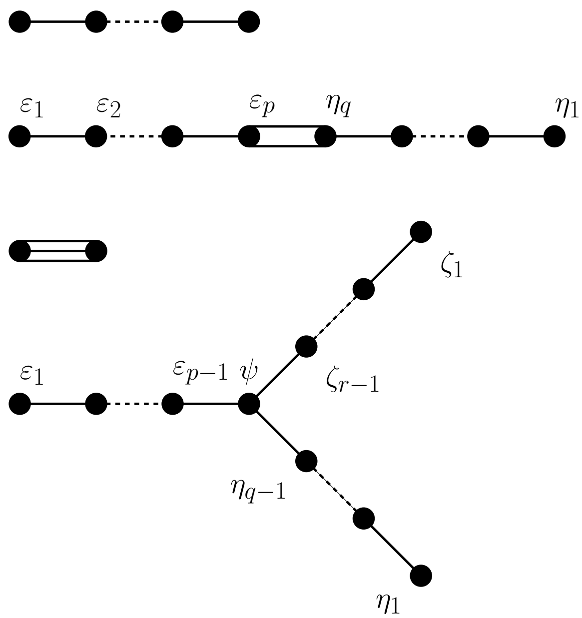
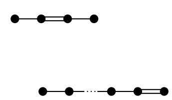


Figure 20: Any connected graph Γ of an admissible set must be one of these.

Proof of 7). If we collapse lines to points as in 6), then we would reach a vertex of degree greater than 3 from each of these graphs which is a contradiction to 4). $\square$

8) Any connected Γ of an admissible set must be of the form shown in Figure 20 9) The only graphs of the second variety are



(type F_4)
&
(type BC_n)

Proof of 9). Define $\varepsilon := \sum_{i=1}^{p} i\varepsilon_i$ and $\eta := \sum_{j=1}^{q} j\eta_j$. Then

$$(\varepsilon, \varepsilon) = \sum_{1}^{p} i^2 - \sum_{1}^{p-1} i(i+1) = \frac{p(p+1)}{2}$$

Similarly,

$$(\eta, \eta) = \frac{q(q+1)}{2}.$$

Now

$$(\varepsilon, \eta)^2 = p^2 q^2 (\varepsilon_p, \eta_q)^2 = \frac{p^2 q^2}{2}$$

By Cauchy-Schwarz:

$$\frac{p^2 q^2}{2} = (\varepsilon, \eta)^2 < (\varepsilon, \varepsilon)(\eta, \eta) = \frac{p(p+1)}{2} \frac{q(q+1)}{2}$$

This implies $(p-1)(q-1) < 2$, so either $p = 1$ or $q = 1$ (Type BC_n), or $p = q = 2$ (Type F_4). $\square$

10) The only Γ of the fourth type is D_n or E_6, E_7, E_8.

Proof of 10). Define $\varepsilon := \sum_{i=1}^{p-1} i\varepsilon_i$, $\eta := \sum_{j=1}^{q-1} j\eta_j$, and $\zeta := \sum_{\ell=1}^{r-1} \ell\zeta_\ell$. Then ε, η, and ζ are necessarily orthogonal and linearly independent, and $\psi \notin \operatorname{span}\{\varepsilon, \eta, \zeta\}$ as in 4) $\cos^2\theta_\varepsilon + \cos^2\theta_\eta +$

$\cos^2\theta_\zeta < 1$ where these are the angles between the projection of $\mathrm{proj}_{\mathrm{span}\{\varepsilon,\eta,\zeta\}}\,\psi$ and the respective roots. On the other hand,

$$\cos^2\theta_\varepsilon = \frac{(\varepsilon,\psi)^2}{|\epsilon|^2} = \frac{(p-1)(\varepsilon_{p-1},\psi)}{|\varepsilon|^2} = \frac{1}{2}(1-\frac{1}{p}).$$

Similarly for θ_η and θ_ζ. This implies that

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r} > 1$$

With out loss of generality, $p \geqslant q \geqslant r$. If $r = 1$ we get A_n, else $r = 2$. Then $2 \leqslant q < 4$. As such we get $(p, 2, 2)$ for type D_n, $(3, 3, 2), (4, 3, 2), (5, 3, 2)$ are E_6, E_7, E_8 respectively. □

This completes the proof. □

33.1 Serre's Theorem

We have seen that

$$\{\text{f.d. s.s. Lie alg}/\mathbb{C}\}_{/\mathrm{iso}} \xrightarrow{\mathcal{D}} \{\text{not necessarily connected Dynkin diagrams}\}$$

by

$$\mathfrak{g} \overset{\text{Choose C.S.A}}{\longmapsto} R \overset{\text{Choose } t\in E_{\mathrm{reg}}}{\longmapsto} R_+ \mapsto \Pi \overset{\text{ordering}}{\longmapsto} A \mapsto D.$$

Serre's Theorem shows that $\mathcal{D}$ is bijective by constructing an explicit inverse $D \mapsto \mathfrak{g}(D)$.

Theorem 33.1 (Chevalley-Serre)**.** *Let $\mathfrak{g}$ be as above, and D its Dynkin Diagram, A = cartan matrix (with respect to some order on vertices). Then $\mathfrak{g}$ is generated by a subset $\{e_i, f_i, h_i\}_{i=1}^r$ satisfying*

$$\begin{array}{ll} \textit{Chevalley 1940's} & \begin{cases} [e_i, f_j] = \delta_{ij} h_i \\ [h_i, e_j] = a_{ij} e_j \\ [h_i, f_j] = a_{ij} f_j \end{cases} \\ \textit{Serre 1960's} & \begin{cases} \mathrm{ad}(e_i)^{1-a_{ij}}(e_j) = 0 \\ \mathrm{ad}(f_i)^{1-a_{ij}}(f_j) = 0 \end{cases} \end{array}$$

Theorem 33.2 (Serre's Theorem)**.** *Let D be a Dynkin diagram and A its cartan matrix. Let $\mathfrak{g}(D)$ be the free Lie algebra on $3r$ symbols $\{e_i, f_i, h_i\}_{i=1}^r$ module the Chevalley-Serre relations.*

Then $\mathfrak{g}(D)$ is a finite dimensional semisimple Lie algebra over $\mathbb{C}$ with Dynkin diagram D. Moreover, $\mathfrak{g} \cong \mathfrak{g}(D)$ for any $\mathfrak{g}$ whose Dynkin diagram is D.

34 Further Reading: Summary of Representation Theory for Lie Groups

34.1 Representations of Lie Groups and Lie Algebras

Definition 34.1. A *representation* of a Lie group is a finite dimensional vector space V together with a morphism $\rho = \rho_V : G \to GL(V)$ of Lie groups.

$$
\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "\rho_V(g)"'] & W \arrow[d, "\rho_W(g)"] \\
V \arrow[r, "f"] & W
\end{tikzcd}
$$

Definition 34.2. A *morphism* between two representation V and W of G is a linear map $f\colon V \to W$ which intertwines the action of G:

$$f \circ \rho_V(g) = \rho_W(g) \circ f$$

for all $g \in G$ i.e.
commutes.

Definition 34.3. V and W are *equivalent* (or *isomorphic*) if there exists invertible $f\colon V \to W$. Denoted $V \cong W$.

Definition 34.4. A *representation* of a Lie algebra $\mathfrak{g}$ is a vector space with a morphism $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ of Lie algebras.

A morphism of Lie algebra reps $f\colon V \to W$ is a defined as for groups. As are $\mathrm{Hom}_{\mathfrak{g}}(V, W)$ and $V \cong W$.

Note 34.5. We always assume that V is a complex vector space. If G is a real Lie group we regard $GL(V)$ as a real Lie group, and want $\rho\colon G \to GL(V)$ to be smooth.

Theorem 34.6. *G Lie group, $\mathfrak{g} = \mathrm{Lie}\,(G)$.*

(1) *Every representation $\rho\colon G \to GL(v)$ gives a representation $\rho_*\colon \mathfrak{gl} \to \mathfrak{gl}(V)$. Every morphism of G-representation is automatically a $\mathfrak{g}$-representation. In other words we have a functor*

$$\begin{aligned} \underline{\mathrm{Rep}}G &\to \underline{\mathrm{Rep}}\,\mathfrak{g} \\ (V, \rho) &\mapsto (V, \rho_*) \end{aligned}$$

(2) *If G is connected and simply connected, then the above is an equivalence of categories. In particular, any representation of $\mathfrak{g}$ can be lifted to a representation of G, and $\mathrm{Hom}_G(V, W) = \mathrm{Hom}_{\mathfrak{g}}(V, W)$.*

Example 34.7.

$$\rho\colon GL(2, \mathbb{C}) \to GL(\mathbb{C}[x, y]_d) \qquad A \mapsto \big(\rho(x, y) \mapsto \rho(ax + by, cx + dy)\big)$$

where $\mathbb{C}[x, y]_d = \bigoplus_{n=0}^{d} \mathbb{C}x^n y^{d-n}$. Also,

$$\rho_*\colon \mathfrak{gl}(2, \mathbb{C}) \to \mathfrak{gl}(\mathbb{C}[x, y]_d) \qquad E_{ij} \mapsto x_i \partial_j$$

with $(x_1, x_2) = (x, y)$.

Example 34.8.

$$\mathrm{Ad}\colon G \to GL(\mathfrak{g}) \qquad \mathrm{Ad}_* = \mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \quad x \mapsto (\mathrm{ad}\, x\colon y \mapsto [x, y]).$$

Remark 34.9. If $\mathfrak{g}$ is a real Lie group then

$$\operatorname{Rep}\mathfrak{g} \cong \operatorname{Rep}\mathfrak{g}_{\mathbb{C}} \quad (V,\rho) \mapsto (V,\rho_{\mathbb{C}})$$

where

$$\rho_{\mathbb{C}}(x+iy) = \rho(x) + i\rho(y)$$

for any representation (V,ρ) of $\mathfrak{g}$ and

$$\operatorname{Hom}_{\mathfrak{g}_{\mathbb{C}}}(V,W) = \operatorname{Hom}_{\mathfrak{g}}(V,W).$$

Note 34.10. The theory of representations in real and/or ∞-dimensional vector spaces is very different.

34.2 Operations on Representations

Let G be a Lie group, $\mathfrak{g}$ a Lie algebra.

Definition 34.11. Let V be a representation of G (respectively $\mathfrak{g}$). A *subrepresentation* of V is a linear subspace $W \subset V$ such that

$$\rho(x)W \subset W$$

for all $x \in G$ (respectively $x \in \mathfrak{g}$).

Definition 34.12. If V is a representation of G (or $\mathfrak{g}$), and $W \subset V$ is a subrepresentation then V/W becomes a representation:

$$\rho_{V/W}(x)(v+W) = \rho_V(x)v + W$$

for all $x \in G$ (or $\mathfrak{g}$) and $v + W \in V/W$.

Definition 34.13. For representations V and W we can define $V \oplus W$ by

$$\rho_{V\oplus W}(g)(v+w) = \rho_V(g)(v) + \rho_W(g)(w)$$

Definition 34.14. For a representation V we can define a representation structure on the dual space V^*. For Lie group case:

$$(\rho_{V^*}(g)\lambda)(v) = \lambda(\rho_V(g^{-1})v)$$

for all $v \in V$, $\lambda \in V^*$, $g \in G$. Lie algebra case:

$$(\rho_{V^*}(x)\lambda)(v) = \lambda(\rho_V(-x)v)$$

for $x \in \mathfrak{g}$.

Definition 34.15. For two representations V and W define a representation structure on $V \otimes W$. For the Lie group case:

$$\rho_{V\otimes W}(g)(v\otimes w) = \rho_V(g)v \otimes \rho_W(g)w$$

For Lie algebras: We compute $(\rho_{V\otimes W})_*$ to find the correct definition. Let $x \in \mathfrak{g}$. Consider $\gamma(t) = \exp(tx)$.

$$\begin{aligned}
(\rho_{V\otimes W})_*(x)(v\otimes w) &= \frac{d}{dt}|_{t=0}\rho_{V\otimes W}(\gamma(t))(v\otimes w)\\
&= \frac{d}{dt}|_{t=0}\rho_V(\gamma(t))v \otimes \rho_W(\gamma(t))w\\
&\overset{\text{Leibniz}}{=} \rho_V(\dot\gamma(0))v \otimes \rho_W(\gamma(0))w + \rho_V(\gamma(0))v \otimes \rho_W(\dot{\gamma(0)})w\\
&= \rho_V(x)v \otimes w + v \otimes \rho_W(x)w.
\end{aligned}$$

This motivates defining

$$\rho_{V\otimes W}(x)(v\otimes w) = \rho_V(x)v \otimes \rho_W(x)w.$$

Exercise 34.16. Check that $\rho_{V\otimes W}$ thus defined is indeed a representation of any Lie algebra $\mathfrak{g}$, given representations ρ_V, ρ_W.

Corollary 34.17. *If V is a representation of G (or $\mathfrak{g}$) then so is $V^{\otimes k}\otimes(V^*)^{\otimes\ell}$.*

Definition 34.18. Let V and W be representations. Then $\mathrm{Hom}(V,W)\cong V^*\otimes W$ by $(v\mapsto\lambda(v)w)\leftarrow\!\shortmid\,\lambda\otimes w$. This gives $\mathrm{Hom}(V,W)$ the structure of a representation with

$$(g.\varphi)(v) = g.\Big(\varphi(g^{-1}.v)\Big) \qquad g\in G$$

or

$$(x.\varphi)(v) = x.\Big(\varphi(v)\Big) - \varphi(x.v) \qquad x\in\mathfrak{g}$$

34.3 Invariants

Definition 34.19. A vector v in a representation V of G (or $\mathfrak{g}$) is *invariant* if $\rho(g)v = v\ \forall g\in G$ ($\rho(x)v = v\ \forall x\in\mathfrak{g}$). $V^G = \{v\in V \mid v \text{ is invariant}\}$ ($V^{\mathfrak{g}} = \{v\in V\mid v\text{ is invariant}\}$).

Example 34.20. $(\mathrm{Hom}(V,W))^G = \mathrm{Hom}_G(V,W)$ (respectively $(\mathrm{Hom}(V,W))^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(V,W)$).

Example 34.21. B be a bilinear form on a representation V.

$$\begin{aligned} B\colon V\times V\to\mathbb{C}\text{ bilinear} &\Leftrightarrow B\colon V\otimes V\to\mathbb{C}\text{ linear}\\ &=\Leftrightarrow B\in(V\otimes V)^* \end{aligned}$$

So G (respectively $\mathfrak{g}$) acts on $(V\otimes V)^*$ via

$$(g.B)(v,w) = B(g^{-1}v, g^{-1}w)$$

respectively

$$(x.B)(v,w) = B(-x.v,w) + B(v,-x.w)$$

so B is invariant iff

$$B(v,w) = B(gv,gw) \qquad \forall g\in G$$

respectively

$$0 = B(x.v,w) + B(v,x.w) \qquad \forall x\in\mathfrak{g}\,.$$

34.4 Irreducible Representations

Definition 34.22. A representaion $V\neq 0$ is *irreducible* (or *simple*) if the only subrepresentations of V are 0 and V. Otherwise V is *reducible.*

Example 34.23. The standard representation $\mathbb{C}^n$ of $SL(n,\mathbb{C})$ is irreducible. (Exercise) Hint: Use $I+E_{ij}$, $i\neq j$ to get $(1,0,\cdots,0)$.

Suppose $V\neq 0$ is reducible. Let $W\subset V$ be a proper nonzero subrepresentation. We get a SES

$$0\to W\to V\to V/W\to 0 \quad (*)$$

Note 34.24. $\dim W$ and $\dim V/W$ are both strictly less than $\dim V$.

(When) does $(*)$ split? i.e $V \cong W \oplus V/W$?

Definition 34.25. A representation V is *completely reducible* (or *semisimple*) if

$$V \cong \oplus_{i=1}^N V_i, \quad V_i \text{ irreducible.}$$

Then:

$$V \cong \oplus_{i=1}^k n_i V_i = \oplus_{i=1}^k V_i^{\oplus n_i}$$

where $V_i \ncong V_j \ \forall i \neq j$. n_i is the *multiplicity* of V_i in V.

Example 34.26. $G = \mathbb{R}$, $\mathfrak{g} = \mathrm{Lie}\,(G) = \mathbb{R}$. A representation of $\mathfrak{g}$ is just a complex finite dimensional vector space V with an $\mathbb{R}$-linear map $\rho\colon \mathbb{R} \to \mathfrak{gl}(V) = End_{\mathbb{C}}(V)$. $\rho(t) = t\rho(1) = t \cdot A$, where $A = \rho(1)$, $A \in End_{\mathbb{C}}(V)$. Conversely any $A \in End_{\mathbb{C}}(V)$ gives a representation $\rho\colon \mathbb{R} \to \mathfrak{gl}(V)$. $V \cong W \Leftrightarrow A_V = TA_WT^{-1}$, some $T \in GL(V)$. This implies that Jordan canonical form classifies up to equivalence all representations of the Lie algebra $\mathbb{R}$. A representation of V is completely reducible iff Av is diagonalizable. V is irreducible iff $\dim V = 1$.

Some Goals of Representation Theory

1) Given G, classify irreducible representations of G.

2) Given a reducible representation, how to decompose it into irreducible representations?

3) For which G are <u>all</u> representations completely reducible?

34.5 Intertwining Operators (=morphisms of representation)

Suppose $A\colon V \to V$ is a diagonalizable intertwining operator:

$$V = \oplus_{\lambda \in \mathbb{C}} V_\lambda, \qquad V_\lambda = \{v \in V \mid Av = \lambda v\}$$

Then $\forall g \in G$, $\forall v \in V_\lambda$:

$$A\rho(g)v = \rho(g)Av = \rho(g)\lambda v = \lambda\rho(g)v.$$

This implies $\rho(g)v \in V_\lambda$, so $\rho(g)V_\lambda \subset V_\lambda$ for all $g \in G$. So $\forall\lambda$: V_λ is a subrepresentation of V.

Corollary 34.27. *If $z \in Z(G)$ such that $\rho(z)$ is diagonalizable, then $V = \oplus V_\lambda$ where V_λ= eigenspace of $\rho(z)$.*

Proof. $\rho(z)\rho(g) = \rho(zg) = \rho(gz) = \rho(g)\rho(z)\ \forall g \in G$. This show that $\rho(z)$ is an intertwining operator. □

Example 34.28. $V = \mathbb{C}^n \otimes \mathbb{C}^n$ representations of $G = GL(n, \mathbb{C})$ $p\colon v \otimes w \mapsto w \otimes v$ commutes with G action implies it is an intertwiner. $V = V_+ \oplus V_-$ eigenspace decomposition.

$$V_+ = \mathrm{Span}\{v \otimes w + w \otimes v\}$$

$$V_- = \mathrm{Span}\{v \otimes w - w \otimes v\}$$

In face, $V_\pm$ are irreducible representations of G.

34.6 Schur's Lemma

Recall all representations are assumed complex and finite dimensional.

Lemma 34.29 (Schur's Lemma)**.**

1) *Let V be an irreducible representation of G. Then*

$$\mathrm{Hom}_G(V, V) = \mathbb{C}\,\mathrm{Id}$$

2) *If V and W are irreducible representations, such that $V \ncong W$. Then* $\mathrm{Hom}_G(V, W) = 0$.

Proof. $\Phi\colon V \to W$ intertwining operator. Then $\ker\Phi$ and $\Im\Phi$ are subrepresentations of V and W respectively. V and W irreducible implies $\Phi = 0$ or an isomorphism. Which shows 2). To get 1), pick any eigenvalue λ of Φ. Then $\Phi - \lambda\,\mathrm{Id}$ is an intertwiner with $\ker(\Phi - \lambda\,\mathrm{Id}) \neq 0$. By V irreducible, we have $V = \ker(\Phi - \lambda\,\mathrm{Id})$. Thus $\Phi = \lambda\,\mathrm{Id}$. □

34.7 Unitary representations

Goal: Show that any representation of a compact real Lie group is completely reducible. Steps:

(1) Any *unitary* representation is completely reducible.

(2) Any representation of a compact real Lie group is unitary.

Definition 34.30. A representation (V, ρ) of G is *unitary* (or *unitarizable*) if there exists a positive definite Hermitian form on V, $(\cdot,\cdot)\colon V \times V \to \mathbb{C}$ that is G-invariant, i.e.

$$(g.u, g.v) = (u, v) \qquad \forall g \in G,\ u, v \in V.$$

A representation V of a Lie algebra $\mathfrak{g}$ is *unitarizable* if $\exists\ (\cdot,\cdot)$ positive definite Hermitian form which is $\mathfrak{g}$-invariant:

$$(x.v, w) + (v, x.w) = 0 \qquad \forall x \in \mathfrak{g},\ v, w \in V.$$

Example 34.31. Let G be a finite group acting on a set X.

$$V = \mathbb{C}^X = \{\text{functions } f\colon X \to \mathbb{C}\}.$$

Define $\rho\colon G \to GL(V)$ by $(\rho(g)f)(x) = f^{-1}(g^{-1}.x)$ $x \in X$, $f \in V$, and $g \in G$. Then (V, ρ) is a rep of G. Define $(f, g) = \sum_{x\in X} f(X)\overline{g(x)}$, $f, g \in V$. Then $(\cdot,\cdot)$ is a G-invariant positive definite Hermitian form hence V is unitarizable.

Theorem 34.32. *Every unitarizable representation is completely reducible.*

Proof. The proof is by induction on $\dim V$. $W \subset V$ nonzero proper subrepresentation. Consider $W^\perp$ with respect to $(\cdot,\cdot)$ on V. We have $\forall v \in W^\perp,\ w \in W$

$$(g.v, w) \overset{*}{=} (v, g^{-1}.w) = 0.$$

Where $*$ is by G-invariance and $g^{-1}.w \in W$ because it is a subrepresentation. This implies that $g.W^\perp \subset W^\perp$, so $W^\perp$ is also a subrepresentation, $V = W \oplus W^\perp$. So $\dim W$, $\dim W^\perp < \dim V$. Proceed by induction. Same idea for for Lie algebras. □

34.7.1 The Haar Measure

Let $G \subset \mathbb{R}^n$ be a real Lie group. Recall $A \subset G$ is *open in* G (respectively. *closed in* G) if $A = B \cap G$ for some open (resp. closed) subset $B \subset \mathbb{R}^n$. Let $\Sigma \subset \mathcal{P}(G)$ be the smallest subset closed under complements and countable $\bigcup$, $\bigcap$, containing all open sets in G. [x]

[x] Σ is called a σ-algebra.

Definition 34.33. A *measure* on G is a map $\mu\colon \Sigma \to [0,\infty]$ such that

i) $\mu\left(\bigsqcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$ ($\bigsqcup$ denotes a disjoint union).

ii) $\mu(\varnothing) = 0$.

Definition 34.34.

$$\int_G c_i \chi_{A_i} d\mu = \sum c_i \mu(A_i).$$

Where $\chi_{A_i} = \begin{cases} 1 & x \in A_i \\ 0 & \text{otherwise} \end{cases}$. Through a limit process we can define $\int_G f d\mu$ *integral of f over G.*

Definition 34.35. A *right Haar measure* on a real Lie group is a Borel measure dg which is invariant under right action of G on itself.

Thus if dg is a Haar measure on G then for an integrable function $f\colon G \to \mathbb{R}$ (i.e. $f \in L^1(G, dg)$)

$$\int_G f(gh)dg = \int_G f(g)dg \qquad \forall h \in G.$$

Example 34.36. Lebesgue measure on $\mathbb{R}$.

$$\int_{\mathbb{R}} f(x+y)dx = \int_{\mathbb{R}} f(x)dx \qquad \forall y \in \mathbb{R}.$$

Example 34.37. The Haar measure on $U(1)$ is given by $\frac{dz}{2\pi i z}$. We have:

$$\int_{U(1)} f(z)\frac{dz}{2\pi i z} = \begin{bmatrix} z = e^{2\pi i\theta} \\ dz = 2\pi i z d\theta \end{bmatrix} = \int_0^1 f(e^{2\pi i\theta})d\theta.$$

Note 34.38. $\forall w = e^{2\pi i\alpha} \in U(1)$ we have

$$\begin{aligned} \int_{U(1)} f(zw)\frac{dz}{2\pi i z} &= \int_0^1 f(e^{2\pi i(\theta+\alpha)})d\theta \\ &= \int_\alpha^{1+\alpha} f(e^{2\pi i\theta})d\theta \\ &= \int_\alpha^1 f(e^{2\pi i\theta})d\theta + \int_1^{1+\alpha} f(e^{2\pi i\theta})d\theta \\ &= \int_\alpha^1 f(e^{2\pi i\theta})d\theta + \int_0^{\alpha} f(e^{2\pi i\theta})d\theta \quad \text{by periodicity of } e^{2\pi i\theta} \\ &= \int_0^1 f(e^{2\pi i\theta})d\theta. \end{aligned}$$

Theorem 34.39. *Let G be a compact real Lie group. Then G has a canonical Borel measure dg which is invariant under*

$$\begin{aligned} g &\mapsto gh \qquad \forall h \in G \\ g &\mapsto hg \qquad \forall h \in G \\ g &\mapsto g^{-1} \end{aligned}$$

and such that $\int_G dg = 1$. This is the Haar measure *on G.*

34.8 Complete Reducibility

We assume that G is a compact real Lie group.

Theorem 34.40. *Any (finite dimensional) representation of G is unitary, hence completely reducible.*

Proof. Let $(\cdot,\cdot)$ be any positive definite Hermitian form on a representation V of G. Define $\langle\cdot,\cdot\rangle\colon V\times V\to\mathbb{C}$ by

$$\langle v,w\rangle=\int_G(\rho(g)v,\rho(g)w)dg.$$

Then $\langle\cdot,\cdot\rangle$ is positive definite and Hermitian. Also $\forall h\in G$,

$$\langle\rho(h)v,\rho(h)w\rangle=\langle v,w\rangle$$

by left invariance of Haar measure. □

34.9 Characters

Definition 34.41. The *character* of a representation V of G is $\chi_V\colon G\to\mathbb{C}$, $\chi_V(g)=\operatorname{Tr}(\rho(g))$.

Lemma 34.42 (Lemma 4.4 in Kirillov)**.**

(1) $\chi_{\text{triv}}=1$ *(Recall:* triv $=\mathbb{C}$ *and* $g.1=1\ \forall g$.

(2) $\chi_{V\oplus W}=\chi_V+\chi_W$

(3) $\chi_{V\otimes W}=\chi_V\cdot\chi_W$

(4) $\chi_V(ghg^{-1})=\chi_V(h)\qquad\forall g,h\in G$

(5) $\chi_{V^*}(g)=\overline{\chi_V(g)}\qquad\forall g\in G.$

Proof. Homework. □

Theorem 34.43 (Orthonormality of characters)**.**

(1) *If V,W are non-isomorphic irreducible representations, then*

$$\int_G\chi_V(g)\overline{\chi_W(g)}dg=0.$$

(2) *If V is any irreducible representation*

$$\int_G|\chi_V(g)|^2=1.$$

Corollary 34.44. *If $V\cong\oplus n_iV_i$ where V_i are nonisomorphic irreducible representations, then $n_i=(\chi_V,\chi_{V_i})\ \forall i$.*

Corollary 34.45. *If V and W are two representations then $V\cong W$ iff $\chi_V=\chi_W$*

Proof. Homework. □

34.10 The Hilbert space $L^2(G, dg)$

Let G be a compact real Lie group. Let

$$L^2(G, dg) = \{f : G \to \mathbb{C} \mid \int_G |f(g)|^2 dg < \infty\}$$

This is a Hilbert space with respect to

$$(f_1, f_2) = \int_G f_1(g)\overline{f_2(g)}dg.$$

Recall: Hilbert $\Rightarrow$ Normed $\Rightarrow$ metric $\Rightarrow$ topological space. Hence we have a notion of closure and denseness.

G acts on the left and right:

$$\begin{aligned}(g.f)(h) &= f(hg) \qquad \forall g, h \in G\\ (f.g)(h) &= f(gh) \qquad \forall f \in L^2(G, dg).\end{aligned}$$

34.11 Matrix Coefficients

Let V be a representation of G. To $(\lambda, v) \in V^* \times V$ we associate a function $\rho_V^{\lambda,v} : G \to \mathbb{C}$ by $\rho_V^{\lambda,v}(g) = \lambda(\rho_V(g)v)$.

Definition 34.46. $\rho_V^{\lambda,v}$ is the *matrix coefficient* corresponding to V, λ, v.

Note 34.47 (Notes)**.**

(1) $\rho_V^{\lambda,v} \in L^2(G, dg)$, because $\rho_V^{\lambda,v}$ is continuous and G is compact.

(2) The right hand side depends bilinearly on (λ, v), therefore we get a map $V^* \otimes V \to L^2(G, dg)$ by $x \mapsto (g \mapsto \rho_V^X(g))$.

(3) Under the isomorphism $V^* \otimes V \to End(V)$ this is equivalent to defining for $\varphi \in End(V)$,

$$\rho_V^\varphi : G \to \mathbb{C}, \qquad \rho_V^\varphi(g) = \mathrm{Tr}(\varphi \circ \rho_V(g))$$

Example 34.48. Fix a basis $\{v_i\}$ for V, let $v_i^* \in V$ be dual basis: $v_i^*(v_j) = \delta_{ij}$. Then $\rho_V^{v_i^* \otimes v_j}(g)$ is just the (i,j) entry of the matrix $\rho_V(g)$ in the basis $\{v_i\}$.

$$G \ni g \xrightarrow{\rho_V} \rho_V(g) \in GL(V) \cong GL(n, \mathbb{C}).$$

Taking $\mathbb{1} = \sum_i v_i^* \otimes v_i$, we get

$$\rho_V^{\mathbb{1}}(g) = \mathrm{Tr}(\rho_V(g)) = \chi_V(g).$$

So matrix coefficients generalize characters.

Theorem 34.49 (Orthogonality of Matrix Coefficients)**.** *Let G be a compact real Lie group.*

1) *Let $V \ncong W$ be irreducible representations. Then*

$$(\rho_V^\varphi, \rho_W^\psi) = 0$$

for all $\varphi \in End(V)$, $\psi \in End(W)$.

2) *Let V be an irreducible representation. Then*

$$(\rho_V^{\varphi_1}, \rho_V^{\varphi_2}) = \frac{\mathrm{Tr}(\varphi_1 \circ \varphi_2)}{\dim V}$$

for all $\varphi_1, \varphi_2 \in End(V)$.

Proof. For all irreducible representations V, W we have:

$$\begin{aligned}(\rho_V^{\varphi_1}, \rho_W^{\psi}) &= \int_G \mathrm{Tr}(\varphi \circ \rho_V(g))\overline{\mathrm{Tr}(\psi \circ \rho_W(g))}dg \\ &= \mathrm{Tr}_{V\otimes W*} \int_G (\rho_V(g) \circ \varphi) \otimes (\psi \circ \rho_W(g))^* dg \\ &= \mathrm{Tr}_{V\otimes W*} \int_G (\rho_V(g) \circ \varphi) \otimes (\rho_{W*}(g) \circ \psi^*) dg \\ &= \mathrm{Tr}_{V\otimes W*} \int_G \rho_{V\otimes W*}(g) \circ (\varphi \otimes \psi^*) dg \\ &= \mathrm{Tr}_{V\otimes W*}(\Phi)\end{aligned}$$

where $\Phi \colon V \otimes W^* \to V \otimes W^*$ is the value-average of $\varphi \otimes \psi$:

$$\Phi = \int_G \rho_{V\otimes W*}(g) \circ (\varphi \otimes \psi^*) dg.$$

The image of Φ is thus contained in $(V \otimes W^*)^G$.

Now if $V \ncong W$ then

$$(V \otimes W^*)^G \cong \mathrm{Hom}_G(W, V) = 0$$

in which case we get $\Phi = 0$. Hence,

$$(\rho_V^{\varphi}, \rho_W^{\psi}) = \mathrm{Tr}\,\Phi = 0.$$

On the other hand if $W = V$ then

$$(V \otimes W^*)^G \cong End_G(V) = \mathbb{C} \cdot \mathrm{Id}_V .$$

Which means dim(range Φ) = 1. So

$$\Phi(\mathrm{Id}_v) = (\mathrm{Tr}\,\Phi) \cdot \mathrm{Id}_V \left(\Phi \sim \begin{bmatrix} \mathrm{Tr}\,\Phi & * & \cdots & * \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{bmatrix} \right) \Rightarrow \mathrm{Tr}_V(\Phi(\mathrm{Id}_V)) = (\mathrm{Tr}\,\Phi) \cdot \dim V.$$

On the other hand,

$$\begin{aligned}\mathrm{Tr}_V(\Phi(\mathrm{Id}_v)) &= \mathrm{Tr}_V \int_G \rho_{V\otimes V*}(g) \circ (\varphi \otimes \psi^*)(\mathrm{Id}_V) dg \\ &= \mathrm{Tr}_V \int_G \rho_{V\otimes V*}(g) \circ \varphi \circ \psi \\ &= \mathrm{Tr}_V \int_G \rho_V(g) \circ \varphi \circ \psi \circ \rho_V(g)^{-1} dg \\ &= \int_G \mathrm{Tr}_V(\rho_V(g) \circ \varphi \circ \psi \circ \rho_V(g)^{-1}) dg \\ &= \int_G \mathrm{Tr}(\varphi \circ \psi) dg \\ &= \mathrm{Tr}(\varphi \circ \psi)\end{aligned}$$

This implies that

$$(\rho_V^{\varphi}, \rho_V^{\psi}) = \mathrm{Tr}_{V\otimes V^*}(\Phi) = \frac{\mathrm{Tr}(\varphi \circ \psi)}{\dim V}$$

□

34.12 The Peter-Weyl Theorem

Let G be a compact real Lie group. The Peter-Weyl Theorem

1) describes $L^2(G, dg)$ as a G-bimodule
2) says that any $f \in L^2(G, dg)$ can be approximated by a linear combination of matrix coefficients coming from irreducible representations.

Let $\hat{G}$ be the set of equivalence classes of irreducible representations of G.
For $[V] \in \hat{G}$, define a G-invariant inner product on $V^* \otimes V \cong End(V)$

$$(\varphi, \psi) = \frac{\mathrm{Tr}(\varphi \circ \psi)}{\dim V}$$

Let $\hat{\bigoplus}_{[V]\in\hat{G}} V^* \otimes V$ denote the Hilbert space completion with respect to this form.

Theorem 34.50 (Peter-Weyl)**.** *The map*

$$m\colon \hat{\bigoplus}_{[V]\in\hat{G}} V^* \otimes V \to L^2(G, dg) \qquad by \qquad V^* \otimes V \ni X \mapsto \rho_V^X$$

is an isometric isomorphism of G-bimodules.

Proof. The onto part requires analysis, we skip the proof. □

Corollary 34.51. *The set of characters $\{\chi_V \mid [V] \in \hat{G}\}$ is an orthonormal Hilbert space basis for $L^2(G, dg)^G$, the space of conjugate-invariant functions on G.*

35 Lecture 31: Universal enveloping algebra

Important tool in representation theory.
Recall If A is an associative $\Bbbk$-algebra can turn A into a Lie algebra, $\mathcal{L}A$, by defining $[x, y] = xy - yx$.
Consider a representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$.

Note 35.1. $\mathfrak{gl}(V) = \mathcal{L}\,\mathrm{End}(V)$.

Replacing $\mathrm{End}(V)$ by an arbitrary associative, we get the notion of an enveloping algebra.

Definition 35.2. An *enveloping algebra* of a Lie algebra $\mathfrak{g}$ is a pair (A, j) where A is an associative algebra and $j\colon \mathfrak{g} \to \mathcal{L}A$ is a Lie algebra morphism, i.e. $j([x, y]) = j(x)j(y) - j(y)j(x)$ and j is linear.

Any representation (V, ρ) gives an enveloping algebra $(\mathrm{End}(V), \rho)$.

Question 35.3. What is the "most general" enveloping algebra?

Definition 35.4. Let $\mathfrak{g}$ be a Lie algebra over a field $\Bbbk$. The *universal enveloping algebra* of $\mathfrak{g}$, denoted $U\,\mathfrak{g}$ (or $U(\mathfrak{g})$), is the associative $\Bbbk$-algebra with 1 generated by symbols $i(x)$ for $x \in \mathfrak{g}$ subject to relations

$$\begin{aligned} i(x+y) &= i(x) + i(y) \qquad \forall x, y \in \mathfrak{g} \\ i(cx) &= ci(x) \qquad \forall x \in \mathfrak{g},\ c \in \Bbbk \end{aligned}$$

and

$$i([x,y]) = i(x)i(y) - i(y)i(x) \qquad \forall x, y \in \mathfrak{g}\,.$$

Remark 35.5. The map

$$\begin{aligned} \mathfrak{g} &\to U\,\mathfrak{g} \\ x &\mapsto i(x) \end{aligned}$$

which we might as well call i, is a Lie algebra morphism

$$i\colon\ \mathfrak{g} \to \mathcal{L}(U\,\mathfrak{g})$$

This is immediate by construction of $U\,\mathfrak{g}$: i is linear and $i([x,y]) = i(x)i(y) - i(y)i(x)$. Abusing notation, we write x instead of $i(x)$. This is ok since we will show $\mathfrak{g} \to U\,\mathfrak{g}$, $x \mapsto i(x)$ is injective.

Remark 35.6. $U\,\mathfrak{g} \cong T\,\mathfrak{g}/I$ where $I = (x \otimes y - y \otimes x - [x,y] \mid x, y \in \mathfrak{g})$ and $T\,\mathfrak{g} = \Bbbk \oplus \mathfrak{g} \oplus \mathfrak{g}^{\otimes 2} \oplus \mathfrak{g}^{\otimes 3} \oplus \cdots$. The isomorphism sends $i(x_1)\cdots i(x_n)$ to $x_1 \otimes \cdots \otimes x_n + I$.

Example 35.7. $\mathfrak{g}$ abelian Lie algebra with basis $\{x_i\}_{i=1}^n$. This implies $U\,\mathfrak{g} \cong S\,\mathfrak{g} \cong \Bbbk[x_1, \ldots, x_n]$ a polynomial algebra. (Recall: $S\,\mathfrak{g} = T\,\mathfrak{g}/(x \otimes y - y \otimes x)$)

Example 35.8. $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Then $U\,\mathfrak{g}$ is the associative $\mathbb{C}$-algebra generated by e, f, h subject to the relations $ef - fe = h,\ he - eh = 2e,\ hf - fh = -2f$.

Note 35.9.

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathfrak{sl}(2, \mathbb{C}) \subset M_2(\mathbb{C}).$$

In $M_2(\mathbb{C})$, $e^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ but in $U\,\mathfrak{g}$, $e^2 \neq 0$ (as we will see).

Example 35.10. The quadratic casimir element of $U(\mathfrak{sl}(2, \mathbb{C}))$ is $c = ef + fe + \frac{1}{2}h^2$. Then $c \in Z(U(\mathfrak{sl}(2, \mathbb{C})))$. Consider the following calculation

$$\begin{aligned} cf &= ef^2 + fef + \frac{1}{2}h^2 f \\ &= (fe + h)f + f(fe + h) + \frac{1}{2}h(fh - 2f) \\ &= fef + fh - 2f + ffe + fh + \frac{1}{2}(fh - 2f)(h - 2) \\ &= f(ef + fe) + f(h - 2 + h + \frac{1}{2}(h-2)^2) \\ &= f(ef + fe + \frac{1}{2}h^2) = fc. \end{aligned}$$

$$
\begin{tikzcd}
\mathfrak{g} \arrow[r, "i"] \arrow[dr, "j"'] & U\,\mathfrak{g} \arrow[d, dashed, "\exists! \widetilde{j}"] \\
& A
\end{tikzcd}
$$

Theorem 35.11 (Universal property of $U\,\mathfrak{g}$)**.** *Let (A, j) be any enveloping algebra of $\mathfrak{g}$. Then there exists a unique morphism $\widetilde{j}\colon U\,\mathfrak{g} \to A$ of associative algebras such that $j = \widetilde{j} \circ i$, i.e. the following diagram commutes*

Corollary 35.12. *Given any representation*

$$\rho\colon\ \mathfrak{g} \to \mathfrak{gl}(V) = \mathcal{L}\,\mathrm{End}(V)$$

there is a unique algebra morphism

$$\widetilde{\rho}\colon U\,\mathfrak{g} \to \mathrm{End}(V)$$

such that $\widetilde{\rho}(i(x)) = \rho(x)\ \forall x \in \mathfrak{g}$.

This makes V into a $U\,\mathfrak{g}$-module: $X.v = \widetilde{\rho}(X)v$. Conversely, given any $U\,\mathfrak{g}$-module V, we get a representation of $\mathfrak{g}$ by

$$\rho\colon\ \mathfrak{g} \to \mathfrak{gl}(V)$$
$$\rho(x)v = i(x)v$$

So $\underline{\mathrm{Rep}}(\mathfrak{g}) \cong U\,\mathfrak{g} - \underline{\mathrm{Mod}}$ is an equivalence of categories.

36 Lecture 32: The Poincaré-Birkhoff-Witt Theorem

Theorem 36.1 (PBW Theorem)**.** *$U(\mathfrak{g})$ is a filtered algebra and its associated graded algebra is congruent to $S(\mathfrak{g})$.*

36.1 Graded Algebras

Definition 36.2. A *gradation* $\mathcal{G}$ on an algebra A is a collection of subspaces $\mathcal{G}_n A$ (or just A_n if the gradation is obvious) such that

i) $A = \bigoplus_{n=0}^{\infty} A_n$;

ii) $A_n A_m \subset A_{n+m}$.

Example 36.3. The tensor algebra on a vector space is naturally graded

$$T(V) = \Bbbk \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \cdots$$

namely $T(V)_n = \begin{cases} V^{\otimes n}, & n > 0 \\ \Bbbk, & n = 0 \end{cases}$.

36.2 Filtered Algebras

Definition 36.4. A *filtration* $\mathcal{F}$ on an algebra A is a collection of subspaces $\mathcal{F}_n A$ (or $A_{(n)}$ if the filtration is obvious) such that

i) $A_{(0)} \subset A_{(1)} \subset A_{(2)} \subset \cdots$;

ii) $\displaystyle\bigcup_{n=0}^{\infty} A_{(n)} = A$;

iii) $A_{(m)}A_{(n)} \subset A_{(m+n)}$.

Any graded algebra A is naturally filtered by $A_{(n)} = A_0 \oplus \cdots \oplus A_n$, but not conversely.

Example 36.5. $ef \in U(\mathfrak{sl}(2,\mathbb{C}))_{(2)}$ Note: $ef = fe + h$, so it is not graded. This filtration is uniquely determined by requiring $x \in U(\mathfrak{g})_{(1)}$ $\forall x \in \mathfrak{g}$.

36.3 Associated Graded Algebra

Think of this as formalizing the idea of a leading term.

Definition 36.6. Given a filtered algebra A the *associated graded algebra* is as follows

$$\text{gr } A = \bigoplus_{n=0}^{\infty} A_{(n)}/A_{(n-1)} \qquad A_{(-1)} = 0 \text{ by convention}$$

Example 36.7. $ef + U(\mathfrak{sl}(2,\mathbb{C}))_{(1)} = fe + U(\mathfrak{sl}(2,\mathbb{C}))_{(1)}$ in $(\text{gr } U(\mathfrak{sl}(2,\mathbb{C})))_2$.

So the PBW theorem says that gr $U(\mathfrak{g}) \cong S(\mathfrak{g})$ as graded algebras. As a word of warning, there is no algebra homomorphism from $A \to \text{gr } A$ when A is filtered.

However there is a function $f\colon A \to \text{gr } A$ defined as follows. Let $a \in A$ and $n \geqslant 0$ be the smallest such that $a \in A_{(n)}$. Then $f(a) = a + A_{(n-1)} \in (\text{gr } A)_n$. We do have that $f(ab) = f(a)f(b)$, but f is not linear.

Proof of PBW thm (sketch).
Step 1: Define $\varphi\colon S(\mathfrak{g}) \to \text{gr}U(\mathfrak{g})$ by

$$\varphi(x_1 \cdots x_n) = x_1 \cdots x_n + U(\mathfrak{g})_{(n-1)}.$$

Is φ well-defined?

$$\varphi(x_1 \cdots x_i x_{i+1} \cdots x_n) - \varphi(x_1 \cdots x_{i+1} x_i \cdots x_n) = x_1 \cdots x_{i-1}[x_i, x_{i+1}]x_{i+2} \cdots x_n + U(\mathfrak{g})_{(n+1)} = 0$$

So we have that φ is well-defined. To show φ is onto, let $\{x_i\}_1^n$ be a basis.

$$\varphi\left(\sum_{k \in \mathbb{N}^n} c_k x_1^{k_1} \cdots x_n^{k_n}\right) = \sum_{\ell \in \mathbb{N}} \sum_{\substack{k \in \mathbb{N}^n \\ \sum k_i = \ell}} c_k x_1^{k_1} \cdots x_n^{k_n} + U(\mathfrak{g})_{(\ell-1)}$$

Using commutators we can reorder the terms, where $\varphi(x_1^{k_1} \cdots x_n^{k_n}) = x_1^{k_1} \cdots x_n^{k_n} + U(\mathfrak{g})_{(n-1)}$ are called *ordered monomials*. We want to show that any element $U(\mathfrak{g})$ can be written as a linear combination of ordered monomials.

It follows by induction in the following way,

$$xy = yx + [x, y]$$

were $x > y$. As an example

$$h^2 e = h(eh + 2e) = heh + 2he.$$

Finialy to show that φ is one-to-one, we show that there exists a unique way to reduce the following: for $x < y < z$ we want to reorder zyx. We can either switch $z \leftrightarrow y$ or $y \leftrightarrow x$. For the first choice we have

$$zyx = yzx + [z, y]x = yxz + y[z, x] + [z, y]x = xyz + [y, x]z + y[z, x] + [z, y]x, \qquad (*)$$

and for the later we have,

$$zyx = zxy + z[y, x] = xzy + [z, x]y + z[y, x] = xyz + x[z, y] + [z, x]y + z[y, x]. \qquad (**)$$

We need to show that the $(*) - (**) = 0$. We notice that the LHS is zero due to the Jacobi identity. □

The following corollary is also known as the PBW theorem:

Corollary 36.8. *Let $\mathfrak{g}$ be a Lie algebra and let $(x_i)_{i \in I}$ be an ordered basis for $\mathfrak{g}$ (I is some index set). Then the set of ordered monomials*

$$\{x_{i_1} \cdots x_{i_k} \mid i_j \in I; i_1 \leqslant \cdots \leqslant i_k\}$$

is a basis for $U(\mathfrak{g})$. When $\mathfrak{g}$ is finite-dimensional, say $I = \{1, 2, \ldots, n\}$, then the basis may be written

$$\{x_1^{a_1} \cdots x_n^{a_n} \mid a_i \in \mathbb{Z}_{\geqslant 0}\}.$$

37 Lecture 33: Highest Weight Theory

Motivation: Recall the representation V_n of $\mathfrak{sl}(2, \mathbb{C})$:

$$\begin{aligned} V_n &= \operatorname{span}\{x^n, x^{n-1}y, \ldots, y^n\} \\ \rho(e) &= x\partial_y \\ \rho(f) &= y\partial_x \\ \rho(h) &= x\partial_x - y\partial_y \end{aligned}$$

V_n contains a special vector $v_0 = x^n$ with three properties:

1) v_0 is a *weight vector*:

$$h.v_0 = (x\partial_x - y\partial_y)(x^n) = nx^n = nv_0.$$

2) V_0 is a *highest weight vector*:

$$e.v_0 = (x\partial_y)(x^n) = 0.$$

3) V_n is generated by v_0:

$$V_n = \mathrm{span}_{\mathbb{C}}\{a_1 \cdot (a_2 \cdot (\cdots (a_k \cdot v_0)) \cdots) \mid k \geqslant 0\ a_i \in \mathfrak{sl}(2, \mathbb{C})\}$$

equivalently, using that

$$a_1 \cdot (a_2 \cdot (\cdots (a_k \cdot v_0)) \cdots) = (a_1 a_2 \cdots a_k) \cdot v_0$$

where $(a_1 a_2 \cdots a_k) \in U(\mathfrak{sl}(2, \mathbb{C}))$, $V_n = U(\mathfrak{sl}(2, \mathbb{C})).v_0$.

Definition 37.1. Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over $\mathbb{C}$, choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and $R_+ \subset R$ a choice of positive roots. A representation of $\mathfrak{g}$ $V = (V, \rho)$ is a *highest weight representation* if $\exists v_0 \in V$ and $\lambda \in \mathfrak{h}^*$ such that

1. v_0 is a *weight vector* of weight λ:

$$h.v_0 = \lambda(h) v_0 \quad \forall h \in \mathfrak{h}\,.$$

2. v_0 is a *highest weight vector*:

$$\mathfrak{n}_+.v_0 = 0$$

where $\mathfrak{n}_+ = \oplus_{\alpha \in R_+} \mathfrak{g}_\alpha$ i.e. $e.v_0 = 0\ \forall e \in \mathfrak{g}_\alpha$, $\forall \alpha \in R_+$.

3. V is generated by v_0

$$V = U(\mathfrak{g}).v_0$$

Theorem 37.2. *Any finite dimensional irreducible representation of $\mathfrak{g}$ is a highest weight representation.*

Proof. $\{\rho(h) \mid h \in \mathfrak{h}\}$ is a family of commuting linear operators on V, hence there is at least one common eigenvector, w, say. Let $\mu \in \mathfrak{h}^*$ be defined by $\rho(h)w = \mu(h)w$. So $V' = \oplus_{\xi \in \mathfrak{h}^*} V_\xi$ is a non-zero subspace of V, where $V_\xi = \{v \in V \mid \rho(h)v = \xi(h)v\}$. So $w \in V_\mu$. But V' is actually a subrepresentation:

$$\mathfrak{g}_\alpha \cdot V_\xi \subset V_{\xi+\alpha}.$$

V irreducible implies that $V = V'$.

Let $\mathrm{Supp}(V) = \{\xi \in \mathfrak{h}^* \mid V_\xi \neq 0\}$ be the support of V. Choose $h \in \mathfrak{h}$ such that $\langle \alpha, h\rangle > 0$ $\forall \alpha \in R_+$ (e.g. $h = \tau^\vee$, where τ defines R_+). Then let $\lambda \in \mathrm{Supp}(V)$ be such that $\langle \lambda, h\rangle$ is maximal. Then $\langle \lambda + \alpha, h\rangle > \langle \lambda, h\rangle$ for all $\alpha \in R_+$. This implies that $\forall \alpha \in R_+$ $\lambda + \alpha \notin \mathrm{Supp}(V)$. Hence, $\mathfrak{g}_\alpha V_\lambda \subset V_{\lambda+\alpha} = 0$ for all $\alpha \in R_+$.

Let v_0 be any nonzero vector in V_λ. Then 1) and 2) hold. that v_0 generates V is obvious since V is irreducible. □

Proposition 37.3. *If V is a finite dimensional irreducible representation of $\mathfrak{g}$ of highest weight λ, then $\lambda(\alpha_i^\vee) \in \mathbb{Z}_{\geqslant 0}$ for ever simple coroot $\alpha_i^\vee = h_i$.*

Proof. Consider the action of $\mathfrak{sl}(2, \mathbb{C})_{\alpha_i} \subset \mathfrak{g}$ on V. Let $v_0 \in V$ be a highest weight vector. Consider

$$V_i = \oplus_{k \in \mathbb{Z}_{\geqslant 0}} V_{\lambda - k\alpha_i}.$$

This is an $\mathfrak{sl}(2, \mathbb{C})_{\alpha_i}$-subrepresentation of V. v_0 satisfies $h_i v_0 = \lambda(h_i) v_0$, $e_i.v_0 = 0$ and v_0 generates a finite dimensional $\mathfrak{sl}(2, \mathbb{C})_{\alpha_i}$-representation.

Let N be minimal such that $f_i^N.v_0 = 0$. Then

$$0 = e_i.(f_i^N.v_0) = (f_i^N e_i).v_0 + [e_i, f_i^N].v_0 = \cdots = (\lambda(h_i) - (N-1)) f_i^{N-1}.v_0$$

Thus $\lambda(h_i) = N - 1$, so we are done. □

So we have a map

$$\{\text{f.d. irreps of } \mathfrak{g}\}_{/iso.} \xrightarrow{\Phi} P_+$$

where $P_+ = \{\lambda \in \mathfrak{h}^* \mid \lambda(\alpha_i^\vee) \in \mathbb{Z}_{\geqslant 0} \, \forall i = 1, 2, \ldots, r\}$ are the *dominant integral weights.*

Goal: Show that we can go back (i.e. there exists an inverse of Φ).

Plan:

1) To any $\lambda \in \mathfrak{h}^*$ construct a universal highest weight representation $M(\lambda)$ of highest weight λ. (Verma Module)

2) Each $M(\lambda)$ has a unique irreducible quotient $L(\lambda)$.

3) Show that $L(\lambda)$ is finite dimensional iff $\lambda \in P_+$.

4) $L(\lambda) \cong L(\mu) \;\Leftrightarrow\; \lambda = \mu$.

38 Lecture 34: Verma Modules

We will need the definition of tensor product of modules over a noncommutative ring.

Definition 38.1. Let R be a ring, M a right R-module, and N a left R-module. Then $M \otimes_R N$ is the abelian group generated by the symbols $m \otimes n$ subject to the relations:

1) $\mathbb{Z}$ bilinearity

$$\begin{aligned} (m_1 + m_2) \otimes n &= m_1 \otimes n + m_2 \otimes n \\ m \otimes n_1 + n_2 &= m \otimes n_1 + m \otimes n_2 \end{aligned}$$

2) R-balanced

$$(m.r) \otimes n = m \otimes (r.m)$$

Moreover, if M is a (S, R)-bimodule, then $M \otimes_R N$ is a left S-module via $s.(m \otimes n) = s.m \otimes n$.

Recall that a *highest weight representation* of $\mathfrak{g}$ is a representation generated by a *highest weight vector*:

1) $hv_0 = \lambda(h)v_0$ for all $h \in \mathfrak{h}$

2) $ev_0 = 0$ for all $e \in \mathfrak{g}_\alpha$ and $\alpha \in R_+$

3) $V = U(\mathfrak{g})v_0$

The following defines a universal highest weight representation associated to any $\lambda \in \mathfrak{h}^*$.

Definition 38.2. Let $\lambda \in \mathfrak{h}^*$. The corresponding *Verma module* $M(\lambda)$ is defined by

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}1_\lambda$$

where $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+ = \mathfrak{h} \oplus_{\alpha \in R_+} \mathfrak{g}_\alpha$, and $\mathbb{C}1_\lambda$ is the 1-dimensional representation of $\mathfrak{b}$ given by:

$$\begin{aligned} h.1_\lambda &= \lambda(h)1_\lambda & \forall h \in \mathfrak{h} \\ x.1_\lambda &= 0 & \forall x \in \mathfrak{n}_+ \end{aligned}$$

and $U(\mathfrak{g})$ is regarded as a right $U(\mathfrak{b})$ module.

Note that, since $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}1_\lambda$ with $\mathbb{C}1_\lambda$ is a left $U(\mathfrak{b})$-module and $U(\mathfrak{g})$ is a $(U(\mathfrak{g}), U(\mathfrak{b}))$-bimodule, the Verma module $M(\lambda)$ is a left $U(\mathfrak{g})$-module.

Example 38.3. $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, $\mathfrak{b} = \mathbb{C}h \oplus \mathbb{C}e$, $\mathbb{C}1_\lambda$ for $\lambda \in \mathfrak{h}^* \cong \mathbb{C}$ such that $h.1_\lambda = \lambda(h)1_\lambda = \lambda' 1_\lambda$ and $e.1_\lambda$. Below we see how we can simplify elements in $M(\lambda)$.

$$\begin{aligned} M(\lambda) \ni hef \otimes 1_\lambda &= h(fe + [e,f]) \otimes 1_\lambda \\ &= (hfe + h^2) \otimes 1_\lambda \\ &= hfe \otimes 1_\lambda + h^2 \otimes 1_\lambda \\ &= hf \otimes 0 + 1 \otimes h^2 1_\lambda \\ &= 1 \otimes (\lambda^1)^2 1_\lambda \\ &= (\lambda')^2 (1 \otimes 1_\lambda) \end{aligned}$$

Recall: The PBW theorem (Thm 36.1) says that $U(\mathfrak{sl}(2,\mathbb{C}))$ has a basis

$$\{f^k h^\ell e^m \mid k, \ell, m \geqslant 0\}$$

So

$$M(\lambda) = \sum_{k,\ell,m \geqslant 0} \mathbb{C} f^k h^\ell e^m \otimes 1_\lambda = \sum_{k,\ell \geqslant 0} \mathbb{C} f^k \otimes h^\ell 1_\lambda = \sum_{k \geqslant 0} \mathbb{C} f^k \otimes 1_\lambda$$

This property holds in general:

Theorem 38.4. *$M(\lambda) \cong U(\mathfrak{n}_-)$ as left $U(\mathfrak{n}_-)$-modules*

Proof. $\varphi \colon U(\mathfrak{n}_-) \to M(\lambda)$ by $x \mapsto x \otimes 1_\lambda$. φ is a surjective using the PBW theorem (thm 36.1):

$$\begin{aligned} M(\lambda) &= U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \otimes_{U(\mathfrak{b})} \mathbb{C}1_\lambda \\ &= U(\mathfrak{n}_-) U(\mathfrak{b}) \otimes_{U(\mathfrak{b})} \mathbb{C}1_\lambda \\ &= U(\mathfrak{n}_-) \otimes \mathbb{C}1_\lambda \\ &\subset \operatorname{im} \varphi. \end{aligned}$$

φ is injective: By PBW theorem $U(\mathfrak{g})$ is free as a right $U(\mathfrak{b})$-module on a basis for $u(\mathfrak{n}_-)$: $U(\mathfrak{g}) \cong U(\mathfrak{n}_-) \otimes_{\mathbb{C}} U(\mathfrak{b})$ by properties of tensors one can show that

$$\begin{aligned} U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}1_\lambda &\cong (U(\mathfrak{n}_-) \otimes_{\mathbb{C}} U(\mathfrak{b})) \otimes_{U(\mathfrak{b})} \mathbb{C}1_\lambda \\ &= U(\mathfrak{n}_-) \otimes_{\mathbb{C}} (U(\mathfrak{b}) \otimes_{U(\mathfrak{b})} \mathbb{C}1_\lambda) \\ &\cong U(\mathfrak{n}_-) \otimes_{\mathbb{C}} \mathbb{C}1_\lambda \\ &\cong U(\mathfrak{n}_-). \end{aligned}$$

$\square$

Corollary 38.5. *The support of $M(\lambda)$ is*

$$\operatorname{Supp}(M(\lambda)) = \lambda - Q_+ = \{\lambda - \sum_{i=1}^r k + i\alpha_i \mid k_i \in \mathbb{Z}_{\geqslant 0}\}$$

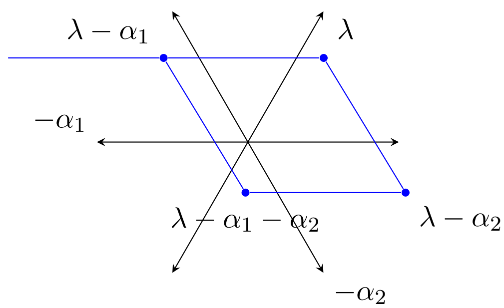
where $\{\alpha_1, \ldots, \alpha_r\} = \Pi$ the set of simple roots. $Q = \mathbb{Z}R = \oplus_1^r \mathbb{Z}\alpha_i$ and $Q_+ = \sum_1^r \mathbb{Z}_{\geqslant 0}\alpha_i$

Proof. $f_{\beta_1}^{k_1} \cdots f_{\beta_n}^{k_n} \otimes 1_\lambda$ where $R_+ = \{\beta_1, \dots, \beta_n\}$. Then

$$\begin{aligned} h.(f_{\beta_1}^{k_1} \cdots f_{\beta_n}^{k_n} \otimes 1_\lambda) &= h f_{\beta_1}^{k_1} \cdots f_{\beta_n}^{k_n} \otimes 1_\lambda \\ &= (f_{\beta_1}^{k_1} \cdots f_{\beta_n}^{k_n} h + [h, f_{\beta_1}^{k_1} \cdots f_{\beta_n}^{k_n}]) \otimes 1_\lambda \\ &\overset{(*)}{=} \lambda(h) f_{\beta_1}^{k_1} \cdots f_{\beta_n}^{k_n} \otimes 1_\lambda - (k_1\beta_1 + \cdots + k_n\beta_n)(h) f_{\beta_1}^{k_1} \cdots f_{\beta_n}^{k_n} \otimes 1_\lambda \\ &= (\lambda - (k_1\beta_1 + \cdots + k_n\beta_n))(h) f_{\beta_1}^{k_1} \cdots f_{\beta_n}^{k_n} \otimes 1_\lambda. \end{aligned}$$

Where $(*)$ is by $[h, f_{\beta_i}] = -\beta_i(h) f_{\beta_i}$ for every i. $\square$

Example 38.6. $\mathfrak{sl}(3, \mathbb{C})$ $\lambda = k_1 w_1 + k_2 w_2$ where $w_i \in \mathfrak{h}^*$ and $w_i(h_j) = \delta_{ij}$ which are the *fundamental weights*. Where the blue lattice is $\mathrm{Supp}(M(\lambda))$.



39 Lecture 35: Classification of Finite-Dimensional Irreducible Representations

Goal: Classify all finite dimensional irreducible represents of a finite dimensional semisimple Lie algebra over $\mathbb{C}$ $\mathfrak{g}$, (V, ρ).

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}1_\lambda$$

Proposition 39.1.

i) *$M(\lambda)$ is a highest weight representation of $\mathfrak{g}$ of highest weight λ.*

ii) *Every highest weight representation of $\mathfrak{g}$ of highest weight λ is a quotient of $M(\lambda)$.*

iii) $M(\lambda)_\lambda = \mathbb{C}(1 \otimes 1_\lambda)$

iv) $\mathrm{Supp}(M(\lambda)) = \lambda - Q_+ = \{\lambda - \sum k_i \alpha_i \mid k_i \in \mathbb{Z}_{\geqslant 0}\}$

v) *$M(\lambda)$ has a unique maximal (proper) submodule $N(\lambda)$. Hence $M(\lambda)$ has a unique irreducible quotient $L(\lambda) = M(\lambda)/N(\lambda)$.*

Proof. i) Put $v_\lambda = 1 \otimes 1_\lambda \in M(\lambda)$,

$$\begin{aligned} h.v_\lambda = h.(1 \otimes 1_\lambda) &= h1 \otimes 1_\lambda \\ &= h \otimes 1_\lambda \\ &= 1 \otimes h.1_\lambda \\ &= 1 \otimes \lambda(h)1_\lambda \\ &= \lambda(h)(1 \otimes 1_\lambda). \end{aligned}$$

This implies v_λ is a weight vector of weight λ. Also $\forall e \in \mathfrak{n}_+$,

$$e.v_\lambda = e \otimes 1_\lambda = 1 \otimes 0 = 0$$

because $\mathfrak{n}_+ \subset \mathfrak{b}$. So v_λ is a highest weight vector.

$$U(\mathfrak{g})v_\lambda = U(\mathfrak{g}).(1 \otimes 1_\lambda) = U(\mathfrak{g}) \otimes 1_\lambda = M(\lambda).$$

ii) Let W be any highest weight representation of $\mathfrak{g}$ of highest weight λ. $w_\lambda \in W$ be a (non-zero) highest weight vector of weight λ.

Consider the map

$$\begin{aligned} \psi \colon U(\mathfrak{g}) \times \mathbb{C}1_\lambda &\to W \\ (a, \xi 1_\lambda) &\mapsto \xi a.w_\lambda \end{aligned}$$

Then

- ψ is $\mathbb{Z}$-bilinear (biadditive)
- Write $U(\mathfrak{b}) = U(\mathfrak{n}_+)U(\mathfrak{h})$. We show that $b \in U(\mathfrak{b})$ $\psi(ab, \xi 1_\lambda = \psi(a, b.\xi 1_\lambda)$. If $b = h \in \mathfrak{h}$, then

$$\begin{aligned} \psi(ah, \xi 1_\lambda) &= \xi a h 1_\lambda \\ &= \xi a \lambda(h) 1_\lambda \\ &= \lambda(h) \xi a 1_\lambda \\ &= \psi(a, \lambda(h)\xi 1_\lambda) \\ &= \psi(a, h.\xi 1_\lambda). \end{aligned}$$

  Similarly for $\mathfrak{b} = e \in \mathfrak{n}_+$. So ψ induces a map from $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}1_\lambda \xrightarrow{\widetilde{\psi}} W$. Since W is generated by w_λ, the map $\widetilde{\psi}$ is surjective. So $W \cong \dfrac{M(\lambda)}{\ker \widetilde{\psi}}$.

v) Let

$$N(\lambda) := \sum_{\substack{S \subsetneq M(\lambda) \\ \text{subrepresentations}}} .$$

We want to show that $N(\lambda) \subsetneq M(\lambda)$.

<u>Facts:</u> Any subrepresentation of a weight representation is a weight representation. This implies all S have a weight decomposition $S = \oplus_{\mu \in \lambda = Q_+} S_\mu$ and so does $N(\lambda)$:

$$N(\lambda) = \oplus_{\mu \in \lambda - Q_+} N(\lambda)_\mu.$$

Since each $S \subset M(\lambda)$ $S_\lambda = 0$. $(S_\lambda \subset M(\lambda)_\lambda \subset \mathbb{C}v_\lambda)$. Hence $N(\lambda)_\lambda = \oplus_{S \subsetneq M(\lambda)} S_\lambda = 0$.

Thus $N(\lambda)$ is proper subrepresentation. $\square$

Lemma 39.2. *If $\lambda, \mu \in \mathfrak{h}^*$ such that $L(\lambda) \cong L(\mu)$, then $\lambda = \mu$.*

Proof. Suppose $\varphi\colon L(\lambda) \to L(\mu)$ is an isomorphism. Then $\varphi(v_\lambda) \in L(\mu)[\lambda]$. Therefore $\lambda \in \mu - \mathbb{Z}_{\geqslant 0}\Pi$, i.e. $\lambda \leqslant \mu$. Switching roles of λ and μ we also have $\mu \leqslant \lambda$. Thus $\lambda = \mu$. □

To summarize:

- Every finite dimensional irreducible representation is a finite dimensional irreducible highest weight representation.
- Every irreducible highest weight representation is $\cong L(\lambda)$ for some unique $\lambda \in \mathfrak{h}^*$.
- Question: When is $\dim L(\lambda) < \infty$?

So $\{L(\lambda) \mid \lambda \in \mathfrak{h}^*\}$ is a complete set of representatives for isoclasses of irreducible highest weight representations of $\mathfrak{g}$.

Theorem 39.3. $\dim L(\lambda) \leqslant \infty$ *iff* $\lambda \in P_+$ *i.e.* $\lambda(\alpha_i^\vee) \in \mathbb{Z}_{\geqslant 0}$ $\forall i = 1, \ldots, r$.

Proof. ($\Rightarrow$): Last time using $\mathfrak{sl}(2, \mathbb{C})_{\alpha_i}$.
($\Leftarrow$): Somewhat lengthy. □

Consequently, we obtain the following theorem, which provides a classification of finite-dimensional irreducible representations of a semisimple Lie algebra $\mathfrak{g}$:

Theorem 39.4. *The set P_+ of dominant integral weights is in bijection with the set of isomorphism classes of finite-dimensional irreducible representations of $\mathfrak{g}$. The bijection is given by $\lambda \mapsto [L(\lambda)]$, where $L(\lambda)$ is the unique irreducible quotient of the Verma module $M(\lambda)$.*

40 Lecture 36: Examples

41 Lecture 37: Central Characters

Definition 41.1. A *character* of an associative algebra A is an algebra homomorphism $A \to \Bbbk$.

Recall that by $Z(\mathfrak{g})$ we mean the center of the universal enveloping algebra of $\mathfrak{g}$. That is, $Z(\mathfrak{g}) = Z(U(\mathfrak{g})) = \{z \in U(\mathfrak{g}) \mid zu = uz\, \forall u \in U(\mathfrak{g})\}$.

Definition 41.2. A *central character* of a Lie algebra $\mathfrak{g}$ is a character of $Z(\mathfrak{g})$.

Example 41.3. Let $\mathfrak{g} = \mathfrak{sl}(2, \Bbbk)$. Choosing the ordered basis (e, h, f), it is easy to check that the dual basis for $\mathfrak{g}$ with respect to the traceform is given by $(f, \frac{1}{2}h, e)$. the $\frac{1}{2}$ comes from the fact that square of the matrix h (in $M_2(\Bbbk)$) has trace 2. Thus the Casimir element is

$$C = ef + fe + \frac{1}{2}h^2.$$

This is an element of $Z(\mathfrak{g})$. In fact, as we shall see, this element generates the center (as an associative algebra). That is, $\mathbb{Z}(\mathfrak{g}) = \Bbbk[C]$. Furthermore, C is algebraically independent over $\Bbbk$ so $\Bbbk[C]$ is a polynomial algebra in one variable.

Thus a central character χ for $\mathfrak{g}$ is determined by the value $\chi(C)$ at the Casimir.

We will show that each finite-dimensional simple $U(\mathfrak{g})$-module gives rise to a central character. We need an instance of Schur's Lemma which can be stated as follows in our case.

Lemma 41.4 (Schur's Lemma)**.** *If V is a simple $U(\mathfrak{g})$-module, then*

$$\mathrm{End}_{U(\mathfrak{g})}(V) = \Bbbk\,\mathrm{Id}_V\,.$$

That is, if $\varphi : V \to V$ is a $U(\mathfrak{g})$-module endomorphism of V, then φ must be a scalar multiple of the identity.

Proof. First we show that any nonzero $U(\mathfrak{g})$-module endomorphism ψ of V has to be invertible. Indeed, since $\ker\psi \neq V$ and V is a simple module, we have $\ker\psi = 0$. Similarly, $\operatorname{im}\psi \neq 0$ so $\operatorname{im}\psi = V$ by simplicity.

Now let $\varphi : V \to V$ be a $U(\mathfrak{g})$-module endomorphism and let $\xi \in \Bbbk$ be an eigenvalue of φ. Such a ξ exists because V is finite-dimensional and $\Bbbk$ is assumed to be algebraically closed. Then consider $\psi = \varphi - \xi\,\mathrm{Id}_V$. Since ξ is an eigenvalue for φ, the map ψ is not injective. Therefore, by the first part, ψ must be identically zero. Thus $\varphi = \xi\,\mathrm{Id}_V$. □

Now we can show the following result.

Proposition 41.5. *Let V be a finite-dimensional simple $U(\mathfrak{g})$-module. Then there exists a central character χ for $\mathfrak{g}$ such that*

$$z.v = \chi(z)v \quad \forall z \in Z(\mathfrak{g}),\ \forall v \in V. \tag{41.1}$$

Proof. Let $z \in Z(\mathfrak{g})$. Then the map $v \mapsto z.v$ is a $U(\mathfrak{g})$-module endomorphism of V: $u.(z.v) = (uz).v = (zu).v = z.(u.v) \forall u \in U(\mathfrak{g}), z \in Z(\mathfrak{g}), v \in V$. Thus, by Schur's Lemma, there exists a scalar $\chi(z)$ such that (41.1) holds. Thus, for any $v \in V$ and $z_1, z_2 \in Z(\mathfrak{g})$,

$$(z_1 z_2).v = z_1.(z_2.v) = z_1.(\chi(z_2)v) = \chi(z_2)z_1.v = \chi(z_1)\chi(z_2)v$$

which, choosing $v \neq 0$, implies that $\chi(z_1 z_2) = \chi(z_1)\chi(z_2)$. Similarly one shows that χ is linear. Thus χ is a central character of $\mathfrak{g}$. □

Notation 41.6. We know that when $\mathfrak{g}$ is semisimple, any finite-dimensional simple $U(\mathfrak{g})$-module is isomorphic to $L(\lambda)$ for some dominant integral weight λ. In this case we denote the associated central character by χ_λ.

Example 41.7. If we go back to the case of $\mathfrak{g} = \mathfrak{sl}_2$, let $V = L(n\omega) = \Bbbk x^n \oplus \Bbbk x^{n-1}y \oplus \cdots \oplus \Bbbk y^n$ be the $n+1$-dimensional simple module ($\omega = \frac{1}{2}\alpha$). Let us describe the corresponding central character $\chi = \chi_{n\omega}$. Since $Z(\mathfrak{g})$ is generated by the Casimir C it suffices to compute $\chi(C)$. Furthermore, since we know C has to act by a scalar, it suffices to compute the action of C on any vector we want. We choose the highest weight vector. This motivates us to rewrite C in the PBW basis coming from the ordered basis (f, h, e), because it is very easy to compute the action on the highest weight vector by a monomial having e on the right: it is zero. We have

$$\begin{aligned} C &= ef + fe + \frac{1}{2}h^2 = \\ &= (fe + h) + fe + \frac{1}{2}h^2 = \\ &= 2fe + \frac{1}{2}(h^2 + 2h) = \\ &= 2fe + \frac{1}{2}\big((h+1)^2 - 1\big). \end{aligned}$$

The last equality is not important right this moment but will become important in what follows. For now, observe that since fe kills the highest weight vector v_+, and $hv_+ = nv_+$, we have

$$Cv_+ = \frac{1}{2}\big((n+1)^2 - 1\big)v_+$$

Thus we conclude

$$\chi_{n\omega}(C) = \frac{1}{2}\big((n+1)^2 - 1\big).$$

Remark 41.8. In physics one would use the parametrization $\ell = n/2$ for the representations, and normalize the Casimir to be $2C$. In those conventions, the value of the Casimir would be $\ell(\ell+1)$.

The example can be generalized. Notice that the procedure of rewriting $z \in Z(\mathfrak{g})$ in the PBW basis and throwing away terms having any positive root vector on the right, is equivalent to applying the Harish-Chandra homomorphism to z.

Proposition 41.9. *Let $\mathfrak{g}$ be a semisimple Lie algebra, and V be a finite-dimensional simple $U(\mathfrak{g})$-module. Then the corresponging central character χ is given by*

$$\chi(z) = \lambda\big(\varphi_{\mathrm{HC}}(z)\big) \tag{41.2}$$

where φ_{HC} denotes the Harish-Chandra homomorphism, and λ has been extended to a character $U(\mathfrak{h}) \to \Bbbk$.

Proof. It once again suffices to check this identity on the highest weight vector v_+ of V. By the PBW theorem, we have a direct sum decomposition $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_+)$. For any $u \in U(\mathfrak{g})$, write $u = u_0 + u_1$ where $u_0 \in U(\mathfrak{h})$ and $u_1 \in \mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_+$. Clearly $u.v_+ = u_0.v_+$ since $\mathfrak{n}_+.v_+ = 0$. Note also that $u_0 = \varphi_{\mathrm{HC}}(u)$. Thus we have for any $z \in Z(\mathfrak{g})$:

$$z.v_+ = z_0.v_+ = \varphi_{\mathrm{HC}}(z_0).v_+$$

Now, the action of an element a of $U(\mathfrak{h})$ on v_+ is given by $\lambda(a)v_+$. Thus we obtain (41.2) $\square$

For an n-dimensional vector space V, let $\Bbbk[V]$ denote the algebra of all functions $f : V \to \Bbbk$ such that when we choose a basis $\{v_i\}_{i=1}^n$ for V, we have $f(x_1v_1 + x_2v_2 + \cdots + x_nv_n) = p(x_1, x_2, \ldots, x_n)$ for some polynomial $p \in \Bbbk[x_1, x_2, \ldots, x_n]$. Notice that the property that f has is independent of the choice of basis, since a linear change of variables map polynomials to polynomials. Thus, in fact, once we fix a basis we have an isomorphism $\Bbbk[V] \to \Bbbk[x_1, x_2, \ldots, x_n]$. But $\Bbbk[V]$ exists as an object independent of choice of basis. The relation between $\Bbbk[V]$ and $\Bbbk[x_1, x_2, \ldots, x_n]$ is exactly analogous to the relation between $\mathrm{End}_{\Bbbk}(V)$ and $M_n(\Bbbk)$.

Lemma 41.10. *Let V be a finite-dimensional vector space. Then there is a canonical isomorphism*

$$S(V^*) \cong \Bbbk[V].$$

Proof. We have an inclusion map $i : V^* \to \Bbbk[V]$. Since $\Bbbk[V]$ is commutative, by the universal property of the symmetric algebra there exists a unique algebra homomorphism $S(V^*) \to \Bbbk[V]$ whose restriction to V is i. Conversely, the inclusion map $V^* \subset S(V^*)$ extends uniquely (using that if $\{x_1, \ldots, x_n\}$ is a basis for V^* then $\Bbbk[V] \simeq \Bbbk[x_1, x_2, \ldots, x_n]$) to an algebra map $\Bbbk[V] \to S(V^*)$ which is inverse to the previous map. $\square$

Notice that $U(\mathfrak{h}) = S(\mathfrak{h}) \cong \Bbbk[\mathfrak{h}^*]$ often one identifies $U(\mathfrak{h})$ with $\Bbbk[\mathfrak{h}^*$. With this in mind, the following theorem describes the center of the universal enveloping algebra of any semisimple Lie algebra.

Theorem 41.11 (Harish-Chandra)**.** *When restricted to the center, the Harish-Chandra homomorphism*

$$\varphi_{\mathrm{HC}} : Z(\mathfrak{g}) \to \Bbbk[\mathfrak{h}^*] \tag{41.3}$$

is injective, with image equal to

$$\Bbbk[\mathfrak{h}^*]^{W\cdot}$$

where W is the Weyl group and $W\cdot$ refers to the dot action, given on $\mathfrak{h}^*$ by

$$w \cdot \lambda = w(\lambda + \rho) - \rho \tag{41.4}$$

where ρ is the Weyl vector $\rho = \frac{1}{2}\sum_{\alpha \in R_+} \alpha$, with induced action on $\Bbbk[\mathfrak{h}^*]$:

$$(w \cdot p)(\lambda) = p(w^{-1} \cdot \lambda)$$

42 Lecture 38: Kac-Moody Algebras

We follow *Introduction to Quantum Groups and Crystal Bases* by Hong and Kang.

Let I be a finite index set.

Definition 42.1. A square matrix $A = (a_{ij})_{i,j \in I}$ with integer entries is a *generalized Cartan matrix (GCM)* if

1. $a_{ii} = 2$, $\forall i \in I$
2. $a_{ij} \leqslant 0$, $i \neq j$
3. $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$.

Definition 42.2. A GCM A is called *symmetrizable* if there exists a diagonal matrix $D = \operatorname{diag}(b_i \mid i \in I)$ with $b_i \in \mathbb{Z}_{>0}$ such that DA is symmetric.

Definition 42.3. A GCM A is *indecomposable* if for all partitions $I = I_1 \sqcup I_2$, $I_i \neq \varnothing$, there is $i \in I_1$, $j \in I_2$ such that $a_{ij} \neq 0$.

For an $I \times I$-matrix, let $\operatorname{corank} A = |I| - \operatorname{rank} A$.

Definition 42.4. A *Cartan datum* is a quintuple $(A, \Pi, \Pi^\vee, P, P^\vee)$ where

1. $P^\vee$ is a free abelian group of rank $|I| + \operatorname{corank} A$ with $\mathbb{Z}$-basis denoted $\{h_i \mid i \in I\} \cup \{d_s \mid s = 1, 2, \ldots, \operatorname{corank} A\}$;
2. $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(P^\vee) \subset \mathbb{Z}\}$ where $\mathfrak{h} = \Bbbk \otimes_{\mathbb{Z}} P^\vee$;
3. $\Pi^\vee = \{h_i \mid i \in I\}$;
4. $\Pi = \{\alpha_i \mid i \in I\}$ is a linearly independent subset of $\mathfrak{h}^*$ satisfying

$$\alpha_j(h_i) = a_{ij} \qquad \alpha_j(d_s) \in \{0, 1\}. \tag{42.1}$$

We call

- $\mathfrak{h}$ the *Cartan subalgebra (CSA)*
- Π the set of *simple roots*,
- $\Pi^\vee$ the set of *simple coroots*,
- P the *weight lattice*,
- $P^\vee$ the *dual weight lattice*

Given a Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ we define the *root lattice* to be

$$Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i \tag{42.2}$$

and the *positive (cone in the) root lattice*

$$Q_+ = \sum_{i \in I} \mathbb{Z}_{\geqslant 0}\alpha_i. \tag{42.3}$$

and set $Q_- = -Q_+$. The *fundamental weights* $\omega_i \in \mathfrak{h}^*$ for $i \in I$ are given by

$$\omega_i(h_j) = \delta_{ij}, \qquad \omega_i(d_s) = 0. \tag{42.4}$$

We have a partial order on $\mathfrak{h}^*$ given by $\lambda \geqslant \mu \Longleftrightarrow \lambda - \mu \in Q_+$.

The *simple reflections* $s_i \in \mathrm{GL}(\mathfrak{h}^*)$, $i \in I$, are given by

$$s_i(\lambda) = \lambda - \lambda(h_i)\alpha_i, \quad \forall \lambda \in \mathfrak{h}^*. \tag{42.5}$$

and the *Weyl group* W is the subgroup of $\mathrm{GL}(\mathfrak{h}^*)$ generated by the set of simple reflections.

Definition 42.5. The Kac-Moody[xi] (KM) algebra $\mathfrak{g}$ associated with a Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is the Lie algebra with generators

$$\{e_i, f_i\}_{i \in I} \cup P^\vee \tag{42.6}$$

subject to relations saying that $P^\vee$ is a subgroup of $\mathfrak{g}$ together with

$$[h, h'] = 0 \quad h, h' \in P^\vee, \tag{42.7}$$
$$[e_i, f_j] = \delta_{ij} h_i, \tag{42.8}$$
$$[h, e_i] = \alpha_i(h) e_i, \tag{42.9}$$
$$[h, f_i] = -\alpha_i(h) f_i, \tag{42.10}$$
$$(\operatorname{ad} e_i)^{1-a_{ij}}(e_j) = 0, \quad i \neq j, \tag{42.11}$$
$$(\operatorname{ad} f_i)^{1-a_{ij}}(f_j) = 0, \quad i \neq j. \tag{42.12}$$

Proposition 42.6. *$\mathfrak{g}$ has a root space decomposition*

$$\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha, \quad \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \; \forall h \in \mathfrak{h}\} \tag{42.13}$$

and a triangular decomposition

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+, \qquad \mathfrak{g}_\pm = \bigoplus_{\alpha \in Q_\pm} \mathfrak{g}_\alpha. \tag{42.14}$$

[xi] Independently discovered in 1967–1968 by three people: V.G. Kac, R. V. Moody and I.L. Kantor

GCMs, hence KM algebras, are classified into three types. For an integer column vector $u = (u_i)_{i \in I}^T$ we write $u > 0$ (resp. $u \geqslant 0$) if $\forall i \in I : u_i > 0$ (resp. $u_i \geqslant 0$).

The following is Theorem 4.3 of Kac's book *Infinite dimensional Lie algebras*

Theorem 42.7. *Let A be an indecomposable GCM. Then exactly one of the following three possibilities hold:*

(Finite Type): $\operatorname{corank} A = 0$, *and* $\exists u > 0 : Au > 0$, *and* $Av \geqslant 0 \Rightarrow (v > 0$ *or* $v = 0)$.

(Affine Type): $\operatorname{corank} A = 1$, *and* $\exists u > 0 : Au = 0$, *and* $Av \geqslant 0 \Rightarrow Av = 0$.

(Indefinite Type): $\exists u > 0 : Au < 0$ *and* $(Av \geqslant 0$ *and* $v \geqslant 0) \Rightarrow v = 0$.

Example 42.8. Examples of GMCs of affine type are

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

An example of a GCM of indefinite type is

$$\begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix}$$

42.1 The Dynkin Diagram of a GCM

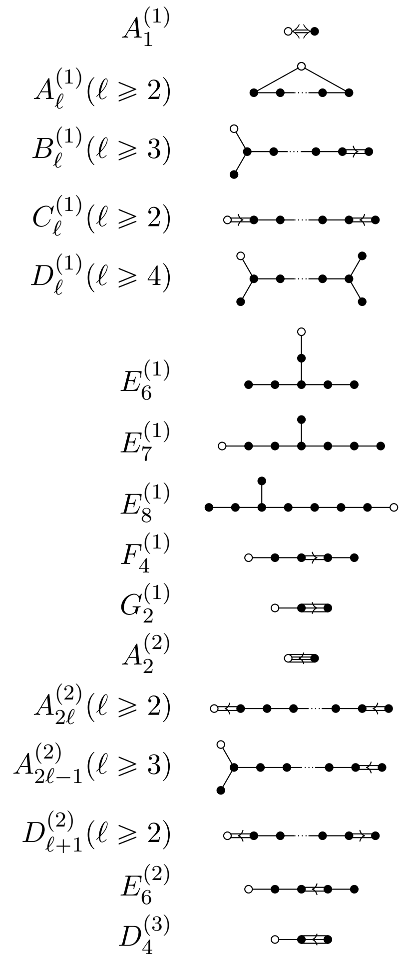
Given a GCM A, we associate a (generalized) Dynkin diagram $D(A)$ as follows. The vertex set is I. For the edges one convention is as follows:

- In the case when $a_{ij}a_{ji} \leqslant 3$ we retain the conventions from finite type (see Definition 31.7).
- When $a_{ij} = a_{ji} = -2$ we depict the edge as $\bullet\!\Leftrightarrow\!\bullet$, and when $(a_{ij}, a_{ji}) = (-4, -1)$ we draw it as $\bullet\!\Lleftarrow\!\!\Rrightarrow\!\bullet$.
- When $a_{ij}a_{ji} > 4$ we draw a bold line between i and j and put the label $(|a_{ij}|, |a_{ji}|)$ above the edge.

The following proposition is not hard to prove, given Kac's theorem above.

Proposition 42.9. *Let A be an indecomposable GCM. Then A is of affine type if and only if* $\det A = 0$ *and all proper subdiagrams of $D(A)$ are of finite type.*

Using this result one can obtain a complete list of the connected Dynkin diagrams of affine type. The white node corresponds to the unique simple root α_0 with $\alpha_0(d) = 1$. The significance of this will be explained in the next theorem.



Example 42.10. The three cartan matrices from the previous example correspond to the affine Dynkin diagrams $A_1^{(1)}$, $A_2^{(2)}$, and $A_2^{(1)}$ respectively.

42.2 Realization of Affine Kac-Moody Algebras as Extensions of Loop Algebras

A very important fact about the affine KM algebras is their realization as extensions of loop algebras.

Theorem 42.11. *The untwisted affine KM algebras are isomorphic to the Lie algebra*

$$\mathring{\mathfrak{g}} \otimes \Bbbk[t, t^{-1}] \oplus \Bbbk c \oplus \Bbbk d \tag{42.15}$$

where $\mathring{\mathfrak{g}}$ is the so called underlying finite type Lie algebra obtained by deleting the white node from the Dynkin diagram, and the Lie bracket is given by

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m\delta_{m+n,0}(x, y)c \tag{42.16}$$

$$[d, x \otimes t^m] = mx \otimes t^m \tag{42.17}$$

$$c \text{ is central} \tag{42.18}$$

The twisted affine KM algebras can be realized as the subalgebra $\mathfrak{g}^\theta$ of all elements fixed by an automorphism θ of order 2 or 3, inside an untwisted affine KM algebra $\mathfrak{g}$. The automorphism comes from a diagram automorphism of the untwisted affine Dynkin diagram.

43 Lecture 39: Hopf Algebras and Quantum Groups

43.1 Vector spaces

We make some remarks about the category of vector spaces over $\Bbbk$.

Bilinear maps are not morphisms in the category of vector spaces. This indicates that they are not the correct thing to look at. By the universal property of tensor products, bilinear maps $U \times V \to W$ are in bijection with linear maps $U \otimes V \to W$.

Selecting an special element $v \in V$, such as an identity element of an algebra, is also not a categorical notion, since we don't have "elements". The solution here is that elements of V are in bijection with linear maps $\Bbbk \to V$.

We also recall that $\otimes$ is a bifunctor, which means that for any two linear maps $f_1 : V_1 \to W_1$ $f_2 : V_2 \to W_2$ we get a linear map $f_1 \otimes f_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$, given by $(f_1 \otimes f_2)(v_1 \otimes v_2) = f_1(v_1) \otimes f_2(v_2)$ and extended linearly.

The category of vector spaces has a special object, $\Bbbk$, which is the *tensor unit object.* We have natural isomorphisms $\Bbbk \otimes V \cong V \cong V \otimes \Bbbk$ for any vector space V.

Lastly, the *flip* $\tau = \tau_{V,W} : V \otimes W \xrightarrow{\cong} W \otimes V$, $v \otimes w \mapsto w \otimes v$, is a natural isomorphism for any vector spaces V, W.

43.2 Algebras

An associative algebra is often defined as a vector space together with a bilinear map $A \times A \to A$, $(a, b) \mapsto ab$ and an element $1_A \in A$ satisfying some axioms. We reformulate this definition in a way that is purely expressed in terms of objects and morphisms in the category of vector spaces.

Definition 43.1. An *algebra* is a triple (A, m, u) where A is a vector space with linear maps

$$m = m_A : A \otimes A \longrightarrow A, \qquad u = u_A : \Bbbk \longrightarrow A$$

such that these diagrams commute:

$$\begin{tikzcd}
 & A \otimes A \otimes A \arrow[dl, "m\otimes 1"'] \arrow[dr, "1\otimes m"] & \\
A \otimes A \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\
 & A &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \otimes A \arrow[r, "u\otimes 1"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "m"] & A \otimes \Bbbk \arrow[l, "1\otimes u"'] \arrow[dl, "\cong"] \\
 & A &
\end{tikzcd}$$

The left diagram is *associativity* the right diagram is *unitality.*

An *algebra map* $f : A \to B$ is a linear map such that these diagrams commute:

$$\begin{tikzcd}
A \otimes A \arrow[r, "m_A"] \arrow[d, "f\otimes f"'] & A \arrow[d, "f"] \\
B \otimes B \arrow[r, "m_B"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[d, "f"] \\
\Bbbk \arrow[ur, "u_A"] \arrow[r, "u_B"'] & B
\end{tikzcd}$$

Example 43.2. $\Bbbk$ is an algebra with $m_\Bbbk$ the natural isomorphism $\Bbbk \otimes \Bbbk \to \Bbbk$ and $u_\Bbbk$ the identity map $\Bbbk \to \Bbbk$.

Example 43.3. If A and B are algebras then the vector space $A \otimes B$ can be equipped with an algebra structure with $m_{A\otimes B}$ and $u_{A\otimes B}$ the unique maps making these diagrams commute:

$$\begin{tikzcd} A \otimes B \otimes A \otimes B \arrow[r, "m_{A\otimes B}"] \arrow[d, "1\otimes\tau\otimes 1"'] & A \otimes B \\ A \otimes A \otimes B \otimes B \arrow[ur, "m_A\otimes m_B"'] & \end{tikzcd} \qquad \begin{tikzcd} \Bbbk \arrow[r, "u_{A\otimes B}"] \arrow[d, "\cong"'] & A \otimes B \\ \Bbbk \otimes \Bbbk \arrow[ur, "u_A\otimes u_B"'] & \end{tikzcd}$$

43.3 Coalgebras

In category theory any notion has a dual notion. For example the dual notion of products is coproducts. Therefore, now that we have formulated the notion of an algebra in categorical terms, we naturally obtain the definition of a *coalgebra*.

Definition 43.4. A *coalgebra* is a triple (C, Δ, ε) where C is a vector space and

$$\Delta : C \to C \otimes C, \qquad \varepsilon : C \to \Bbbk$$

are linear maps so that these diagrams commute:

$$\begin{tikzcd} & C \otimes C \otimes C & \\ C \otimes C \arrow[ur, "\Delta\otimes 1"] & & C \otimes C \arrow[ul, "1\otimes\Delta"'] \\ & C \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] & \end{tikzcd} \qquad \begin{tikzcd} \Bbbk \otimes C & C \otimes C \arrow[l, "\varepsilon\otimes 1"'] \arrow[r, "1\otimes\varepsilon"] & C \otimes \Bbbk \\ & C \arrow[ul, "\cong"] \arrow[u, "\Delta"'] \arrow[ur, "\cong"'] & \end{tikzcd}$$

A *coalgebra map* $f : C \to D$ is a linear map such that these diagrams commute:

$$\begin{tikzcd} C \otimes C \arrow[d, "f\otimes f"'] & C \arrow[l, "\Delta_C"'] \arrow[d, "f"] \\ D \otimes D & D \arrow[l, "\Delta_D"'] \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "\varepsilon_A"'] \arrow[dd, "f"] \\ \Bbbk & \\ & D \arrow[ul, "\varepsilon_D"] \end{tikzcd}$$

Example 43.5. Let A be a finite-dimensional algebra. Then the dual space A^* is a coalgebra with $\Delta = m^*$ and $\varepsilon = u^*$. (For this to make sense we must use that the natural injective map $A^* \otimes A^* \to (A \otimes A)^*$ that we always have is surjective in the case that A is finite-dimensional.)

43.4 Bialgebras

Definition 43.6. A *bialgebra* is a quintuple $(B, m, u, \Delta, \varepsilon)$ such that (B, m, u) is an algebra and (B, Δ, ε) is a coalgebra and one of the following equivalent conditions hold:

(i) Δ and ε are algebra maps,

(ii) m and u are coalgebra maps.

A *bialgebra map* $f : B \to C$ is a linear map which is both an algebra map and a coalgebra map.

Example 43.7. Let M be a finite monoid, and $B = \Bbbk^M$ be the set of all functions $\xi : M \to \Bbbk$. This is an algebra with respect to pointwise operations. To define the comultiplication, we note that $\Bbbk^M \otimes \Bbbk^M \cong \Bbbk^{M\times M}$. We define Δ by $\Delta(\xi)(m,n) = \xi(mn)$. The counit is given by $\varepsilon(\xi) = \xi(1_M)$.

Example 43.8. The universal enveloping algebra $U(\mathfrak{g})$ of any Lie algebra $\mathfrak{g}$ is not just an algebra but a bialgebra. We show there are algebra maps $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ and $\varepsilon : U(\mathfrak{g}) \to \Bbbk$ satisfying

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \qquad \varepsilon(x) = 0 \qquad \forall x \in \mathfrak{g}. \tag{43.1}$$

Define $\Delta' : \mathfrak{g} \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ by $\Delta'(x) = x \otimes 1 + 1 \otimes x$ for $x \in \mathfrak{g}$. This is a linear map, hence by the universal property of the tensor algebra, Δ' induces an algebra map $\widetilde{\Delta} : T(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$. One checks that $\widetilde{\Delta}(I) \subset I$, where I is the ideal defining $U(\mathfrak{g})$. Thus $\widetilde{\Delta}$ descends to an algebra map $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$. Similarly one proves the existence of the counit $\varepsilon : U(\mathfrak{g}) \to \Bbbk$.

Warning: The rule $\Delta(x) = x \otimes 1 + 1 \otimes x$ only holds for x in the Lie algebra $\mathfrak{g}$, not for all $x \in U(\mathfrak{g})$. For example, for $x, y \in \mathfrak{g}$, $\Delta(xy)$ is defined to be $\Delta(x)\Delta(y) = (x\otimes 1+1\otimes x)(y\otimes 1+1\otimes y) = (xy) \otimes 1 + x \otimes y + y \otimes x + 1 \otimes (xy)$. A similar remark applies to the counit.

43.5 Quantum groups

A quantum group is a certain bialgebra (in fact, Hopf algebra, see next lecture) that we associate to any Cartan datum.

Definition 43.9. Let $\mathbb{F} = \Bbbk(q)$ be the field of rational functions in an indeterminate q. Let $(A, \Pi, \Pi^\vee, P, P^\vee)$ be a Cartan datum where A is a symmetrizable GCM. The associated *(Drinfeld-Jimbo) quantum group*, denoted $U_q(\mathfrak{g})$ is the $\mathbb{F}$-algebra with generators $\{e_i, f_i\}_{i\in I}$ and $\{q^h\}_{h\in P^\vee}$ satisfying

$$q^0 = 1, \quad q^{h+h'} = q^h q^{h'}, \qquad \forall h, h' \in P^\vee, \tag{43.2}$$

$$q^h e_i q^{-h} = q^{\alpha(h)} e_i, \qquad \forall h \in P^\vee, \tag{43.3}$$

$$q^h f_i q^{-h} = q^{-\alpha(h)} f_i, \qquad \forall h \in P^\vee, \tag{43.4}$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \tag{43.5}$$

$$\sum_{n=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij}, \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0, \qquad \forall i \neq j, \tag{43.6}$$

$$\sum_{n=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij}, \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0, \qquad \forall i \neq j. \tag{43.7}$$

where $K_i = q^{b_i h_i}$ and $q_i = q^{b_i}$. The last two relations are the *quantum Serre relations*. The *q-binomial coefficients* are elements of $\mathbb{F}$ defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_1!}, \qquad [n]_q! = [n]_q \cdot [n-1]_1 \cdots [1]_q, \qquad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{43.8}$$

The comultiplication and counit on $U_q(\mathfrak{g})$ are given by

$$\Delta(q^h) = q^h \otimes q^h, \qquad \forall h \in P^\vee, \tag{43.9}$$
$$\Delta(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i, \tag{43.10}$$
$$\Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i, \tag{43.11}$$
$$\varepsilon(q^h) = 1, \qquad \forall h \in P^\vee, \tag{43.12}$$
$$\varepsilon(e_i) = 0, \tag{43.13}$$
$$\varepsilon(f_i) = 0. \tag{43.14}$$

44 Lecture 40: Hopf algebras contd.

44.1 Convolution Algebras

If C is a coalgebra and A is an algebra, then the space $\mathrm{Hom}_{\Bbbk}(C, A)$ becomes an associative algebra as follows. For $f, g \in \mathrm{Hom}_{\Bbbk}(C, A)$ we define $f * g$ to be the composition

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{m} A. \tag{44.1}$$

The identity element of $\mathrm{Hom}_{\Bbbk}(C, A)$ is the composition of the maps

$$C \xrightarrow{\varepsilon} \Bbbk \xrightarrow{u} A.$$

44.2 Hopf Algebras

Definition 44.1. A *Hopf algebra* H is a bialgebra $(H, m, u, \Delta, \varepsilon)$ such that the identity map $H \to H$ is invertible in the convolution algebra $\mathrm{End}_{\Bbbk}(H)$. That is, if there exists a linear map

$$S : H \longrightarrow H \tag{44.2}$$

such that this diagram commutes:

$$\begin{tikzcd}
 & H \arrow[dl, "\Delta"'] \arrow[dd, "\varepsilon"] \arrow[dr, "\Delta"] & \\
H \otimes H \arrow[dd, "S \otimes 1"'] & & H \otimes H \arrow[dd, "1 \otimes S"] \\
 & \Bbbk \arrow[dd, "u"] & \\
H \otimes H \arrow[dr, "m"'] & & H \otimes H \arrow[dl, "m"] \\
 & H &
\end{tikzcd}$$

(Note that this diagram is self-dual.) S is called the *antipode*.

Abusing notation one often denotes a Hopf algebra by $(H, m, u, \Delta, \varepsilon, S)$ even though S is not an additional piece of data, rather it is a property of the data already present.

44.3 Examples

1. Let G be a group and let $\Bbbk G$ be the group algebra defined as the vector space with basis G and multiplication extended bilinearly from the multiplication in G. This is a Hopf algebra with
$$\Delta(g) = g \otimes g, \qquad \varepsilon(g) = 1, \qquad S(g) = g^{-1}$$
for all $g \in G$. Note that these formulas have to be extended linearly to all of $\Bbbk G$. For example $\Delta(g + h) = \Delta(g) + \Delta(h) = g \otimes g + h \otimes h \neq (g + h) \otimes (g + h)$.

2. Let $\mathfrak{g}$ be a Lie algebra and $U(\mathfrak{g})$ its universal enveloping algebra. Then $U(\mathfrak{g})$ is a Hopf algebra with
$$\Delta(x) = x \otimes 1 + 1 \otimes x, \qquad \varepsilon(x) = 0, \qquad S(x) = -x$$
for all $x \in \mathfrak{g}$. This time these formulas have to be extended multiplicatively to all of $U(\mathfrak{g})$. For example $\Delta(xy) = \Delta(x)\Delta(y) = (x \otimes 1 + 1 \otimes x)(y \otimes 1 + 1 \otimes y) = xy \otimes 1 + x \otimes y + y \otimes x + 1 \otimes xy$.

3. Let G be a finite (resp. compact) group, and $\mathcal{F}(G)$ be the algebra of all (resp. continuous) functions from G to $\Bbbk$ (resp. $\mathbb{R}$). It is known that $\mathcal{F}(G \times G) \cong \mathcal{F}(G) \otimes \mathcal{F}(G)$. Thus we may define a comultiplication on $\mathcal{F}(G)$ by
$$\Delta(f)(g, h) = f(gh), \quad \forall f \in \mathcal{F}(G),\ \forall g, h \in G.$$
The counit and antipode are given by
$$\varepsilon(f) = f(1), \qquad S(f)(g) = f(g^{-1}).$$

44.4 Sweedler Notation

Let C be a coalgebra. For every $x \in C$ we have that $\Delta(x) \in C \otimes C$. Therefore there are $N_x \in \mathbb{Z}_{\geqslant 0}$ and elements $x^i_{(1)}, x^i_{(2)} \in C$ such that
$$\Delta(x) = \sum_{k=1}^{N_x} x^i_{(1)} \otimes x^i_{(2)}. \tag{44.3}$$
However, neither N_x nor the elements $x^i_{(j)}$ are unique, due to the bilinear nature of $C \otimes C$.

Example 44.2. Let $C = \Bbbk G$ be the group algebra of a group G and let $g \in G$. Then, by definition of the comultiplication,
$$\Delta(g) = g \otimes g$$
With this choice of writing it, we have $N_g = 1$ and $g^1_{(1)} = g^1_{(2)} = g$. However we may also write
$$\Delta(g) = (g - 1) \otimes g + 1 \otimes g$$
where $1 \in G$ is the identity element of G. With this choice, $N_g = 2$ and $g^1_{(1)} = g - 1, g^1_{(2)} = g, g^2_{(1)} = 1, g^2_{(2)} = g$.

Due to this non-uniqueness, we have to make sure that any map out of $C \otimes C$ is actually well-defined.

For example, the map
$$f : C \to C, \qquad f(x) = \sum_{k=1}^{N_x} x^k_{(1)} + x^k_{(2)}$$
is not well-defined, because the expression is not bilinear in $(x^k_{(1)}, x^k_{(2)})$. Concretely, the two different choices of writing $\Delta(g)$ in the example above would give different results (check!).

Similarly,
$$f : C \to C, \qquad f(x) = x^1_{(1)}\varepsilon(x^1_{(2)})$$
is not well-defined, because it only involves the two factors from the first term in $\Delta(x)$, which is not unique.

Sweedler notation uses more efficient notation, while at the same time making it harder to write down ill-defined expressions. Instead of (44.3) we write

$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} \tag{44.4}$$

With the superscripts suppressed, it is impossible to "select" any particular term, and that is a good thing because they are not well-defined anyway.

There is also a "summation-less"[xii] Sweedler notation, wherein we simply write

$$\Delta(x) = x_{(1)} \otimes x_{(2)} \tag{44.5}$$

The summation is implied, because of the subscripts (1) and (2) appearing.

Let us practice by writing out the relevant Hopf algebra axioms in summationless Sweedler notation:

Coassociativity reads

$$x_{(1)(1)} \otimes x_{(1)(2)} \otimes x_{(2)} = x_{(1)} \otimes x_{(2)(1)} \otimes x_{(2)(2)} \tag{44.6}$$

while counit axiom is

$$\varepsilon(x_{(1)})x_{(2)} = x = x_{(1)}\varepsilon(x_{(2)}) \tag{44.7}$$

and the antipode axiom becomes

$$S(x_{(1)})x_{(2)} = u\varepsilon(x) = x_{(1)}S(x_{(2)}). \tag{44.8}$$

In an algebra known to be associative, we don't need to use parenthesis when writing an iterated product such as xyz. The dual version of this is that with a coassociative comultiplication, we write each side of (44.6) simply as

$$x_{(1)} \otimes x_{(2)} \otimes x_{(3)}. \tag{44.9}$$

We may iterate this too and write

$$\Delta^n(x) = x_{(1)} \otimes x_{(2)} \otimes \cdots \otimes x_{(n+1)} \tag{44.10}$$

which means we have applied Δ n times to various tensor factors; which ones we chose doesn't matter by coassociativity.

44.5 Opposites and Co-opposites; A Property of the Antipode

Recall the flip $\tau : x \otimes y \mapsto y \otimes x$.

If A is an algebra, the *opposite* algebra, denoted A^{op} is (A, m^{op}, u) where $m^{\mathrm{op}} = m \circ \tau$.

Dually, if C is a coalgebra, the *co-opposite* coalgebra, denoted C^{cop} is $(C, \Delta^{\mathrm{op}}, \varepsilon)$ where $\Delta^{\mathrm{op}} = \tau \circ \Delta$.

Theorem 44.3. *Let H be a Hopf algebra. Then S is a bialgebra map $H^{\mathrm{op,cop}} \to H$. That is,*

(i) *$S(xy) = S(y)S(x)$ and $S \circ u = u$.*

(ii) *$S(x)_{(1)}S(x)_{(2)} = S(x_{(2)})S(x_{(1)})$ and $\varepsilon \circ S = \varepsilon$.*

[xii] analogous to Einstein's summation convention in tensor calculus

Proof. We only prove $S(xy) = S(y)S(x)$, leving the other statements as an exercise.

Define $\lambda, \rho \in \mathrm{Hom}_{\Bbbk}(H \otimes H, H)$ in a way that corresponds to the left and right hand sides:

$$\lambda(x \otimes y) = S(xy), \qquad \rho(x \otimes y) = S(y)S(x).$$

Here we regard $H \otimes H$ as a coalgebra and H as an algebra, thus $\mathrm{Hom}_{\Bbbk}(H \otimes H, H)$ is an associative algebra with respect to the convolution product. To show that $\lambda = \rho$ it suffices to show that

$$\lambda * m = u\varepsilon_{H\otimes H} = m * \rho$$

Indeed, then

$$\lambda = \lambda * u\varepsilon_{H\otimes H} = \lambda * m * \rho = u\varepsilon_{H\otimes H} * \rho = \rho$$

Using that

$$\Delta_{H\otimes H}(x \otimes y) = (1 \otimes \tau \otimes 1)\Delta(x) \otimes \Delta(y) = x_{(1)} \otimes y_{(1)} \otimes x_{(2)} \otimes y_{(2)}$$

we have by definition of the convolution product $*$,

$$\begin{aligned}
(\lambda * m)(x \otimes y) &= \lambda(x_{(1)} \otimes y_{(1)})m(x_{(2)} \otimes y_{(2)}) \\
&= S(x_{(1)}y_{(1)})x_{(2)}y_{(2)} \\
&= S((xy)_{(1)})(xy)_{(2)} \qquad \text{since } \Delta \text{ is an algebra map} \\
&= u\varepsilon(xy) \qquad \text{by antipode axiom for } H \\
&= u\varepsilon(x)\varepsilon(y) \\
&= u\varepsilon_{H\otimes H}(x \otimes y).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(m * \rho)(x \otimes y) &= m(x_{(1)} \otimes y_{(1)})\rho(x_{(2)} \otimes y_{(2)}) \\
&= x_{(1)}y_{(1)}S(y_{(2)})S(x_{(2)}) \\
&= x_{(1)}u\varepsilon(y)S(x_{(2)}) \qquad \text{by antipode axiom} \\
&= u\varepsilon(x)u\varepsilon(y) \qquad \text{since } u\varepsilon(y) \in \Bbbk \text{ and using antipode axiom} \\
&= u\varepsilon_{H\otimes H}(x \otimes y).
\end{aligned}$$

This finishes the proof. □

45 Lecture 41: Tensor Products of Modules

The following principle will be key: If $f : A \to B$ is an algebra map, and V is a B-module, then V becomes an A-module by

$$a.v = f(a).v$$

Next we make three observations about left modules over algebras:

(i) If V is an A-module and W is a B-module, then $V \otimes W$ is an $A \otimes B$-module via

$$(a \otimes b).(v \otimes w) = (a.v) \otimes (b.w)$$

(ii) $\Bbbk$ is obviously a $\Bbbk$-module.

(iii) If V is a left A-module then the dual space V^* is a left A^{op}-module by $(a.\xi)(v) = \xi(a.v)$.

As a direct corollary we get the following statement about modules over Hopf algebras.

Lemma 45.1. *Let H be a Hopf algebra.*

(i) *If V and W are H-modules (regarding H as just an algebra), then $V \otimes W$ is an H-module with action*

$$h.(v \otimes w) = (h_{(1)}.v) \otimes (h_{(2)}.w)$$

(ii) *The vector space $\nVdash = \mathfrak{k}$ is an H-module with action*

$$h.1 = \varepsilon(h)$$

(iii) *If V is an H-module then V^* is an H-module with action*

$$(h.\xi)(v) = \xi\big(S(h).v\big)$$

Proof. (i) Use that the comultiplication $\Delta : H \to H \otimes H$ is an algebra map together with the two observations above.

(ii) Use that the counit $\varepsilon : H \to \Bbbk$ is an algebra map.

(iii) Use that the antipode is an algebra map $S : H^{\mathrm{op}} \to H$ □

46 Lie Superalgebras

47 Reduction Algebras and Extremal Projectors

48 Gelfand-Tsetlin Bases for Representations of $\mathfrak{gl}_n$

49 Crystal Bases

50 Category $\mathcal{O}$

51 The Weyl Character Formula

Let $V = \oplus_{\lambda \in \mathfrak{h}^*} V_\lambda$. Then we define the following, $\operatorname{ch} V := \sum (\dim V_\lambda) e^\lambda \in \mathbb{Z}[\mathfrak{h}^*]$ with $e^\lambda e^\mu = e^{\lambda+\mu}$ and $e^0 = 1$.

Theorem 51.1.

1) $\operatorname{ch}(V \oplus W) = (\operatorname{ch} V) + (\operatorname{ch} W)$

2) $\operatorname{ch}(V \otimes W) = (\operatorname{ch} V)(\operatorname{ch} W)$

3) *If V, W are finite dimensional, then $V \cong W$ iff $\operatorname{ch} V = \operatorname{ch} W$.*

Theorem 51.2 (Weyl Character Formula)**.** *$\lambda \in P_+$ then*

$$\operatorname{ch}(L(\lambda)) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)}}{\prod_{\alpha \in R_+} (e^{\alpha/2} - e^{-\alpha/2})}$$

where $\rho = \dfrac{1}{2} \sum_{\alpha \in R_+} \alpha$

52 Appendix: A Brief Introduction to Category Theory

52.1 Classes

A *class* is like a set but can be bigger. Every set is a class but not all classes are sets. A class which is not a set is a *proper class*. Just like with sets we can form the cartesian product of classes, consider functions between classes and so on.

The main reason that we need classes is so that we can talk about things like *the class of all sets* because there is no set that contains all sets. (Likewise there is no class that contains all classes, but somehow we don't need to really worry about that!)

52.2 Partial binary operations

A *partial binary operation* $*$ on a class X is a function from some subclass of $X \times X$ to X. We write $* : X \times X \dashrightarrow X$ to indicate that the domain of $*$ may not be all of $X \times X$.

52.3 Definition of category

Definition 52.1. A *category* $\mathcal{C}$ is a quintuple $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, s, t, \circ)$ where

- $\mathcal{C}_0$ is a class whose elements are called the *objects* of $\mathcal{C}$,
- $\mathcal{C}_1$ is a class whose elements are called the *morphisms* of $\mathcal{C}$,
- $s : \mathcal{C}_1 \to \mathcal{C}_0$ is a map called the *source map*,
- $t : \mathcal{C}_1 \to \mathcal{C}_0$ is a map called the *target map*,
- $\circ : \mathcal{C}_1 \times \mathcal{C}_1 \dashrightarrow \mathcal{C}_1$ is a partial binary operation called *composition (of morphisms)* so that $\alpha \circ \beta$ is defined for any morphisms $\alpha, \beta \in \mathcal{C}_1$ with $t(\beta) = s(\alpha)$,

subject to the following two axioms:

(i) (identity) for every object $x \in \mathcal{C}_0$ there exists a morphism $1_x \in \mathcal{C}_1$ with

$$s(1_x) = t(1_x) = x$$

$$\alpha \circ 1_x = \alpha \quad \text{for all morphisms } \alpha \in \mathcal{C}_1 \text{ with } s(\alpha) = x$$

$$1_x \circ \beta = \beta \quad \text{for all morphisms } \beta \in \mathcal{C}_1 \text{ with } t(\beta) = x$$

(ii) (associativity) we have

$$(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$$

for any morphisms $\alpha, \beta, \gamma \in \mathcal{C}_1$ with $t(\gamma) = s(\beta)$ and $t(\beta) = s(\alpha)$.

Notation 52.2. $\mathcal{C}_0$ and $\mathcal{C}_1$ are sometimes denoted $\operatorname{Ob} \mathcal{C}$ and $\operatorname{Mor} \mathcal{C}$ respectively. You should think of the source and target maps as giving the domain and codomain of a morphism. In this spirit, if $\alpha \in \mathcal{C}_1$ is a morphism with $s(\alpha) = x$ and $t(\alpha) = y$ we write $\alpha : x \to y$.

52.4 Examples

To specify a category we have to say what the objects and morphisms are. The source, target and $\circ$ are almost always the domain, codomain and usual composition.

Example 52.3. 1) The category of sets and functions Set. This means that by definition Set_0 is the class of all sets, and Set_1 is the class of all functions between sets.

2) The category of abelian groups and group homomorphisms Ab.

3) For any ring R the category of left R-modules and R-module homomorphisms R-Mod.

4) If L and R are rings the category of (L, R)-bimodules and (L, R)-bimodule homomorphisms is denoted by L-Mod-R.

5) The category of topological spaces and continuous functions Top.

52.5 Functors

Just like a group homomorphism is a structure preserving function between groups, a functor is a structure preserving function between categories. Since a category has two underlying classes, a functor needs to be a pair of functions.

Definition 52.4. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *(covariant) functor* F from $\mathcal{C}$ to $\mathcal{D}$ is a pair of maps $F = (F_0, F_1)$ where $F_i : \mathcal{C}_i \to \mathcal{D}_i$ for $i = 0, 1$ such that

(i) $F_1(1_x) = 1_{F_0(x)}$ for all $x \in \mathcal{C}_0$

(ii) if $\alpha : x \to y$ then $F_1(\alpha) : F_0(x) \to F_0(y)$

(iii) $F_1(\alpha \circ \beta) = F_1(\alpha) \circ F_1(\beta)$ for all morphisms $\alpha, \beta \in \mathcal{C}_1$ with $t(\beta) = s(\alpha)$.

A *contravariant functor* F from $\mathcal{C}$ to $\mathcal{D}$ is the same things as a covariant functor except it reverses the direction of morphisms in the sense that (ii) and (iii) are replaced by

(ii') if $\alpha : x \to y$ then $F_1(\alpha) : F_0(y) \to F_0(x)$

(iii') $F_1(\alpha \circ \beta) = F_1(\beta) \circ F_1(\alpha)$ for all morphisms $\alpha, \beta \in \mathcal{C}_1$ with $t(\beta) = s(\alpha)$.

Notation 52.5. Usually we write Fx for $F_0(x)$ and $F\alpha$ for $F_1(\alpha)$ if no confusion can arise.

52.6 Examples

All examples will be a covariant functors. The following examples are related to the universal property of free R-modules (see next section).

Example 52.6. 1) The forgetful functor $\mathcal{O}_R : R\text{-}\mathsf{Mod} \to \mathsf{Set}$ (where $\mathcal{O}$ stands for oblivion) sends any left R-module M to the underlying set M, and any R-module homomorphism to itself (now regarded as just a function).

2) The free functor $\mathcal{F}_R : \mathsf{Set} \to R\text{-}\mathsf{Mod}$ sends any set X to the free left R-module on the set X, denoted $\mathcal{F}_R X$. And if $\alpha : X \to Y$ then $\mathcal{F}_R\alpha : \mathcal{F}_R X \to \mathcal{F}_R Y$ is the morphism induced by the composition $X \to Y \to \mathcal{F}_R Y$.

The next two examples are important in the context of tensor products (see next section on adjoint functors). Let L, S, R be rings with 1 and fix an (S, R)-bimodule B.

Example 52.7. 1)

$$- \otimes_S B : L\text{-}\mathsf{Mod}\text{-}S \to L\text{-}\mathsf{Mod}\text{-}R$$

is the functor that sends an (L, S)-bimodule A to the (L, R)-bimodule $A \otimes_S B$, and sends an (L, S)-bimodule morphism $\alpha : A \to A'$ to the (L, R)-bimodule morphism $\alpha \otimes 1_B : A \otimes_S B \to A' \otimes_S B$.

2) In the opposite direction we have the following functor:

$$\mathrm{Hom}_R(B, -) : L\text{-}\mathsf{Mod}\text{-}R \to L\text{-}\mathsf{Mod}\text{-}S$$

which sends an (L, R)-bimodule A to $\mathrm{Hom}_R(B, A)$, the set of right R-module maps $B \to A$. $\mathrm{Hom}_R(B, A)$ is an (L, S)-bimodule through

$$(\ell \cdot \varphi)(b) = \ell \cdot (\varphi(b)) \quad \ell \in L, b \in B, \varphi \in \mathrm{Hom}_R(B, A)$$

$$(\varphi \cdot s)(b) = \varphi(s \cdot b) \quad \forall s \in S, b \in B, \varphi \in \mathrm{Hom}_R(B, A)$$

On morphisms the functor $\mathrm{Hom}_R(B, -)$ takes $\alpha : A \to A'$ to the map $\widetilde{\alpha} : \mathrm{Hom}_R(B, A) \to \mathrm{Hom}_R(B, A')$ given by post-composition (push forward): $\widetilde{\alpha}(\varphi) = \alpha \circ \varphi$.

52.7 Pairs of adjoint functors

52.8 Definition

Definition 52.8. Given categories $\mathcal{C}$ and $\mathcal{D}$, and covariant functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ we say that F *is left adjoint to* G *and* G *is right adjoint to* F if there is a natural bijection

$$\mathrm{Hom}_{\mathcal{C}}(x, Gy) \xrightarrow{\eta_{x,y}} \mathrm{Hom}_{\mathcal{D}}(Fx, y)$$

for all $x \in \mathcal{C}_0$ and $y \in \mathcal{D}_0$. Here $\mathrm{Hom}_{\mathcal{C}}(a, b)$ denotes the class of morphisms in $\mathcal{C}$ from an object a to an object b. That the family $(\eta_{x,y})_{x \in \mathcal{C}_0, y \in \mathcal{D}_0}$ is "natural" means that whenever $\alpha : x \to x'$ and $\beta : y \to y'$ are morphisms in $\mathcal{C}_1$ and $\mathcal{D}_1$ respectively the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(x', Gy) \arrow[r, "\eta_{x',y}"] \arrow[d, "-\circ\alpha"] & \mathrm{Hom}_{\mathcal{D}}(Fx', y) \arrow[d, "-\circ F\alpha"] \\
\mathrm{Hom}_{\mathcal{C}}(x, Gy) \arrow[r, "\eta_{x,y}"] \arrow[d, "\mathcal{F}(\beta)\circ-"] & \mathrm{Hom}_{\mathcal{D}}(Fx, y) \arrow[d, "\beta\circ-"] \\
\mathrm{Hom}_{\mathcal{C}}(x, Gy') \arrow[r, "\eta_{x,y'}"] & \mathrm{Hom}_{\mathcal{D}}(Fx, y')
\end{tikzcd}$$

The commutativity of this diagram makes mathematically precise the vague statement that $\eta_{x,y}$ should be defined "the same way" regardless of the objects x and y. There is an analogous definition for contravariant functors.

52.9 Examples

Many universal properties can be expressed in terms of adjoint functors.

Example 52.9. 1) In example 52.6, the free functor $\mathcal{F}_R : \mathsf{Set} \to R\text{-}\mathsf{Mod}$ is left adjoint to the forgetful functor $\mathcal{O}_R : R\text{-}\mathsf{Mod} \to \mathsf{Set}$ because

$$\mathrm{Hom}_{\mathsf{Set}}(X, \mathcal{O}_R M) \cong \mathrm{Hom}_R(\mathcal{F}_R X, M)$$

for any set X and R-module M. In words, any set map $X \to M$ extends uniquely to an R-module morphism $\mathcal{F}_R X \to M$. The naturality is a tedious but straightforward exercise.

2) In Example 52.7, the functor $-\otimes_S B$ is left adjoint to $\mathrm{Hom}_R(B, -)$. Let ${}_L\mathrm{Hom}_R(X, Y)$ denote the set of (L, R)-bimodule homomorphisms between (L, R)-bimodules X and Y. Then what we are saying is that there is a natural bijection

$${}_L\mathrm{Hom}_S(A, \mathrm{Hom}_R(B, C)) \cong {}_L\mathrm{Hom}_R(A \otimes_S B, C).$$

Taking $L = R = \mathbb{Z}$ the left hand side can be identified with the set of S-balanced maps $A \times B \to C$, so this expresses precisely the universal property of the tensor product.