

# Math 620X Lecture Notes

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# Introduction

Geometry is a good way of dealing with infinite sets. Thus, to study infinite groups, we want to impose some kind of geometric structure on the underlying sets. There are many ways to do this: topological groups, algebraic groups, group schemes, etc. The nicest kind of geometric object, is that of a manifold. Their properties most closely resemble our everyday intuition about curves and surfaces.

A *Lie group* is a group which is also a manifold. Tangent spaces may be viewed as linear approximations of manifolds. The tangent space of a Lie group (at the identity element) can be given extra structure (coming from the group operation) making it into a *Lie algebra*. In other words, *Lie algebras are linear approximations of Lie groups*.

Therefore, to properly understand the origin of Lie algebras we must first understand something about Lie groups, and therefore we begin by studying manifolds.

## 1 Lecture 1

### 1.1 Manifolds

Reference: Spivak, *Calculus on Manifolds*, Ch. 5.

Throughout, we say that a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is *differentiable* (of class  $C^{\infty}$ ) if all of the partial derivatives of  $f$  exist to any order. A function  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $F(x) = (F_1(x), \dots, F_m(x))$  is differentiable if all the component functions  $F_i$  are. Another word for differentiable is *smooth*.

**Definition 1.1.** Let  $U, V \subseteq \mathbb{R}^n$  be open sets. Then a map  $f : U \rightarrow V$  is a *diffeomorphism* if  $f$  is differentiable and invertible, and  $f^{-1}$  is differentiable.

**Definition 1.2** (Manifold in  $\mathbb{R}^n$ ). A subset  $M$  of  $\mathbb{R}^n$  is a  $k$ -dimensional *manifold* if  $\forall x \in M$  the following condition holds:  $\exists$  open sets  $U_x, V_x \subseteq \mathbb{R}^n$  and a diffeomorphism  $h_x : U_x \rightarrow V_x$  such that  $x \in U_x$  and (see Figure 1)

$$h_x(U_x \cap M) = V_x \cap \{y \in \mathbb{R}^n \mid y_k = 0\}.$$

**Remark 1.3.** 1. As we have defined it, the manifold (given as a certain subset of  $\mathbb{R}^n$ ) does not depend on the choices of  $(U_x, V_x, h_x)$  at each  $x$ . Such triples just have to exist.

2. The restriction of  $h_x$  to  $U_x \cap M$  followed by projection  $\pi_k$  to the first  $k$  components provides each point in this portion of the manifold with *coordinates*:  $\phi_k = \pi_k \circ h_x \circ \text{id} : U_x \cap M \rightarrow \mathbb{R}^k$  for some  $a_k \in \mathbb{R}$ . We call the pair  $(U_x \cap M, \phi_k)$  a *coordinate chart*.

3. An abstract manifold is defined differently, (as a second countable topological space with a smooth atlas). However, the *Whitney Embedding Theorem* says that any abstract manifold of dimension  $m$  can be embedded into  $\mathbb{R}^{2m}$ . Thus the “concrete” definition given here is equivalent to the abstract one.

**Example 1.4.** Any open subset of  $\mathbb{R}^n$  is an  $n$ -dimensional manifold.

**Example 1.5.** Any singleton  $\{x\}$  is a zero-dimensional manifold.

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<sup>1</sup>“Lie” (pronounced *LEE*) refers to the Norwegian mathematician Sophus Lie, who himself called them *continuous groups*.

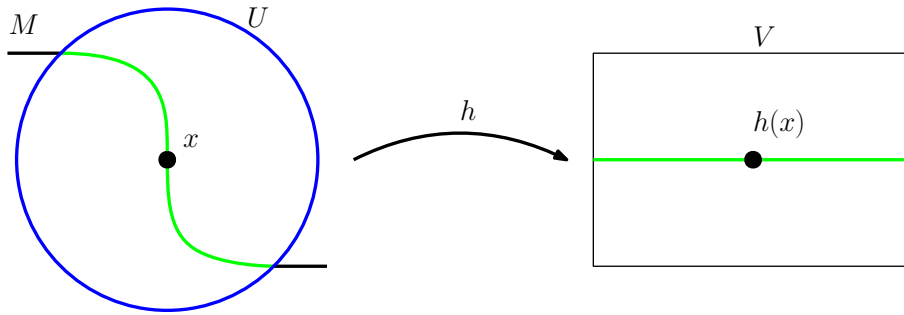


Figure 1: Manifold visualization

**Example 1.6.** Any linear subspace  $V$  of  $\mathbb{R}^n$  is a  $k$ -dimensional manifold, where  $k = \dim_{\mathbb{R}} V$ .

**Exercise 1.7.** Prove that if  $M$  is a  $k$ -dimensional manifold in  $\mathbb{R}^m$  and  $N$  is an  $l$ -dimensional manifold in  $\mathbb{R}^n$  then  $M \times N$  is naturally a  $(k + l)$ -dimensional manifold in  $\mathbb{R}^{m+n}$ .

How do we prove that something more interesting is a manifold?

**Theorem 1.8** (Implicit Function Theorem). *Let  $n \geq p \geq 0$  be integers. Let  $A \subseteq \mathbb{R}^n$  be an open subset and  $g: A \rightarrow \mathbb{R}^p$  be a differentiable function such that  $g^1(x) = \begin{pmatrix} g_1 \\ \vdots \\ g_p \end{pmatrix}$  has rank  $p$  whenever  $g(x) = 0$ . Then  $g^{-1}(0)$  is an  $(n - p)$ -dimensional manifold.*

For a vector  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  we write  $|x| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$  for the standard Euclidean norm.

**Example 1.9.** The  $n$ -sphere  $S^n = \{x \in \mathbb{R}^{n+1} \mid |x|^2 = 1\}$  is a manifold. To prove it, let  $g: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  defined by  $g(x) = |x|^2 - 1$ . Then  $g^1(x) = (2x_1, 2x_2, \dots, 2x_{n+1})$ . If  $g^1(x) = 0$  (the zero vector) then  $x_i = 0$  for all  $i$ , hence  $g(x) = -1$ . So whenever  $g(x) = 0$  we have  $g^1(x) \neq 0$  (thus, viewed as a  $1 \times (n+1)$ -matrix, has rank 1). By Theorem 1.8,  $g^{-1}(0) = S^n$  is an  $n$ -dimensional manifold.

**Exercise 1.10.** The *special linear group*, denoted  $SL_2(\mathbb{R})$ , is the set of all real  $2 \times 2$  matrices of determinant one. Show that  $SL_2(\mathbb{R})$  is a 3-dimensional manifold. (The set of all real  $2 \times 2$ -matrices can be identified with  $\mathbb{R}^4$ .) Can you generalize this to  $SL_3(\mathbb{R})$ ?  $SL_n(\mathbb{R})$ ?

**Definition 1.11.** A *morphism of manifolds* (or *differentiable map*, or *smooth map*)

$$f: M \rightarrow N,$$

where  $M$  and  $N$  are manifolds of dimension  $k$  and  $\ell$  respectively, is a function such that  $x \in M$ , the function (see Figure 2)

$$f_x = h_{f(x)} \circ f \circ h_x^{-1}: V_x \times \mathbb{R}^k \rightarrow V_{f(x)} \times \mathbb{R}^\ell$$

is differentiable. If furthermore  $f$  is invertible and its inverse is a morphism of manifolds, then  $f$  is an *isomorphism of manifolds* (or *homeomorphism*). When such an  $f$  exists,  $M$  and  $N$  are called *homeomorphic*.

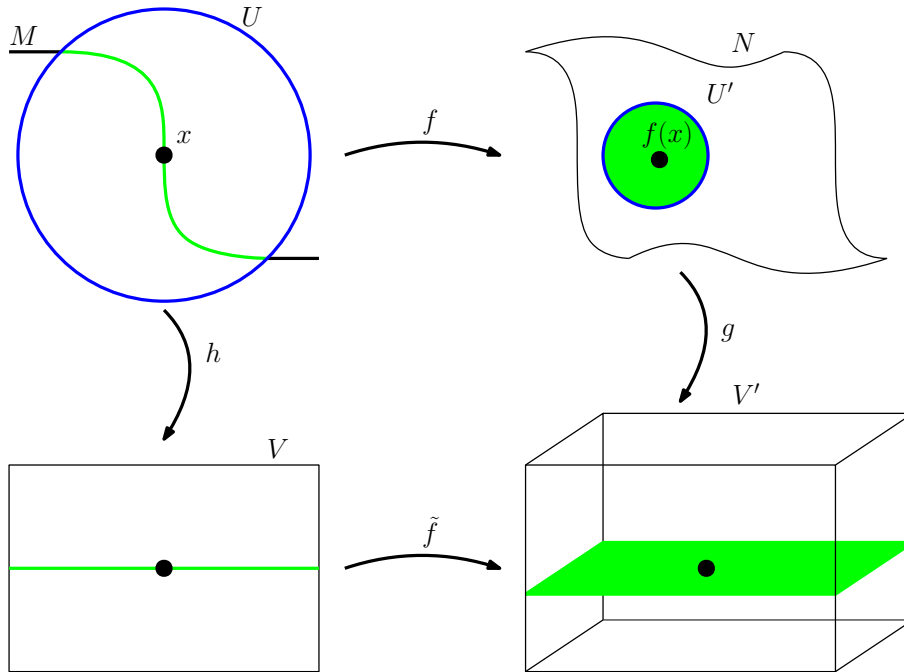


Figure 2: Morphism visualization

**Definition 1.12.** A Lie group  $G$  is a group which is also a manifold such that the maps,

$$G \times G \rightarrow G \text{ by } (g, h) \mapsto gh$$

and

$$G \rightarrow G \text{ by } g \mapsto g^{-1}$$

are differentiable maps (i.e morphisms of manifolds). A map  $\phi: G \rightarrow H$  is a *morphism of Lie groups* if it is a morphism of manifolds and a group homomorphism.

**Example 1.13.**  $\mathbb{R}^n$ ,  $\mathbb{Q}$  is a Lie group.

**Example 1.14.**  $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$  is a 1-dimensional (real) Lie group with respect to multiplication.

**Example 1.15.**  $GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\}$  is an open subset of  $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ , so it is a manifold. Matrix multiplication (respectively, matrix inverse) is given by polynomials (respectively, rational functions) of the entries and are therefore smooth maps. Thus  $GL(n, \mathbb{R})$  is a Lie group.

Why is it open?  $\det: \mathbb{R}^{n^2} \rightarrow \mathbb{R}$  is continuous which implies that  $\det^{-1}(\mathbb{R} \setminus \{0\})$  is open in  $\mathbb{R}^{n^2}$ .

**Example 1.16.**  $SU(2) = \{A \in M_2(\mathbb{C}) \mid A^* = -A, \det A = 1\}$  is a 3-dimensional Lie group diffeomorphic to  $S^3$  (see Example 2.5(5) in Kirillov, Jr.).

**Remark 1.17.** For the definition of *complex Lie Groups*, replace "differentiable" with "complex analytic". Unless otherwise emphasized, any theorem about Lie groups holds in both the real and complex cases.



## 2 Lecture 2

### 2.1 Connectedness

Let  $M$  be a manifold. Define a binary relation  $\sim$  on  $M$  by  $x, y \in M$ :

$$x \sim y \iff \exists \text{ continuous map } \gamma: [0, 1] \rightarrow M \text{ with } \gamma(0) = x, \gamma(1) = y$$



Figure 3: An example and nonexample

If  $x \sim y$  we say that  $x$  is connected to  $y$ .

**Exercise 2.1.** Prove  $\sim$  is an equivalence relation.

The equivalence classes

$$\{x \in M \mid y \sim x\}$$

are *connected components*. The set of equivalence classes of  $M$  is denoted by  $\pi_0(M)$ . If  $x \sim y$  for all  $x, y \in M$  we say that  $M$  is *connected*.

**Example 2.2.** The  $n$ -sphere  $S^n$  is a connected manifold for any  $n \neq 1$ .

*Proof.* Let  $x, y \in S^n, x \neq y$ . We will show  $x \sim y$ . By transitivity of connectedness, it suffices to do this in the case  $x, y \neq -x$ , that is, for non-antipodal pairs of points. For such  $x, y$  the straight line in  $\mathbb{R}^{n+1}$  through  $x$  and  $y$  does not pass through the origin. Define  $\gamma: [0, 1] \rightarrow S^n$  by

$$\gamma(t) = \frac{(1-t)x + ty}{\|(1-t)x + ty\|}.$$

The numerator defines a line segment in  $\mathbb{R}^{n+1}$ . Dividing by the norm forces  $\|\gamma(t)\| = 1$  so that  $\gamma(t) \in S^n$  for all  $t$  (see Figure 4). Then  $\gamma$  is a continuous function such that  $\gamma(0) = x$  and  $\gamma(1) = y$ . Thus  $x \sim y$ .  $\square$

The following simple fact can be used to show that a manifold is disconnected.

**Lemma 2.3.** Suppose  $f: M \rightarrow N$  is a surjective continuous map between manifolds. If  $M$  is connected, then  $N$  is connected. (Equivalently, if  $N$  is disconnected, then  $M$  must be disconnected.)

*Proof.*  $f \circ \gamma$  is a continuous map from  $[0, 1]$  to  $N$  connecting  $f(x)$  and  $f(y)$  whenever  $\gamma: [0, 1] \rightarrow M$  is a continuous map connecting  $x, y \in M$ .  $\square$

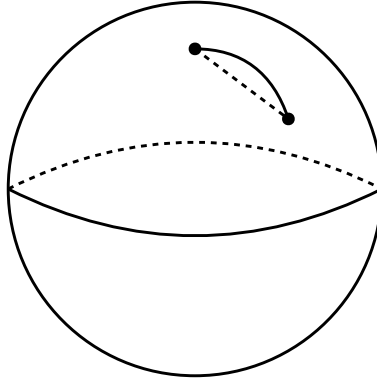


Figure 4: Visualization of  $\gamma$  used to prove  $S^n$  is connected

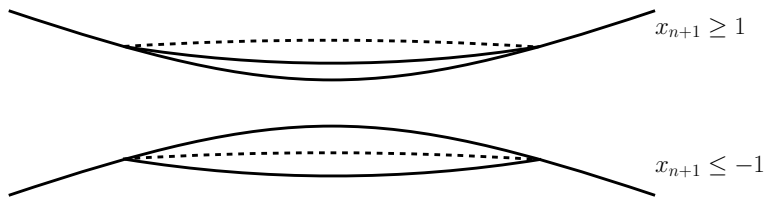


Figure 5: The  $n$ -dimensional hyperboloid  $H^n$

**Exercise 2.4.** The  $n$ -dimensional hyperboloid  $H^n = \{x \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_n^2 - x_{n+1}^2 = 1\}$  is an  $n$ -dimensional manifold by the implicit function theorem applied to the function  $g(x) = x_1^2 + \dots + x_n^2 - x_{n+1}^2 - 1$ . Prove that  $H^n$  has two connected components. (See Figure 5.) *Hint:* The projection map  $\mathbb{R}^{n+1} \rightarrow \mathbb{R}$  onto the last coordinate is a continuous map. Likewise, the projection map  $\mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$  onto the first  $n$  coordinates is continuous.

**Definition 2.5.** A subset  $N$  of a manifold  $M \subseteq \mathbb{R}^n$  is an *open submanifold* if there is an open subset  $U$  of  $\mathbb{R}^n$  such that  $N = U \cap M$ .

**Proposition 2.6.** Let  $M \subseteq \mathbb{R}^n$  be a manifold and  $x \in M$ . Then the connected component  $\text{rc}_x$  is an open submanifold.

*Proof.* For any  $y \in \text{rc}_x$  pick an open set  $U_y \subseteq \mathbb{R}^n$  as in the definition of a manifold. Then any  $z \in M \cap U_y$  is connected to  $y$ , hence to  $x$  (see Figure 6). So  $M \cap U_y \subseteq \text{rc}_x$ . Let  $U = \bigcup_{y \in \text{rc}_x} U_y$ . Then  $U$  is open in  $\mathbb{R}^n$  and  $U \cap M = \text{rc}_x$ .  $\square$

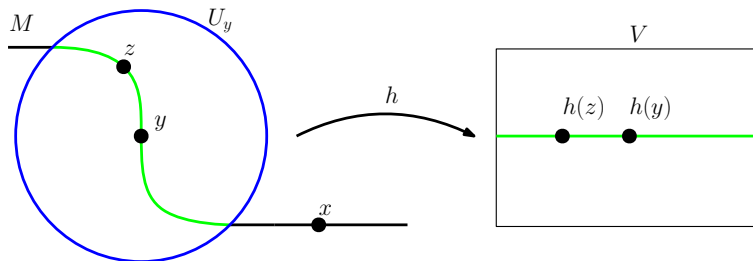


Figure 6: Visual justification that for any  $z \in M \cap U_y$ ,  $z \in \text{rc}_x$

## 2.2 Connected Lie Groups

**Definition 2.7.** A Lie group  $G$  is *connected* if it is connected as a manifold.

**Notation 2.8.** If  $G$  is a Lie group then  $G^0$  denotes the connected component of  $G$  that contains the identity element.  $G^0 = \tau e_G$ .

**Remark 2.9.** By Proposition 2.6 above,  $G^0$  is an open submanifold of  $G$ .

**Example 2.10.** Any finite group  $G$  can be viewed as a 0-dimensional Lie group by placing its elements on the real line:

$$e$$

In this case the connected component  $G^0 = \tau e$  only contains the identity element.

**Example 2.11.** The *orthogonal group*  $Opn, Rq$  is the group of orthogonal matrices:

$$Opn, Rq = \{A \in M_{np}Rq \mid A^T A = I = AA^T\}.$$

One can show, using the implicit function theorem, that  $Opn, Rq$  is a manifold of dimension  $npn - 1/2$ . Since matrix multiplication and inverse are smooth,  $Opn, Rq$  is a Lie group. This group has two connected components consisting of orientation-preserving and orientation-reversing transformations respectively. In more detail, the determinant map  $\det : Opn, Rq \rightarrow \tau \pm 1$  sends an orthogonal matrix  $A$  to  $\det A$  which is  $\pm 1$  since  $1 = \det I = \det AA^T = \det A \det A^T = \det A^2$ . Since  $\det$  is surjective and continuous and  $\tau \pm 1$  is disconnected,  $Opn, Rq$  must have at least two connected components by Lemma 2.3. Moreover  $Opn, Rq^0 = \{A \in Opn, Rq \mid \det A = 1\} = SOpn, Rq$  which is the *special orthogonal group*.

In fact,  $SOpn, Rq$  is connected, hence  $Opn, Rq^0 = SOpn, Rq$ . Furthermore, the matrix  $g = \text{diag}(1, 1, 1, \dots, 1)$  is orthogonal of determinant  $-1$ . Multiplication by  $g$  provides a diffeomorphism between the two connected components.

**Theorem 2.12.** Let  $G$  be a Lie group. Then  $G^0$  is a normal subgroup and is itself a Lie group. The quotient group  $G/G^0$  is discrete, meaning each coset  $gG^0$  is an open submanifold of  $G$ .

*Proof.*  $e \in G^0$  by definition. If  $f : M \rightarrow N$  is continuous then  $x \sim y$  in  $M$  iff  $f(x) = f(y)$  in  $N$  (Exercise). In particular,  $f(x) = f(y)$  implies  $f(x) = f(y)$ . Apply to  $i : G \rightarrow G$ ,  $i(g) = g^{-1}$  gives  $i(G^0) \subseteq G^0$ . Similarly,  $m : G \rightarrow G$ ,  $m(g) = gh^{-1}$  gives  $m(G^0) \subseteq G^0$ . Lastly, fix  $g \in G$ ,  $g \sim ghg^{-1}$ . Then  $gG^0 = ghg^{-1}G^0$ . Thus  $G^0$  is a normal subgroup. Since  $gG^0 = \tau g$  each coset in  $G/G^0$  is an open submanifold by Proposition 2.6, so  $G/G^0$  is discrete.  $\square$

**Example 2.13.** Let  $G = Opn, Rq$ . Then the determinant map  $\det : G \rightarrow \tau \pm 1$  is surjective with kernel equal to  $G^0$ . Thus  $G/G^0 = \tau \pm 1$ .

## 3 Lecture 3

### 3.1 Simple Connectedness

Let  $M$  be a connected manifold and fix  $x_0 \in M$ , called a *base point*.

**Definition 3.1.** A *path* in  $M$  is a continuous map  $\gamma : [0, 1] \rightarrow M$  such that  $\gamma(0) = x_0$ .  $\gamma$  is a *loop* if  $\gamma(0) = \gamma(1)$ . The *constant loop*  $\gamma_0$  is given by  $\gamma_0(t) = x_0$  for  $t \in [0, 1]$ .

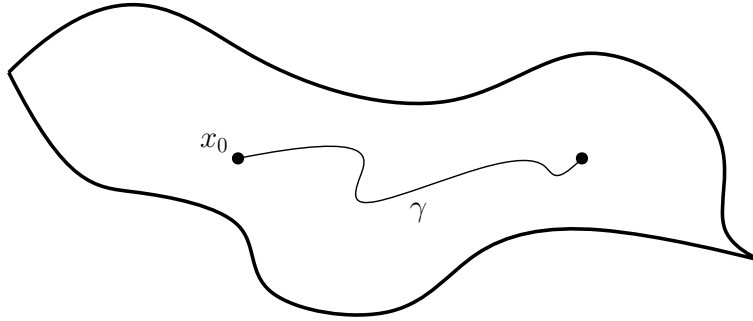


Figure 7: Example of a path  $\gamma$  with base point  $x_0$

Two paths are homotopic if one can be continuously deformed into the other. The precise definition is as follows.

**Definition 3.2.** Let  $x_0$  and  $x_1$  be points in  $M$ . Two paths  $\gamma, \delta$  in  $M$  with  $\gamma(0) = \delta(0) = x_0$  and  $\gamma(1) = \delta(1) = x_1$  are *homotopic* if  $\exists$  continuous map  $h: [0, 1]^2 \rightarrow M$  such that

$$h(0, s) = x_0 \text{ and } h(1, s) = x_1 \text{ for all } s \in [0, 1],$$

and

$$h(t, 0) = \gamma(t) \text{ and } h(t, 1) = \delta(t) \text{ for all } t \in [0, 1].$$

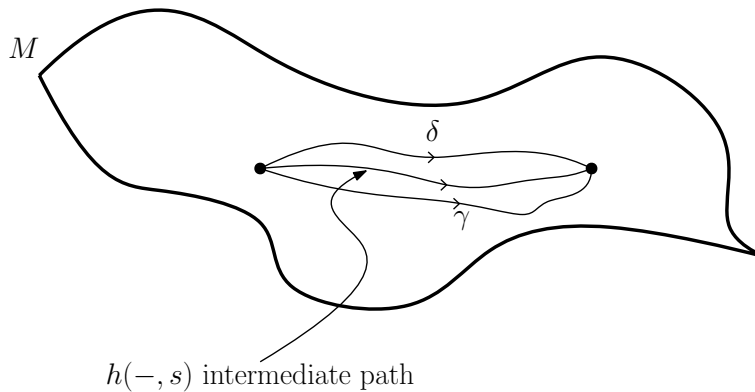


Figure 8: Example of homotopic paths

**Definition 3.3.**  $M$  is *simply connected* if every loop in  $M$  is homotopic to the constant loop. (Note: this is independent of the choice of  $x_0$ )

**Example 3.4.** In figure 9 we see that  $\mathbb{R}^2$  is a simply connected manifold, while  $S^1$  is not.

**Example 3.5.** The *projective plane*  $\mathbb{R}P^2$  (in the Poincaré model) is  $D^2$  but opposite points on  $S^1$  identified:  $D^2/\sim$  where  $x \sim y$  iff  $|x| = |y| = 1$  &  $x = -y$ . Then  $\mathbb{R}P^2$  is not simply connected. See figures 10a and 10b.

**Definition 3.6.** The *product* of two loops  $\gamma, \delta$  in  $M$  is  $\gamma \# \delta: [0, 1] \rightarrow M$

$$\gamma \# \delta(t) = \begin{cases} \gamma(2t) & 0 \leq t \leq \frac{1}{2} \\ \delta(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

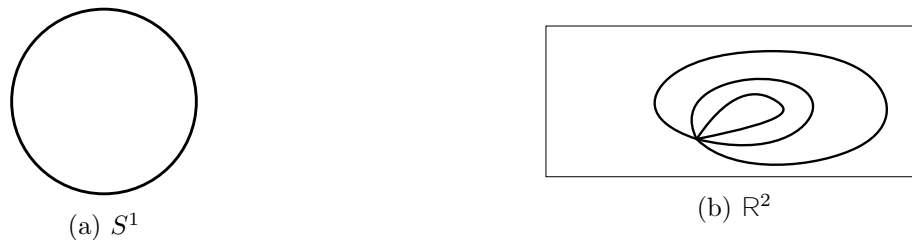
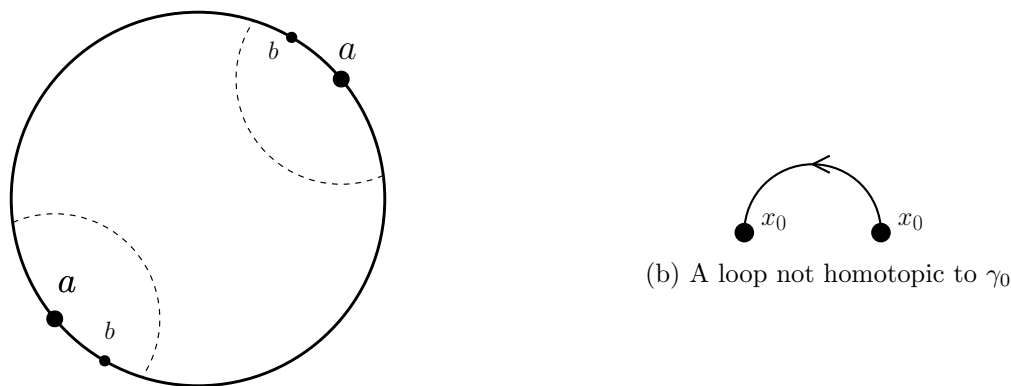


Figure 9: An example and non-example of simply connected manifolds



(a) Depiction of  $\mathbb{RP}^2$  with identical points labeled

(b) A loop not homotopic to  $\gamma_0$

Figure 10

**Exercise 3.7.** Homotopy defines an equivalence relation on the set of loops in  $M$ . The set of equivalence classes is denoted  $\pi_1(M, x_0)$ .

**Exercise 3.8.**  $\pi_1(M, x_0)$  is a group with respect to the operation:

$$(\gamma \circ \delta) \circ \epsilon = \gamma \circ (\delta \circ \epsilon).$$

$\pi_1(M, x_0)$  is the (1st) *fundamental group* of  $M$  (a.k.a. *Poincare group* of  $M$ ).

**Exercise 3.9.**  $\pi_1(M, x_0) = \pi_1(M, y_0)$  for any  $x_0, y_0 \in M$  (Recall: we assume  $M$  connected).

**Example 3.10.**  $\pi_1(\mathbb{P}^2) \cong \mathbb{Z}/2\mathbb{Z}$ , notice  $\gamma \circ \gamma^2 = \gamma_0$  in figure 11

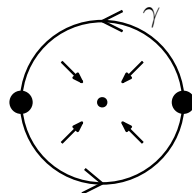


Figure 11

**Example 3.11.**  $\pi_1(S^1) \cong \mathbb{Z}$ . The correspondence is the winding number.

### 3.2 The Universal Cover

Some convenient terminology:

- A subset  $U \in M$  of a manifold  $M$  in  $\mathbb{R}^n$  is *open in  $M$* , or simply *open* (when no confusion can arise), if  $U = M \times A$  for some open set  $A \in \mathbb{R}^n$ .
- A *neighborhood (abbreviated nbh) of  $x$*  is a set containing  $x$ .

**Definition 3.12.** Let  $M$  be a connected manifold. A *cover*  $(\tilde{M}, p)$  (covering space) for  $M$  is a connected manifold  $\tilde{M}$  together with a morphism  $p: \tilde{M} \rightarrow M$  such that:  $\forall x \in M$ ,  $\exists$  connected open neighborhood  $U \in M$  of  $x$  such that every connected component of  $p^{-1}(U)$  diffeomorphically onto  $U$ .  $(\tilde{M}, p)$  is a *universal cover* if it is simply connected. Often we just write  $\tilde{M}$  for  $(\tilde{M}, p)$ .

**Example 3.13.**  $(\mathbb{R}, p)$  is a universal cover for  $S^1$  where  $p: x \mapsto e^{ix}$ .

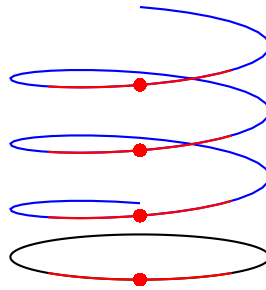


Figure 12: Visualization of  $(\mathbb{R}, p)$  as covering space over  $S^1$

**Theorem 3.14.** *Every connected manifold has a universal cover. Moreover, it is unique up to diffeomorphism.*

*Proof (sketch).* Pick a base point  $x_0 \in M$ . Define  $\tilde{M}$  to be the set of homotopy classes of paths in  $M$  starting at  $x_0$  (see Figure 13). Define  $p: \tilde{M} \rightarrow M$ ,  $p[\gamma] = \gamma(1)$ . One can show that this is a universal cover.  $\square$

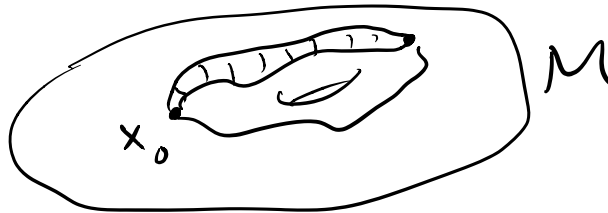


Figure 13: Visualization of Homotopy Classes

**Theorem 3.15.** *Any morphism of connected manifolds  $f: M \rightarrow N$  can be lifted to a morphism of their respective universal covers  $\tilde{f}: \tilde{M} \rightarrow \tilde{N}$ .*

**Theorem 3.16.** *If  $G$  is a connected Lie group, then its universal cover  $\tilde{G}$  has a canonical structure of a Lie group such that*

i)  $p: \mathbb{C} \rightarrow \mathbb{N} G$  is a morphism of Lie groups

ii)  $\ker p = \pi_1(pG, eq)$ .

Moreover,  $\ker p$  is a discrete subgroup of  $\mathbb{C}$ , and  $\ker p \in Z(p\mathbb{C})$  the center of  $\mathbb{C}$ .

**Example 3.17.**  $G = S^1 \times S^1 \times \mathbb{Z}$ . The connected component at the identity element is  $G^0 = S^1 \times S^1$ . The universal cover of  $G^0$  is  $\mathbb{R}^2$ . See Figure 14.

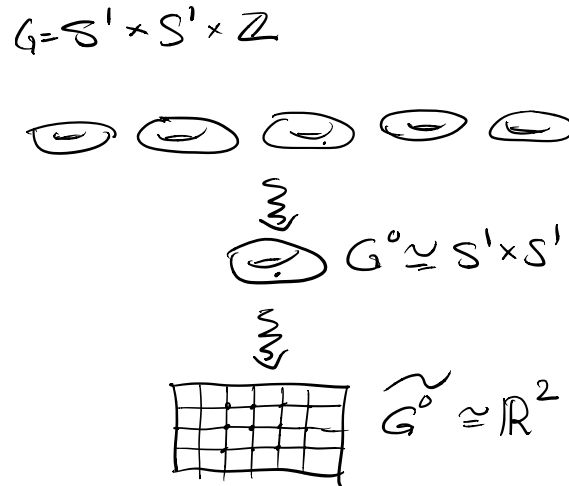


Figure 14:  $G = S^1 \times S^1 \times \mathbb{Z}$

## 4 Lecture 4

### 4.1 Coordinate Systems

A coordinate system is a local parametrization of a manifold. Coordinate systems are used to construct tangent spaces.

Let  $M$  be a manifold and  $x \in M$ . Let  $(U_x, V_x, h_x)$  be a triple as in the definition of a manifold. Let  $W_x$  be the image of the projection of  $V_x \times \mathbb{R}^k$  to the first  $k$  components.

**Definition 4.1.** The map  $f_x: W_x \rightarrow M$  given by  $f_x(a_1, a_2, \dots, a_k) = h_x(a_1, \dots, a_k, 0, \dots, 0)$  (with  $n - k$  trailing zeros) is called a *coordinate system* around  $x$ .

Since  $h_x$  is a diffeomorphism, the derivative  $h_{x,y}$  has rank  $n$  at all  $y \in V_x$ . Consequently  $f_x$  has rank  $k$  at any point in  $W_x$ . Furthermore  $f_x^{-1}: F_x(W_x) \rightarrow W_x$  is continuous. The following theorem shows that the data  $(U_x, V_x, h_x)$  can in fact be replaced by the data  $(W_x, f_x)$ .

**Theorem 4.2** (See Spivak, Theorem 5-2). A subset  $M$  of  $\mathbb{R}^n$  is a manifold if  $\forall x \in M$  there is an open set  $U \subseteq \mathbb{R}^n$ ,  $x \in U$ , and an open set  $W \subseteq \mathbb{R}^k$  and an injective differentiable function  $f: W \rightarrow \mathbb{R}^n$  such that

- 1)  $f(W) = M \cap U$
- 2)  $f|_W$  has rank  $k$   $\forall y \in W$
- 3)  $f^{-1}: f(W) \rightarrow W$  is continuous.

## 4.2 Tangent Space

**Definition 4.3.** Let  $M$   $k$ -dimensional manifold,  $x \in M$ ,  $f: W \rightarrow \mathbb{R}^n$  be a coordinate system around  $x$ , and  $a = f^{-1}(x)$ . Since  $f|_W$  has rank  $k$ , the image of  $f|_W: \mathbb{R}^k \rightarrow \mathbb{R}^n$  given by the matrix

$$\begin{matrix} B_1 f|_W & B_2 f|_W & \dots & B_k f|_W \\ \vdots & \vdots & & \vdots \\ B_1 f|_W & B_2 f|_W & \dots & B_k f|_W \end{matrix}$$

is a  $k$ -linear subspace of  $\mathbb{R}^n$  called the *tangent space of  $M$  at the point  $x$* , denoted  $T_x M$ .

Often we draw/think of  $T_x M$  as the affine space  $x + f|_W \mathbb{R}^k$ .

**Note 4.4.** By the chain rule,  $T_x M$  is independent of the choice of coordinate system  $f$ .

**Example 4.5.** Let us describe the tangent space of  $S^2$  at  $(0, 0, 1)$ . Let  $W = \{(s, t) \mid s^2 + t^2 = 1\}$  and  $f: W \rightarrow \mathbb{R}^3$ ,  $f(s, t) = (s, t, \sqrt{1 - s^2 - t^2})$ . Then  $f$  is a coordinate system around  $x = (0, 0, 1)$ . Let  $a = (0, 0)$  (notice  $f(a) = x$ ).

$$f|_W = \begin{matrix} B_s f|_W & B_t f|_W \\ B_s f|_W & B_t f|_W \\ B_s f|_W & B_t f|_W \end{matrix} = \begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix}$$

so  $T_x S^2 = f|_W \mathbb{R}^2 = \mathbb{R} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mathbb{R} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ .

**Example 4.6.** (See Figure 15.) A coordinate system for the 2-dimensional torus  $M$  embedded in  $\mathbb{R}^3$  is given by (here  $R_i, r_i > 0$  are the two radii):

$$f(s, t) = (R_1 \cos s \cos t, R_1 \cos s \sin t, R_1 \sin s), \quad (s, t) \in [0, 2\pi) \times [0, 2\pi)$$

This is a coordinate system around any point except for the point  $f(\pi, \pi) = (-R_1, 0, 0)$ . To get a coordinate system around that point one can use the same expression for  $f$  but change the domain to  $[0, 2\pi)$ . We can use this coordinate system to find the tangent space for a point  $p = f(s, t)$ . As an example, take  $s = \pi/3$  and  $t = \pi/4$ . The derivative (matrix) of  $f$  is

$$f|_W = \begin{matrix} B_s f|_W & B_t f|_W \\ B_s f|_W & B_t f|_W \\ B_s f|_W & B_t f|_W \end{matrix} = \begin{matrix} -R_1 \sin s \cos t & -R_1 \cos s \sin t \\ -R_1 \sin s \sin t & R_1 \cos s \cos t \\ R_1 \cos s & 0 \end{matrix}$$

Thus the tangent space  $T_p M$  at  $p = f(\pi/3, \pi/4) = (R_1 \sqrt{3}/2, R_1 \sqrt{2}/2, R_1 \sqrt{3}/2)$  equals

$$T_p M = \mathbb{R} \begin{pmatrix} -R_1 \sqrt{3}/2 \\ -R_1 \sqrt{2}/2 \\ R_1 \sqrt{3}/2 \end{pmatrix} + \mathbb{R} \begin{pmatrix} -R_1 \sqrt{2}/2 \\ R_1 \sqrt{3}/2 \\ 0 \end{pmatrix}$$



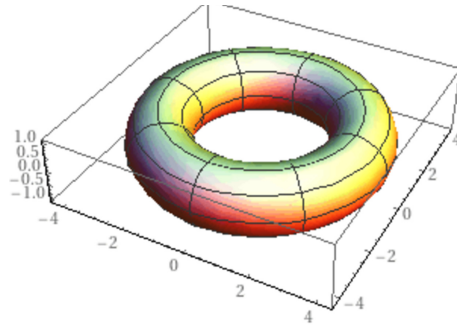


Figure 15: Torus with  $R = 3$  and  $r = 1$ .

### 4.3 Vector Fields

**Definition 4.7.** A *vector field* on  $M$  is a map  $F: M \rightarrow \mathbb{R}^n$  such that  $F(x) \in T_x M$  for all  $x \in M$ .  $F$  is called *differentiable* if for every coordinate system  $f: W \rightarrow \mathbb{R}^n$  at any  $x \in M$ , a  $\tilde{F}: f^{-1}(x) \rightarrow \mathbb{R}^n$  is a differentiable vector field on  $W$ .

### 4.4 The Derivative (Differential) of a Morphism

If  $f: M \rightarrow N$  is a morphism of manifolds there is an induced linear map  $df_x = f_{*p}: T_p M \rightarrow T_p N$  given as follows

$$\begin{aligned} \alpha: U \rightarrow \mathbb{R}^n & \quad \alpha(p) = p \\ \beta: V \rightarrow \mathbb{R}^n & \quad \beta(p) = f(p) \\ T_p M & \quad \alpha^{-1}(p) \cong \mathbb{R}^n \\ T_{f(p)} N & \quad \beta^{-1}(p) \cong \mathbb{R}^n \end{aligned}$$

Define  $f_{*p}$  by the formula

$$f_{*p}^{-1} \circ f_{*} \circ \alpha^{-1}(p) = \beta^{-1}(p) \circ f_{*p} \circ \alpha^{-1}(p).$$

### 4.5 Vector Fields on Lie Groups

Fix  $g \in G$ . There are three maps  $G \rightarrow G$ :

$$\begin{aligned} L_g: h & \rightarrow gh \\ R_g: h & \rightarrow hg^{-1} \\ \text{Ad}_g: H & \rightarrow ghg^{-1} \end{aligned}$$

They are morphisms of manifolds  $\tilde{f}$  have differentials.

**Example 4.8.**  $dL_g: T_1 G \rightarrow T_g G$ ,  $x \in T_1 G$

**Definition 4.9.** A vector field  $F$  on a Lie group  $G$  is *left-invariant* if  $g \cdot F(x) = F(gx)$  for all  $g, x \in G$ .

---

<sup>ii</sup>this mapping is single-valued

**Definition 4.10.** The *Lie algebra* of a Lie group  $G$  is defined by

$$\mathfrak{g} = \text{Lie } G = \text{all left-invariant v.fields on } G \mathfrak{u}.$$

**Theorem 4.11.**  $v \in \mathfrak{g}$  is a linear isomorphism of  $\mathfrak{g}$  with  $T_1 \mathfrak{p}G \mathfrak{q}$ .

*Proof.* Let  $x \in T_1 \mathfrak{p}G \mathfrak{q}$ . Define  $v \mathfrak{p}g \mathfrak{q} = g \cdot x$ . Then  $v$  is left invariant. Uniqueness is obvious.  $\square$

## 5 Lecture 5

### 5.1 Classical groups

The following is an assortment of Lie groups called *classical groups*. These are the most important but not an exhaustive list; there are many more. (Once we classify simple complex Lie algebras we can give a general definition of classical Lie group: A connected Lie group is *classical* if its complexified Lie algebra is a classical Lie algebra.)

- The *general linear group*:

$$GL(n, \mathbb{R}) = \text{all invertible } n \times n \text{ matrices with real entries} \mathfrak{u},$$

- The *special linear group*:

$$SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid \det A = 1\} \mathfrak{u}.$$

- The *orthogonal group*:

$$\begin{aligned} O(n) &= \{A \in GL(n, \mathbb{R}) \mid A^T A = A A^T = I_n\} \mathfrak{u} \\ &= \{A \in GL(n, \mathbb{R}) \mid \langle Ax, Ay \rangle = \langle x, y \rangle \ \forall x, y \in \mathbb{R}^n\} \mathfrak{u}, \end{aligned}$$

where  $\langle \cdot, \cdot \rangle$  is the standard dot product: the unique (up to change of basis) positive definite, non-degenerate, symmetric, bilinear form on  $\mathbb{R}^n$ .

- The *special orthogonal group*:

$$SO(n) = O(n) \cap SL(n, \mathbb{R}).$$

- The *orthogonal group of signature*  $(p, q)$ :

$$O(p, q) = \{A \in GL(p+q, \mathbb{R}) \mid \langle Ax, Ay \rangle_{p,q} = \langle x, y \rangle_{p,q}\} \mathfrak{u},$$

where  $\langle \cdot, \cdot \rangle_{p,q}$  is the (unique up to change of basis) nondegenerate, symmetric, bilinear form on  $\mathbb{R}^{p+q}$  of signature  $(p, q)$ :  $\langle x, y \rangle_{p,q} = \sum_{i=1}^p x_i y_i - \sum_{i=p+1}^{p+q} x_i y_i$ .

- The corresponding “special” version is as expected:

$$SO(p, q) = O(p, q) \cap SL(p+q, \mathbb{R}).$$

- The *symplectic group*:

$$Sp(2n, \mathbb{R}) = \{A \in GL(2n, \mathbb{R}) \mid \omega(Ax, Ay) = \omega(x, y) \forall x, y \in \mathbb{R}^{2n}\}$$

$$= \{A \in GL(2n, \mathbb{R}) \mid A^T J A = J\}$$

where  $\omega$  is the (unique up to change of basis) non-degenerate, skew-symmetric, bilinear form on  $\mathbb{R}^{2n}$  given by  $\omega(x, y) = \sum_{i=1}^n (x_i y_{i+n} - y_i x_{i+n})$ , and  $J$  is the Gram matrix of the form given by  $2n \times 2n$ -matrix  $\begin{pmatrix} 0_n & I_n \\ I_n & 0_n \end{pmatrix}$ .

- The *unitary group*:

$$U(n) = \{A \in GL(n, \mathbb{C}) \mid A^* = -A, AA^* = I_n\}$$

where  $A^* = \overline{A^T}$  hermitian adjoint (conjugate-transpose).

- The *special unitary group*:

$$SU(n) = U(n) \cap SL(n, \mathbb{C})$$

**Exercise 5.1.** Special symplectic group? Show that  $Sp(2n, \mathbb{R}) \in SL(2n, \mathbb{R})$ , so it is already "special".

## 5.2 The Exponential Map

How can one prove that the above matrix groups are actually Lie groups? There are three methods.

Method 1: Implicit function theorem (see Theorem 1.8). For example,  $O(n) = \{A \in GL(n, \mathbb{R}) \mid AA^T = -I_n\}$  is defined by  $n(n-1)/2$  equations in  $n^2$  (since  $AA^T$  is symmetric). Compute Jacobian of this system and show it has full rank (i.e. rank  $n(n-1)/2$ ). This can be computationally difficult.

Method 2: Observe that  $O(n)$  forms a closed subset of  $GL(n, \mathbb{R})$  and use a theorem about closed Lie subgroups (next time).

Method 3: Using the so called *the exponential map* as a coordinate system around the identity element of  $G$ .

To define this exponential map, first recall that  $GL(n, \mathbb{R})$  is an open subset of  $\mathbb{R}^{n^2}$ . Hence the identity map  $GL(n, \mathbb{R}) \rightarrow \mathbb{R}^{n^2}$  is a coordinate system around the identity map. Its derivative is the identity matrix. Therefore the Lie algebra of  $GL(n, \mathbb{R})$ , identified with the tangent space at the identity, is all of  $\mathbb{R}^{n^2}$ . In other words,

$$\mathfrak{gl}(n, \mathbb{R}) : \text{Lie algebra of } GL(n, \mathbb{R}) = T_1 GL(n, \mathbb{R}) = \text{all } n \times n \text{ matrices}$$

The power series

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges absolutely in matrix norm<sup>iii</sup> for any  $n \times n$  matrix  $x$ . So it defines a smooth (in fact, analytic) map of manifolds:

$$\exp: \mathfrak{gl}(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R})$$

<sup>iii</sup>The matrix norm is  $\|x\| = \sup_{\|v\|=1} \|xv\|$

Furthermore,

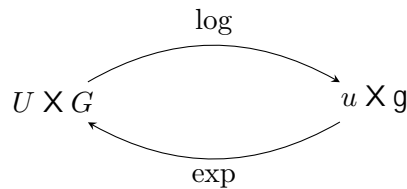
$$\log(1 + x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$$

is smooth (in fact analytic) near  $1 \in \mathfrak{gl}(n, \mathbb{R})$ , where “near” means in the matrix norm.

**Theorem 5.2.**

- 1)  $\log(\exp(x)) = x$  and  $\exp(\log(x)) = x$  whenever defined.
- 2)  $\exp(0) = 1, \exp'(0) = \text{Id}$ , where  $\exp'(0)$  denotes the derivative of  $\exp$  at 0.
- 3) If  $xy = yx$  then  $\exp(x+y) = \exp(x)\exp(y)$ . If  $xy \neq yx$  then  $\log(xy) = \log(x) + \log(y)$  near 1.
- 4) For fixed  $x \in \mathfrak{gl}(n, \mathbb{R})$ ,  $t \mapsto \exp(tx)$  is a morphism of Lie groups  $\mathbb{R} \rightarrow \text{GL}(n, \mathbb{R})$  (a one-parameter subgroup).
- 5)  $\exp(Ax)A^{-1} = A \exp(x)A^{-1}$  and  $\exp(x^t) = \exp(x)^t$

**Theorem 5.3** (Thm 2.30 in Kirillov, Jr.). For each classical group  $G \in \text{GL}(n, \mathbb{R})$  there is a vector space  $\mathfrak{g} \in \mathfrak{gl}(n, \mathbb{R})$  such that for some neighborhood  $U$  of 1 in  $\text{GL}(n, \mathbb{R})$  and some neighborhood  $u$  of 0 in  $\mathfrak{gl}(n, \mathbb{R})$  the logarithm and exponential maps restrict to diffeomorphisms



*Proof.* This is done case by case. We will look at a few cases in the next lecture. □

**Corollary 5.4** (Corollary 2.31 in Kirillov, Jr.). Each classical group  $G \in \text{GL}(n, \mathbb{R})$  is a Lie group with tangent space  $T_1G = \mathfrak{g}$  and  $\dim G = \dim \mathfrak{g}$ .

*Proof.*  $\exp: u \times \mathfrak{g} \rightarrow U \times G$  is a coordinate system around  $1 \in G$ , by Theorem 5.3. Furthermore,  $\exp'(0) = \text{Id}$  which has full rank, hence has full rank when restricted to  $u \times \mathfrak{g}$ . Let  $g \in G$  be arbitrary. Then the composition

$$L_g \circ \exp: u \times \mathfrak{g} \rightarrow U \times G$$

is a coordinate system around  $g$ . So every  $g \in G$  has a coordinate system so by Theorem 4.2,  $G$  is a manifold. The multiplication map  $G \times G \rightarrow G$  is the restriction of the multiplication in  $\text{GL}(n, \mathbb{R})$  hence is smooth. Similarly the inverse map  $G \rightarrow G, g \mapsto g^{-1}$ , is smooth. Thus  $G$  is a Lie group.

For the second part,

$$\begin{array}{ccc}
 \exp : & T_0 \mathfrak{g} & \xrightarrow{\sim} & T_1 G \\
 & \downarrow & & \downarrow \\
 & \mathfrak{g} & & \mathfrak{g} \\
 \exp'(0) & & & \\
 & \downarrow & & \downarrow \\
 & \text{Id} & & 
 \end{array}$$

This gives an isomorphism between  $\mathfrak{g}$  and  $T_1G$ . In particular  $\dim G = \dim T_1G = \dim \mathfrak{g}$ . □

Similarly one can show that the classical groups  $G \in \text{GL}(n, \mathbb{C})$  are Lie groups.

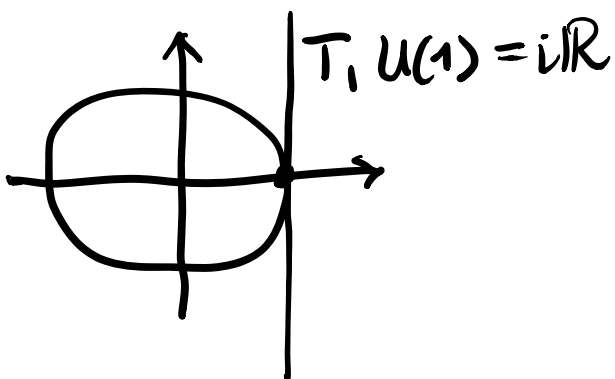
## 6 Lecture 6

### 6.1 Classical Groups (continued)

First we begin with two examples with tangent spaces.

**Example 6.1.**  $T_a \mathbb{R}^k \cong \mathbb{R}^k$  because  $\text{Id}: \mathbb{R}^k \xrightarrow{\sim} \mathbb{R}^k$  is a coordinate system around  $a$ .  $\text{Id}(x_1, \dots, x_k) = (x_1, \dots, x_k)$  and  $\text{Id}'(a) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = I_k$  and  $I_k(\mathbb{R}^k) \cong \mathbb{R}^k$ . Similarly  $T_a \mathbb{C}^k \cong \mathbb{C}^k$ .

**Example 6.2.**  $T_1 U(1) \cong i\mathbb{R} \subset \mathbb{R}$ .  $U(1) = \{z \in \mathbb{C} \mid |z| = 1\} = \{e^{it} \mid t \in \mathbb{R}\}$ . Now  $\varphi: \mathbb{R} \xrightarrow{\sim} U(1)$  with  $\varphi(t) = e^{it}$  is a coordinate system around  $1 \in U(1)$ ,  $\varphi'(0) = 1$ ,  $\varphi'(0) = 2\pi i \cdot 1 = 2\pi i$  matrix.  $\varphi'(0) \in \mathbb{R} \xrightarrow{\sim} 2\pi i \in \mathbb{R} \xrightarrow{\sim} i\mathbb{R}$ .



**Definition 6.3.** A *neighborhood* (nbg) of a point  $x \in \mathbb{R}^n$  is an open subset  $U \subset \mathbb{R}^n$  such that  $x \in U$ .

Recall from last time, Theorem 5.3.

*Proof of thm 5.3.*

$G = GL(n, \mathbb{C})$

By properties of exp and log.

$G = SL(n, \mathbb{C})$

For any  $x \in \mathfrak{gl}(n, \mathbb{C})$  we have the identity

$$\exp(\text{Tr} x) = \det(\exp x) \quad (6.1)$$

For any  $A \in GL(n, \mathbb{C})$ ,

$$\exp(Ax) = A \exp(x) A^{-1}$$

*Proof of Equation 6.1.* So 6.1 holds for  $x$  iff it holds for  $Ax$ . Find  $A \in GL(n, \mathbb{C})$  such that  $Ax = \begin{pmatrix} s & & \\ & \ddots & \\ & & n \end{pmatrix}$  where  $s, n \in \mathbb{C}$ ,  $s$  diagonal,  $n$  strictly upper triangular,  $sn = ns$ . Then use that

$$\exp(s) = \exp(s) \exp(n).$$

Easy to check 6.1 for  $s$  and  $n$  holds for  $s + n$  holds for  $x$ . □

Thus for  $X \in \mathfrak{gl}(n, \mathbb{C})$  near 1,  $X = \exp x$  for some  $x \in \mathfrak{gl}(n, \mathbb{C})$ .

$$\det X = \det(\exp x) = \exp(\text{Tr} x).$$

So  $\det X = 1 \iff \text{Tr} x = 0$ , so statement is true with  $\mathfrak{g} = \{x \in \mathfrak{gl}(n, \mathbb{C}) \mid \text{Tr} X = 0\}$ .

**Notation 6.4.**  $\text{sl}(n, \mathbb{q})$ .

$G = \text{O}(n, \mathbb{q})$   $\text{tX P GL}(n, \mathbb{q}) | XX^t = I$   
 For  $\text{X P GL}(n, \mathbb{q})$  near 1 write  $X = \exp(x), x \in \mathfrak{gl}(n, \mathbb{q})$  by properties of exp. Then

$$\begin{aligned} XX^t &= I \iff I - X^{-1} \rho(x^t) X^{-1} = \rho(X^t X) = 1 \\ &\iff X^t X - XX^t = I \\ &\iff 0 = \log I = \log X + \log X^t = x + x^t \end{aligned}$$

Conversely if  $x + x^t = 0$  then  $x$  and  $x^t$  commute  $\rho(x) \rho(x^t) = \rho(x + x^t) = \rho(0) = I$ .  
 So statement true with  $\mathfrak{g} = \mathfrak{so}(n, \mathbb{q}) = \text{tX P gl}(n, \mathbb{q}) | x + x^t = 0$  set of skew-symmetric matrices.

$G = \text{SO}(n, \mathbb{q})$   
 $\mathfrak{g} = \mathfrak{so}(n, \mathbb{q}) = \text{sl}(n, \mathbb{q}) \times \mathfrak{o}(n, \mathbb{q})$ . However!

$$x + x^t = 0 \iff \text{Tr } x = 0$$

So actually  $\mathfrak{so}(n, \mathbb{q}) = \mathfrak{o}(n, \mathbb{q})$ . Which makes sense because  $\text{SO}(n, \mathbb{q}) = \text{O}(n, \mathbb{q})^0$  the connected component of  $I$ .

$G = \text{U}(n, \mathbb{q}), \text{SU}(n, \mathbb{q})$

$\mathfrak{u}(n, \mathbb{q}), \mathfrak{su}(n, \mathbb{q})$

$G = \text{Sp}(n, \mathbb{q})$

$\mathfrak{sp}(n, \mathbb{q})$

$G = \text{Sp}(n, \mathbb{q})$

$\mathfrak{sp}(n, \mathbb{q})$

Read about the previous few in the book. □

## 7 Lecture 7

### 7.1 Submanifolds

- Open
- Immersed
- Embedded

Recall:

A subset  $N$  of a manifold  $M \in \mathbb{R}^n$  is an *open submanifold* if  $N = M \cap U$  for some open subset  $U$  of  $\mathbb{R}^n$ .

**Example 7.1.**  $M = S^1 \in \mathbb{R}^2, U = \{ (x, y) | y > 0 \}$ ,  $N = M \cap U$  is an open submanifold of  $S^1$ .

**Example 7.2.**  $\text{GL}(n, \mathbb{q})$  is an open submanifold of  $\mathbb{q}^{n^2}$ .

**Example 7.3.** Every connected component of a manifold is an open submanifold. In particular  $G^0$  is an open submanifold of  $G$ .

**Definition 7.4.** A morphism of manifolds  $f: X \rightarrow Y$  is an *immersion* if  $f_* : T_x X \rightarrow T_{f(x)} Y$  has full rank ( $= \dim X$ ) for every  $x \in X$ .

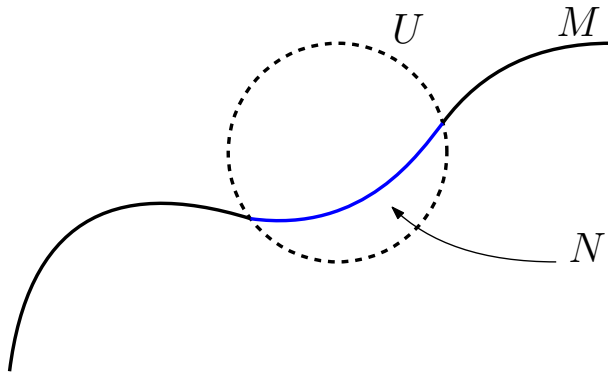


Figure 16: Visualization of an open submanifold

**Example 7.5.**  $f: S^1 \rightarrow \mathbb{R}^2$ ,  $f(\cos \theta, \sin \theta) = (\cos \theta, \sin 2\theta)$   
 $\varphi: \theta \in \mathbb{R} \rightarrow (\cos \theta, \sin \theta) \in S^1$ ,  $\varphi^{-1}(p, q) = \arctan\left(\frac{\sin a}{\cos a}\right) + 2\pi k$ ,  $T_x S^1 \cong \mathbb{R}$ ,  $f_*: T_x S^1 \rightarrow T_{f(x)} \mathbb{R}^2 \cong \mathbb{R}^2$  is given  
 by  $f_* \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\sin a \\ 2 \cos 2a \end{pmatrix}$  for  $t = 0$ .

*Proof of  $\varphi^{-1}$ .*

$\varphi^{-1}: M \rightarrow \mathbb{R} \cong N \cong \mathbb{R}^2$   
 $\alpha: \mathbb{R} \rightarrow \mathbb{R}^2$ ,  $\beta: \mathbb{R}^2 \rightarrow \mathbb{R}$   
 $W \cong \mathbb{R} \cong W^1 \cong \mathbb{R}^2$   
 $\varphi^{-1} \circ f \circ \alpha^{-1} = \beta^{-1} \circ f \circ \alpha^{-1}$ , so  $\beta^{-1} \circ f \circ \alpha^{-1} = f^{-1} \circ \alpha^{-1}$ . Now since  $\beta = \text{Id}$ ,  $\varphi \circ \alpha^{-1} = f^{-1} \circ \alpha^{-1}$

$$\begin{pmatrix} \sin a \\ 2 \cos 2a \end{pmatrix} = f^{-1} \begin{pmatrix} \sin a \\ \cos a \end{pmatrix}$$

□

**Definition 7.6.** The pair  $(\varphi, X)$  is an *immersed submanifold* of  $Y$ . By abuse of terminology we sometimes say  $f(X)$  is an immersed submanifold.

**Definition 7.7.** If  $f: X \rightarrow Y$  is an immersion such that

- 1)  $f$  is injective
- 2)  $f: X \rightarrow f(X)$  is a homeomorphism,

then  $f$  is an *embedding* and we say  $f(X)$  is an (*embedded*) *submanifold* of  $Y$ .

**Example 7.8.**  $f: \mathbb{R} \rightarrow S^1 \subset \mathbb{R}^2$  is an immersion

**Example 7.9.**  $f: \mathbb{R} \rightarrow \mathbb{R}^2$ ,  $f(t) = (t, \sin t)$  is an embedding. The graph of  $y = \sin x$  is a submanifold of  $\mathbb{R}^2$ .

**Example 7.10.**  $f: \mathbb{R} \rightarrow \mathbb{R}^2$

$$f(t) = \begin{cases} (0, t) & , t \leq 1 \\ (t, \sin t) & , 1 < t < 2 \\ (t^3, \sin \pi t) & , 2 \leq t \end{cases}$$

making  $f$  differentiable and injective. Then  $f$  is not an embedding since  $f: \mathbb{R} \rightarrow f(\mathbb{R})$  is not a homeomorphism.

**Example 7.11.**  $f: \mathbb{R} \times \mathbb{S}^1 \rightarrow \mathbb{S}^1$ ,  $f(p, t) = pe^{ati} + e^{bti}$  where  $a, b \in \mathbb{R}$  such that  $a/b$  is irrational. Then  $f$  is an injective immersion, but  $f(\mathbb{R} \times \mathbb{S}^1)$  is dense in  $\mathbb{S}^1$  so  $f: \mathbb{R} \times \mathbb{S}^1 \rightarrow \mathbb{S}^1$  is not a homeomorphism and  $f$  is not an embedding.

**Definition 7.12.** A closed Lie subgroup  $H$  of a Lie group  $G$  is a subgroup which is also a submanifold.

**Example 7.13.** Any linear subspace  $U$  of a vector space  $V$  is a closed Lie subgroup.

**Example 7.14.**  $G^0$  is a closed Lie subgroup of  $G$  using the identity map embedding  $G^0 \hookrightarrow G$  by  $x \mapsto x$ .

**Example 7.15.** If  $G_1$  and  $G_2$  are Lie groups then  $G_1 \times \{1\}$  and  $\{1\} \times G_2$  are closed Lie subgroups.

**Theorem 7.16** (Thm 2.9 in Kirillov).

- 1) Let  $H$  be a closed Lie subgroup of Lie group  $G$ . Then  $H \cong V \times G$  for some closed subset  $V \in \mathbb{R}^n$ . (i.e.  $H$  is closed in  $G$ )
- 2) Conversely, any subgroup  $H$  of a Lie group  $G$  such that  $H$  is closed in  $G$ , is a closed Lie subgroup.

*Proof.* Skipped. □

**Example 7.17.**  $Sp(2n, \mathbb{C})$  is a closed Lie subgroup of  $GL(2n, \mathbb{C})$ .

*Proof.* Know  $Sp(2n, \mathbb{C})$  subgroup.  $A \in GL(2n, \mathbb{C})$ :  $A \in Sp(2n, \mathbb{C})$  iff  $\omega(Ax, Ay) = \omega(x, y)$  for  $x, y \in \mathbb{C}^{2n}$ .  $\omega(x, y) = x^t J y$  where  $J = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$ . So let  $V = \{A \in M_{2n}(\mathbb{C}) \mid A^t J A = J\}$ . Then  $V$  is a closed subset of  $M_{2n}(\mathbb{C}) \cong \mathbb{R}^{4n}$ , and  $Sp(2n, \mathbb{C}) = V \cap GL(2n, \mathbb{C})$ . Similar for other classical Lie groups. □

## Further Reading

### 7.2 Quotient Groups and Homogeneous Spaces

[Corollary 2.10, read yourselves]

**Definition 7.18.** Let  $F$  be a manifold. A morphism  $p: T \rightarrow B$  of manifolds is a fiber bundle over  $B$  with fiber  $F$  if

- 1)  $p$  is surjective,
- 2)  $p$  is locally trivial: Each  $x \in B$  has a neighborhood  $U$  and a diffeomorphism  $T_U: p^{-1}(U) \rightarrow U \times F$  called a local trivialization, such that

$$\begin{array}{ccc}
 U \times F & \xleftarrow{T_U} & p^{-1}(U) \subset T \\
 \searrow p \circ pr_1 & & \swarrow p \\
 & U &
 \end{array}$$

commutes i.e.  $p|_{p^{-1}(U)} = p \circ pr_1 \circ T_U$ .



- 3) At every  $x \in M$ , the map  $T_{U,x}: p^{-1}(p(x)) \rightarrow F$  given by  $T_{U,x}(p(x), f) = f$  is a homeomorphism.
- 4) Whenever  $U \times V \rightarrow H$ , for any  $x \in U \times V$ , the map  $T_{V,x}^{-1} \circ T_{U,x}: F \rightarrow F$  is smooth.

$B$ =base space,  $F$ =fiber,  $T$ =total space.

**Example 7.19.** Let  $F, B$  any manifolds  $p: F \rightarrow B \cong B$ ,  $(p(x), y) \in F$  is the *trivial fiber bundle* over  $B$  with fiber  $F$ .

**Example 7.20.** Tangent bundle on  $S^2$

$$TS^2 = \{(p(x), v) \mid x \in S^2, v \in T_x S^2\}, \quad p: TS^2 \rightarrow S^2, \quad (p(x), v) \in TS^2$$

is a fiber bundle with fiber  $\mathbb{R}^2$ .

**Example 7.21.**  $M$  any manifold. *Tangent bundle* on  $M$ :

$$TM = \{(p(x), v) \mid x \in M, v \in T_x M\}, \quad p: TM \rightarrow M, \quad (p(x), v) \in TM$$

is a fiber bundle with fiber  $\mathbb{R}^k$ ,  $k = \dim M$ .

**Theorem 7.22** (Thm 2.11 in Kirillov).

- 1) Let  $G$  be a Lie group of dimension  $n$ , and  $H \in G$  a closed Lie subgroup of dimension  $k$ . Then the set of left cosets  $G/H = \{gH \mid g \in G\}$  has a natural structure of a manifold of dimension  $n - k$ , such that the canonical map  $p: G \rightarrow G/H, g \mapsto gH$  is a fiber bundle with fiber  $H$ . Also  $T_p G/H \cong T_1 G / T_1 H \cong (\mathfrak{g} / \mathfrak{h})$ .
- 2) If  $H$  normal, closed Lie subgroup of a Lie group  $G$ , then  $G/H$  has a canonical structure of a Lie group, and  $p: G \rightarrow G/H$  gives an isomorphism  $T_p G/H \cong T_1 G / T_1 H$ .

*Proof.* Beyond scope of class. □

2.3 Homomorphism Thm. Read yourselves.

### 7.3 Homogeneous Spaces

Let  $M$  be a manifold. Let  $\text{Diff}(M)$  be the group of diffeomorphisms  $\varphi: M \rightarrow M$ .

**Definition 7.23.** An *action* of a Lie group  $G$  on  $M$  is a group homomorphism  $\rho: G \rightarrow \text{Diff}(M)$  such that the map  $G \times M \rightarrow M, (g, x) \mapsto \rho(g)x$  is a morphism of manifolds

**Notation 7.24.**  $g.x := \rho(g)x$

**Example 7.25.**  $GL(n, \mathbb{R})$  acts on  $\mathbb{R}^n$

**Example 7.26.**  $O(n, \mathbb{R})$  acts on  $S^{n-1}$ .

**Example 7.27.**  $G$  acts on  $G$  via  $\text{Ad}, L, R, g \mapsto \text{Ad} g$ .

**Example 7.28.**  $GL(n, \mathbb{R})$  acts on the set of *ags* in  $\mathbb{R}^n$ :

$$F_n(\mathbb{R}^n) = \{(p(V_0) \in V_1 \in \dots \in V_n \mid V_d \text{ subspace of } \mathbb{R}^n, \dim V_d = d)$$

**Note 7.29.**  $F_2(\mathbb{R}^2) \cong \mathbb{R}P^2$  and  $g.p(V_0) \in V_1 \in \dots \in V_n := \rho(g)V_0 \in gV_1 \in \dots \in gV_n \in g \in GL(n, \mathbb{R})$ .

**Theorem 7.30** (Thm 2.20 in Kirillov). Let  $M$  be a manifold with an action of a Lie group  $G$ . Then  $\pi^{-1}(m) \subset M$ , the stabilizer  $\text{Stab}_G(m) = \{g \in G \mid g \cdot m = m\}$  is a closed Lie subgroup of  $G$  and the map

$$G/\text{Stab}_G(m) \rightarrow \pi^{-1}(m)$$

is an injective immersion.

*Proof.* Future (?) □

**Corollary 7.31** (Cor 2.21 in Kirillov). Each orbit  $O_m = \{g \cdot m \mid g \in G\}$  is an immersed submanifold of  $M$ , and  $T_m O_m = T_1 G \cdot T_1 G_m$ . If  $O_m$  is a submanifold then  $G/\text{Stab}_G(m) \rightarrow O_m$  is a diffeomorphism.

**Definition 7.32.** A  $G$ -homogeneous space is a manifold with a transitive action of  $G$ .

**Corollary 7.33.** Let  $M$  be a  $G$ -homogeneous space and  $x \in M$ . Then  $G/\text{Stab}_G(x) \rightarrow M$  is a fiber bundle with fiber  $\text{Stab}_G(x)$ , and  $M \cong G/\text{Stab}_G(x)$  as  $G$ -homogeneous spaces.

**Example 7.34.**  $\text{SO}(n-1, \mathbb{R})/\text{Stab}_G(m) \cong \text{SO}(n, \mathbb{R})/\text{Stab}_G(m) \cong S^{n-1}$ , where  $\text{SO}(n-1, \mathbb{R})$  is the stabilizer of  $m = (0, 0, \dots, 1)$ . So

$$S^{n-1} \cong \frac{\text{SO}(n, \mathbb{R})}{\text{SO}(n-1, \mathbb{R})}$$

**Example 7.35.**  $GL(n, \mathbb{R})$  acts transitively on  $F_n(\mathbb{R}^n)$ . Pick standard flag

$$V^{\text{st}} = \{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 = \dots = x_{n-1} = 0 \}$$

Then the stabilizer  $\text{Stab}_{GL(n, \mathbb{R})}(V^{\text{st}}) = B(n, \mathbb{R})$  all invertible upper triangular matrices so

$$F_n(\mathbb{R}^n) \cong \frac{GL(n, \mathbb{R})}{B(n, \mathbb{R})}$$

which equips  $F_n(\mathbb{R}^n)$  with the structure of a manifold of dimension  $n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$

## 8 Lecture 8

### 8.1 The General Exponential Map

Let  $G$  be a Lie group (not necessarily classical) and  $\mathfrak{g} = T_1 G$  its Lie algebra.

Goal: Define a map  $\exp: \mathfrak{g} \rightarrow G$  which generalizing matrix exponential map.

$$\mathfrak{gl}(n, \mathbb{R}) \cong \mathfrak{gl}(n, \mathbb{R}) \rightarrow G \quad x \mapsto e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

The problem is  $x^k$  has no meaning in general.

**Definition 8.1.** A one-parameter subgroup of  $G$  is a morphism of Lie groups  $\gamma: \mathbb{R} \rightarrow G$ .

Let  $\gamma: \mathbb{R} \rightarrow G$  be a one-parameter subgroup of  $G$ . In particular  $\gamma(0) = 1$ , the identity element in the group  $G$ . So, the derivative at zero is a linear map  $\gamma'(0): T_0 \mathbb{R} \rightarrow T_1 G$ . Since  $T_0 \mathbb{R}$  can be identified with  $\mathbb{R}$  and  $T_1 G$  can be identified with the Lie algebra of  $G$ , we obtain a linear map

$\mathfrak{g}$ . But a linear map  $\mathbb{R} \rightarrow \mathfrak{g}$  is determined by the value at 1  $\in \mathbb{R}$  and is really the same thing as an element of  $\mathfrak{g}$ . In this way we regard  $\mathfrak{g}$  as an element of  $\mathfrak{g}$ . Thus we have a map

$$\text{one-parameter subgroups of } G \xrightarrow{\cong} \mathfrak{g} \quad (8.1)$$

In fact, this map is a bijection. This gives a third way to think about the Lie algebra of  $G$ , the previous two being the space of all left-invariant vector field (the official definition), and the tangent space at the identity of  $G$ .

To prove it, we need the following theorem from differential equations, which we will not prove here.

**Theorem 8.2** (Local Integrability of Vector Fields on Manifolds). *Let  $v$  be a vector field on a manifold  $M$  and let  $p \in M$ . Then there exists an open interval  $I \subseteq \mathbb{R}$  containing 0 and a smooth map  $\gamma : I \rightarrow M$  satisfying the differential equation:*

$$\begin{aligned} \dot{\gamma}(t) &= v(\gamma(t)) \\ \gamma(0) &= p \end{aligned} \quad (8.2)$$

Moreover, if  $\eta : J \rightarrow M$  is another local solution to that differential equation, then  $\gamma = \eta$  on  $I \cap J$ .

We can now prove that (8.1) is a bijection.

**Proposition 8.3** (Prop 3.1 in Kirillov). *Let  $G$  be a Lie group,  $\mathfrak{g} = T_1G$ , and  $x \in \mathfrak{g}$ . Then there exists a unique one-parameter subgroup  $\gamma_x$  of  $G$  such that  $\dot{\gamma}_x(0) = x$ .*

*Proof.* Let  $v_x$  be the left invariant vector field on  $G$  with  $v_x(1) = x$ . Let  $\gamma : I \rightarrow G$  be a solution to the system

$$\begin{aligned} \dot{\gamma}(t) &= v_x(\gamma(t)) \\ \gamma(0) &= 1 \end{aligned} \quad (8.3)$$

We claim that

$$\gamma(s+t) = \gamma(s)\gamma(t) \quad \text{when } s, t, s+t \in I. \quad (8.4)$$

To show this, fix  $s$  and let

$$\alpha(t) = \gamma(s)\gamma(t), \quad \beta(t) = \gamma(s+t).$$

Then

$$\begin{aligned} \dot{\alpha}(t) &= L_{\gamma(s)} \dot{\gamma}(t) \\ &= L_{\gamma(s)} v_x(\gamma(t)) \quad \text{by the chain rule} \\ &= L_{\gamma(s)} v_x(\gamma(t)) \\ &= v_{\gamma(s)\gamma(t)} \quad \text{by left invariance of } v_x \\ &= v_{\beta(t)}. \end{aligned}$$

On the other hand

$$\dot{\beta}(t) = v_x(\gamma(s+t)) = v_x(\beta(t)).$$

Thus  $\alpha$  and  $\beta$  satisfy the same differential equation and  $\alpha(0) = \beta(0) = \gamma(s)$ . By the uniqueness part of Theorem 8.2,  $\alpha(t) = \beta(t)$ . This proves the claim (8.4). We leave it as an exercise to show that  $\gamma$  extends uniquely to a one-parameter group  $\mathbb{R} \rightarrow G$ . Taking  $t = 0$  in (8.3) we have  $\dot{\gamma}(0) = v_x(1) = x$ .  $\square$

**Definition 8.4.** The exponential map  $\exp: \mathfrak{g} \rightarrow G$  is defined by  $\exp(x) = \gamma_x(1)$ .

**Example 8.5.**  $G = U(1) \cong \mathbb{C} \setminus \{0\} / \sim$  is real 1-dim Lie group. We have seen  $\mathfrak{g} = i\mathbb{R}$ . Given  $x \in \mathfrak{g}$ , the corresponding 1-parameter subgroup is

$$\gamma_x(t) = e^{tx}$$

because  $\gamma_x'(s) = e^{sx} = e^{sx}x$  and  $\gamma_x'(0) = x$ . So the exponential map in this case is  $\exp: i\mathbb{R} \rightarrow U(1)$  by  $x \mapsto \gamma_x(1) = e^x$  i.e. the usual one.

**Example 8.6.**  $\mathfrak{so}(3)$  Lie algebra  $\mathfrak{so}(3)$  has a basis  $\{J_x, J_y, J_z\}$ :

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

One can show that

$$\gamma_{J_x}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & \sin t & \cos t \end{pmatrix}.$$

Indeed it's a morphism of Lie groups and  $\gamma_{J_x}'(0) = J_x$ . So  $\exp(tJ_x) = \gamma_{J_x}(t)$ .

### 8.1.1 Properties of the Exponential Map

**Exercise 8.7.** Let  $G$  be a Lie group and  $x \in T_1G$ . Then for any real numbers  $s, t$  we have

$$\gamma_{sx}(t) = \gamma_x(st).$$

(Hint: Take  $\frac{d}{dt}|_{t=0}$  on both sides and use uniqueness.)

**Theorem 8.8** (thm 3.7 in Kirillov). *The general exponential map satisfies the following:*

- 1)  $\exp(0) = 1$  is the identity map
- 2)  $\exp$  is a diffeomorphism between some neighborhood of 0 in  $\mathfrak{g}$  and some neighborhood of 1 in  $G$ . The inverse is denoted by  $\log$ .
- 3)  $\exp(st) = \exp(t) \exp(s)$  if  $s, t \in \mathbb{R}$ ,  $x \in \mathfrak{g}$
- 4) If  $\varphi: G_1 \rightarrow G_2$  is a morphism of Lie groups, then

$$\exp(\varphi(x)) = \varphi(\exp(x)) \quad \text{for } x \in \mathfrak{g}_1 \quad (Recall: \varphi'(0) = d\varphi_0)$$

- 5) For  $X \in \mathfrak{g}$ ,  $y \in G$   $\exp(y)X^{-1} = \exp(\text{Ad } X)y$

*Proof.*

- 1) By Exercise 8.7, we have  $\exp(tx) = \gamma_x(t)$  for  $x \in \mathfrak{g}$  and  $t \in \mathbb{R}$ . Now differentiate with respect to  $t$  and take  $t = 0$  to get  $\exp'(0)x = x$ .
- 2) Immediate by inverse function theorem and part 1).
- 3) Follows from the fact that  $t \mapsto \exp(tx)$  is a one-parameter subgroup.
- 4) Follows from the uniqueness of the one-parameter subgroup: Let  $x \in \mathfrak{g}_1$ , consider  $\gamma: \mathbb{R} \rightarrow G_1 \rightarrow G_2$  by  $t \mapsto \gamma_x(t) = \varphi(\gamma_x(t))$ . This is a one-parameter subgroup of  $G_2$ . Hence  $\gamma'(0) = \varphi'(0)x$ . So by uniqueness of a one-parameter subgroup of  $G_2$  with  $\gamma'(0) = \varphi'(0)x$ . So  $\gamma = \varphi \circ \gamma_x$  in  $G_2$ . This shows that  $\varphi \circ \gamma_x = \varphi' \circ \gamma_x$ . Now evaluate both sides at  $t = 1$ .
- 5) Follows from 4) by  $G \rightarrow G$  by  $Y \mapsto XYX^{-1} = \text{Ad } X(Y)$ . □

## Further Reading

### 8.2 Classes of Manifolds

- (1) *Complex manifold*  $M \in \mathbb{C}^n$ : Coordinate systems  $\varphi: U \xrightarrow{\sim} M$  where  $U \in \mathbb{C}^d$  are *holomorphic* functions.
- (2)  $C^k$ -*manifolds*  $M \in \mathbb{R}^n$ : Coordinate systems  $\varphi: U \xrightarrow{\sim} M$  where  $U \in \mathbb{R}^d$ , are differentiable of class  $C^k$ .

$C^0$	continuous functions
$C^1$	1st order differentiable functions
$C^k$	all partial derivatives up to order $k$ exist and are continuous
$C^\infty$	all partial derivatives exist ( $\mathbb{R}$ continuous) <i>smooth</i> functions
$C^\omega$	all real analytic functions (functions with Taylor series expansions)

One could imagine a theory of  $C^k$ -Lie groups for  $k \in \mathbb{N} \cup \{\infty, \omega\}$ , in the real case. However we have:

**Theorem 8.9** (Deep Theorem (see Remark 2.2)). *Let  $G$  be a real Lie group of class  $C^0$ . Then there is a Lie group  $G^1$  of class  $C^1$  such that  $G \cong G^1$  as  $C^0$ -Lie groups. Moreover,  $G^1$  is unique up to isomorphism of  $C^1$ -Lie groups.*

The conclusion to be drawn from this is "every real Lie group is a  $C^1$ -Lie group". it suffices to consider

- Complex Lie groups
- Real Lie groups

We may WLOG assume all maps involved have Taylor series expansions.

## 9 Lecture 9: The Bracket (Commutator)

Let  $G$  be a Lie group and  $\mathfrak{g} = T_1G$ . The vector space  $\mathfrak{g}$  is equipped with a canonical bilinear operation, denoted  $[x, y]$  for  $x, y \in \mathfrak{g}$  and is called the *bracket* on  $\mathfrak{g}$ .

There are two equivalent ways to introduce it.

### 9.1 First Method to Introduce the Bracket: via the Logarithm

Recall  $\exp$  is locally a diffeomorphism.

$$\exp: \mathfrak{g} \xrightarrow{\sim} G$$

Let  $U \in \mathfrak{g}$  be a neighborhood of 0 and  $V \in G$  neighborhood of 1 such that  $\exp|_U: U \xrightarrow{\sim} V$  is a diffeomorphism with an inverse denoted by  $\log$ . consider the map

$$\varphi: \mathfrak{g} \times \mathfrak{g} \xrightarrow{\sim} G \quad \text{by} \quad (x, y) \mapsto \exp(x)\exp(y).$$

Then for  $\rho x, y \in \mathfrak{g}$   $\mu(\rho x, y) = \log(\exp(\rho x) \exp(\rho y))$  for some  $\mu(\rho x, y) \in \mathfrak{g}$ . Explicitly,

$$\mu(\rho x, y) = \log(\exp(\rho x) \exp(\rho y))$$

defined for  $\rho x, y \in \mathfrak{g}$   $\rho V \in \mathfrak{g}$ .  $\mu$  is a real analytic (or holomorphic) function and thus has a Taylor series at  $(0, 0)$ .

**Lemma 9.1.**  $\mu(\rho x, y) = x + y + \frac{1}{2}[\rho x, \rho y] + \dots$  where,  $\lambda: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  is a skew-symmetric bilinear map.

*Proof.* General Taylor series:

$$f(\rho x, \rho y) = \frac{1}{0!} L_0 + \frac{1}{1!} L_1(\rho x, \rho y) + \frac{1}{2!} L_2(\rho x, \rho y) + \dots$$

Where  $L_k$  is a multilinear function of  $k$  variables. In our case

$$L_k(\rho x, \rho y) = \frac{1}{k!} \sum_{i_1, \dots, i_k} \lambda_{i_1, \dots, i_k}(\rho x, \rho y) \rho^{i_1} \dots \rho^{i_k}$$

So

$$\mu(\rho x, \rho y) = c_0 + \alpha_1(\rho x) + \alpha_2(\rho y) + \frac{1}{2} Q_1(\rho x, \rho x) + \lambda(\rho x, \rho y) + Q_2(\rho y, \rho y) + \dots$$

for some linear functions  $\alpha_1, \alpha_2$  and bilinear functions  $Q_1, \lambda, Q_2$ . Observe

$$\mu(\rho x, 0) = \log(\exp(\rho x) \exp(0)) = \rho x$$

So  $c_0 = 0$ ,  $\alpha_1(\rho x) = \rho x$ , and  $Q_1(\rho x, \rho x) = 0$ . Similarly  $\mu(0, \rho y) = \rho y$ , which gives  $\alpha_2(\rho y) = \rho y$  and  $Q_2(\rho y, \rho y) = 0$ . Lastly,

$$\mu(\rho x, \rho x) = \log(\exp(\rho x) \exp(\rho x)) = \log(\exp(2\rho x)) = 2\rho x.$$

So  $\lambda(\rho x, \rho x) = 0$   $\rho x \in \mathfrak{g}$  which implies  $\lambda(\rho x, \rho y) = \lambda(\rho y, \rho x) = 0$   $\rho x, \rho y \in \mathfrak{g}$ . Now since  $\lambda$  is bilinear we have,  $\lambda(\rho x, \rho y) = -\lambda(\rho y, \rho x) = 0$ .  $\square$

**Definition 9.2.** The skew-symmetric bilinear function  $\lambda: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  as introduced above is called the *commutator* (or *bracket*).

**Notation 9.3.**  $[\rho x, \rho y] = \lambda(\rho x, \rho y)$ .

**Proposition 9.4.**

1) Let  $\varphi: G_1 \rightarrow G_2$  be a morphism of Lie groups, and  $d\varphi_x: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  its differential. Then

$$d\varphi_x([\rho x, \rho y]) = [\varphi(\rho x), \varphi(\rho y)]$$

for any  $\rho x, \rho y \in \mathfrak{g}_1$ .

2)  $\text{Ad}_g([\rho x, \rho y]) = [\text{Ad}_g(\rho x), \text{Ad}_g(\rho y)]$  for any  $g \in G, \rho x, \rho y \in \mathfrak{g}$ .

3) Let  $X = \exp(\rho x)$  and  $Y = \exp(\rho y)$ . Then the group commutator  $[X, Y] = XYX^{-1}Y^{-1}$  equals  $[\rho x, \rho y] + \dots$  terms of higher order.

*Proof.* 1) Recall for  $x \in \mathfrak{g}_1$

$$\exp(\varphi) \rho(x) \varphi = \varphi \rho(\exp(x))$$

hence

$$\begin{aligned} \exp(\mu\rho) \rho(x, \varphi) \rho(y) &= \exp(\rho) \rho(x) \exp(\rho) \rho(y) \\ &= \varphi \rho(\exp(x)) \exp(\rho) \rho(y) \\ &= \varphi \rho(\exp(\mu x), y) \\ &= \exp(\rho) \rho(\mu x, y). \end{aligned}$$

Now apply  $\log$  to both sides.

2)  $\text{Ad}_G: G \rightarrow G$  is a morphism of Lie groups. Apply 1).

3) Explicit calc. □

**Corollary 9.5.** *If  $G$  is a commutative Lie group then  $\tau(x, y) = 0$  for all  $x, y \in \mathfrak{g}$ .*

*Proof.* Use 3) from above. □

**Example 9.6.** Let  $G \in GL(n, \mathbb{R})$  and  $\mathfrak{g} \in \mathfrak{gl}(n, \mathbb{R})$ . Then,

$$\exp(x) \rho^{-1}(x) \varphi = \exp(y) \rho^{-1}(y) \varphi$$

This implies that

$$\exp(x) \exp(y) \exp(\rho^{-1}(x) \varphi) \exp(\rho^{-1}(y) \varphi) = \exp(\rho^{-1}(x) \varphi) \exp(\rho^{-1}(y) \varphi) \exp(x) \exp(y)$$

So  $\tau(x, y) = xy - yx$ .

## 9.2 Second way to Introduce the Bracket: via Derivations

### 9.2.1 Derivations on an Algebra

If  $A$  is an (not necessarily associative) algebra, a *derivation* on  $A$  is a linear map  $D: A \rightarrow A$  satisfying  $D(ab) = D(a)b + aD(b)$  for all  $a, b \in A$ . Let  $\text{Der} A$  denote the set of all derivations of  $A$ . It is easy to see that  $\text{Der} A$  is a subspace of  $\text{End} A$  (the space of all linear maps from  $A$  to itself). If  $D$  and  $E$  are two derivations on  $A$  we define their commutator as

$$[D, E] = D \circ E - E \circ D.$$

**Lemma 9.7.**  $[D, E] \in \text{Der} A$  for all  $D, E \in \text{Der} A$ .

*Proof.*  $\rho D \circ E \rho^{-1} = D \circ E \rho^{-1} \rho = D \circ E \rho^{-1} \rho = D \circ E \rho^{-1} \rho = D \circ E \rho^{-1} \rho = D \circ E \rho^{-1} \rho = D \circ E \rho^{-1} \rho$  so when we switch  $D$  and  $E$  and subtract the middle two terms cancel, giving the result. □

One checks the identities

$$\begin{aligned} [D, E] &= -[E, D] \quad \text{skew-symmetry} \\ [D, [E, F]] + [E, [F, D]] + [F, [D, E]] &= 0 \quad \text{Jacobi identity} \end{aligned}$$

### 9.2.2 Vector Fields as Derivations

Let  $M$  be a manifold,  $C^\infty(M)$  the algebra of smooth real-valued functions on  $M$  with pointwise operations,  $\text{Vect}(M)$  be the vector space of smooth vector fields on  $M$ .

**Lemma 9.8.** *There is an isomorphism of vector spaces*

$$\text{Vect}(M) \cong \text{Der } C^\infty(M), \quad X \mapsto D_X$$

where

$$(D_X f)(p) = \left. \frac{d}{dt} f(\gamma(t)) \right|_{t=0} = f'(p) X_p,$$

and  $\gamma : \mathbb{R} \rightarrow M$  is any smooth map with  $\gamma(0) = p$  and  $\dot{\gamma}(0) = X_p$ . (Here  $X_p \in T_p M$  denotes the tangent vector of the field  $X$  at  $p$ .)

**Remark 9.9.** As can be seen from the last expression, the result does not depend on the choice of  $\gamma$  satisfying the given conditions.

*Proof.* That  $X \mapsto D_X$  is a linear map is easy to see. Suppose that  $D_X = D_Y$  for some vector fields  $X, Y$ . Then

$$f'(p) X_p = f'(p) Y_p \tag{9.1}$$

for all smooth functions  $f$  on  $M$  and all  $p \in M$ . Here we can think of  $f'(p)$  as the gradient of  $f$  at  $p$ , which is a row vector, while  $X_p$  and  $Y_p$  are column vectors upon which  $f'(p)$  acts via matrix multiplication (dot product). By varying the smooth function  $f$  appropriately, we can get any row vector to appear as  $f'(p)$ . Then (9.1) implies that  $u \cdot X_p - Y_p = 0$  for any row vector  $u$ , hence  $X_p = Y_p$ .

(We skip the proof of surjectivity) □

### 9.2.3 Bracket on Vector Fields

Combining these results we may define a bilinear operation on  $\text{Vect}(M)$  as follows: For two vector fields  $X, Y \in \text{Vect}(M)$ , define  $[X, Y]$  to be the unique vector field such that

$$D_{[X, Y]} = [D_X, D_Y]$$

**Exercise 9.10.** If  $X$  and  $Y$  are left invariant vector fields on a Lie group  $G$ , then the vector field  $[X, Y]$  is also left invariant.

### 9.2.4 Explicit Form of Bracket on $T_1 G$ Using Exponential Map and Differentiation

Let  $G$  be a Lie group and in this section put  $\mathfrak{g} = T_1 G$ , the tangent space at identity. We know that  $X \mapsto X_p$  gives a bijection between left invariant vector fields on  $G$  and  $T_1 G$ . Thus we should be able to define a bracket on  $T_1 G$ .

Recall that if  $u \in T_1 G$  is a tangent vector, the corresponding left invariant vector field  $X_u$  is defined by  $X_u(p) = pL_g u$ .

Thus the corresponding derivation  $D_u = D_{X_u}$  on  $C^\infty(M)$  satisfies

$$(D_u f)(p) = \left. \frac{d}{dt} f(\exp ptu) \right|_{t=0} = f'(p) u.$$

Therefore  $[u, v]$  must be the unique tangent vector that satisfies

$$p[D_u, D_v]f(p) = f'(p) [u, v]$$



for all  $f \in C^\infty(M)$ .

To find an explicit expression for  $[u, v]$  we compute:

$$\begin{aligned} \langle D_u D_v f, \cdot \rangle &= \frac{d}{dt} \Big|_{t=0} \langle D_v f \rangle_{\exp(tu)} \\ &= \frac{d}{dt} \Big|_{t=0} \langle \exp(tu)^{-1} \cdot D_v f \rangle_{p_1} \quad \text{by } G\text{-action on } C^\infty(M) \\ &= \frac{d}{dt} \Big|_{t=0} \langle D_v \exp(-tu) \cdot f \rangle_{p_1} \quad \text{by left invariance of } D_v \\ &= \frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} \langle \exp(-ts) \cdot f \rangle_{\exp(sv)} \\ &= \frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} f \cdot \exp(tu) \exp(sv) \\ &= \frac{d}{dt} \Big|_{t=0} f^1 \cdot \exp(tu) \cdot \frac{d}{ds} \Big|_{s=0} \exp(tu) \exp(sv) \\ &= f^2 \cdot \frac{d}{dt} \Big|_{t=0} \exp(tu) \cdot \frac{d}{ds} \Big|_{s=0} \exp(sv) - f^1 \cdot \frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} \exp(tu) \exp(sv) \end{aligned}$$

The second derivative term cancels when we switch  $u$  and  $v$  and subtract, giving:

$$\langle [D_u, D_v] f, \cdot \rangle = f^1 \cdot \left( \frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} \exp(tu) \exp(sv) - \frac{d}{ds} \Big|_{s=0} \exp(sv) \exp(tu) \right)$$

for all  $f \in C^\infty(M)$ . We conclude that the Lie bracket on  $T_1G$  can be computed as follows:

$$[u, v] = \frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} \exp(tu) \exp(sv) - \frac{d}{ds} \Big|_{s=0} \exp(sv) \exp(tu), \quad @u, v \in T_1G. \quad (9.2)$$

## 10 Further Reading

### 10.1 Computing Differentials Using Curves

Let  $\varphi: M \rightarrow N$  be a morphism of manifolds. Let  $p \in M$  and  $x \in T_pM$ . To find  $d\varphi_p(x)$ , also denoted  $\varphi_*(x)$ , let  $\gamma: \mathbb{R} \rightarrow M$  be any morphism with  $\gamma(0) = p$ , and  $d\gamma_0 = x$ <sup>iv</sup>. Then  $d\varphi_p(x) = d\varphi_p(d\gamma_0) = d\varphi_p(x)$ , where the first equality is by the chain rule. On the other hand

$$d\varphi_p(x) = \frac{d}{dt} \Big|_{t=0} \varphi(\gamma(t))$$

This is particularly useful for Lie groups: If  $x \in \mathfrak{g}$  then  $\gamma_x: \mathbb{R} \rightarrow G$ ,  $\gamma_x(t) = \exp(tx)$  is a natural curve through  $1 \in G$  with derivative  $x$ . So given a Lie group  $G$ , a manifold  $M$  and a morphism of manifolds  $\varphi: G \rightarrow M$  its differential at  $1$  can be computed as follows:

$$\varphi_* = d\varphi_1: \mathfrak{g} \rightarrow T_1M = \varphi_*(x) = \frac{d}{dt} \Big|_{t=0} \varphi(\exp(tx)).$$

### 10.2 Differential of Ad

Recall  $\text{Ad}_g: G \rightarrow G$  by  $x \mapsto gxg^{-1}$ . Its differential is

$$\text{Ad}_g: \mathfrak{g} \rightarrow \mathfrak{g}.$$

So  $\text{Ad}_g \in GL(\mathfrak{g})$ , and  $\text{Ad}: G \rightarrow GL(\mathfrak{g})$ . So its differential is  $\text{ad}: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ .

<sup>iv</sup>Here we identify  $\text{Hom}_{\mathbb{R}}(\mathbb{R}; T_pM)$  with  $T_pM$

**Lemma 10.1.**

1)  $\text{ad } x.y = [x, y]$

2)  $\text{Ad}(\exp X) = \exp(\text{ad } X)$

*Proof.* 1) For  $g \in G$  consider  $\text{Ad } g: G \rightarrow G$ . By formula for  $\varphi$  its differential is given by  $\text{Ad } g: \mathfrak{g} \rightarrow \mathfrak{g}$  by

$$\text{Ad } g \cdot \frac{d}{dt} \Big|_0 \text{Ad } g \cdot \exp tY = \frac{d}{dt} \Big|_0 \exp tY \cdot \text{Ad } g$$

By the same formula again,  $\text{ad}: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$  is given by

$$\text{ad } x.y = \frac{d}{ds} \Big|_0 \frac{d}{dt} \Big|_0 \exp sX \exp tY \exp -sX = \frac{d}{ds} \Big|_0 \frac{d}{dt} \Big|_0 \exp tY \cdot \text{Ad } \exp sX \cdot [x, y] = [x, y].$$

Where the second last equality is by Proposition 9.4 part 3).

2) Immediate by  $X \exp tY X^{-1} = \exp \text{Ad } X.Y$  which was proved earlier. □

**Theorem 10.2** (Jacobi Identity). *Let  $G$  be a Lie group and  $\mathfrak{g} = T_1G$ . Then the skew-symmetric bilinear map  $r, s: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  satisfies the Jacobi Identity:*

$$r(x, r(y, z)) + r(y, r(z, x)) + r(z, r(x, y)) = 0.$$

*This identity can also be written (using skew-symmetry and bilinearity):*

$$r(x, r(y, z)) + r(x, y, z) + r(y, r(x, z))$$

$$= \text{ad } x.r(y, z) + r(\text{ad } x.y, z) + r(y, \text{ad } x.z)$$

$$= \text{ad } r(x, y) + \text{ad } x \cdot \text{ad } y + \text{ad } y \cdot \text{ad } x$$

*Proof.* Since  $\text{Ad}: G \rightarrow \text{GL}(\mathfrak{g})$  is a morphism of Lie groups, its differential preserves the commutator by Proposition 9.4 1). But the commutator on  $\mathfrak{gl}(\mathfrak{g})$  is  $[A, B] = AB - BA$  by example 9.6. So

$$\text{ad } r(x, y) = r(\text{ad } x, \text{ad } y) = \text{ad } x \cdot \text{ad } y - \text{ad } y \cdot \text{ad } x.$$

Applying both sides to  $z \in \mathfrak{g}$  we get

$$r(x, r(y, z)) + r(x, y, z) + r(y, r(x, z))$$

The other forms left as an exercise. □

## 11 Lecture 10: Lie Algebras

### 11.1 Lie Algebras and Homomorphisms

**Definition 11.1.** A *Lie algebra*  $\mathfrak{g}$  over a field  $\mathbb{F}$  is a vector space  $\mathfrak{g}$  together with a bilinear map

$$r, s: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$$

satisfying

i)  $r(x, y) = -r(y, x)$  (*skew-symmetry*)<sup>v</sup>

<sup>v</sup>If  $\text{char } \mathbb{F} = 2$  this condition is replaced by  $r(x, x) = 0$ .

ii)  $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$  (*Jacobi identity*)

A homomorphism of Lie algebras  $\psi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  is a linear map satisfying

$$\psi([x, y]) = [\psi(x), \psi(y)].$$

If  $\psi$  is moreover bijective, it is an *isomorphism*. Two Lie algebras  $\mathfrak{g}_1, \mathfrak{g}_2$  are *isomorphic*, written  $\mathfrak{g}_1 \cong \mathfrak{g}_2$ , if there exists an isomorphism  $\varphi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ .

**Example 11.2.** Let  $A$  be any associative algebra over  $\mathbb{F}$  (i.e. a ring containing  $\mathbb{F}$ ). Then define

$$[a, b] = ab - ba.$$

This operation turns  $A$  into a Lie algebra, denoted  $L(A)$ .

**Exercise 11.3.** Show that  $L(A)$  defined above is indeed a Lie algebra.

**Definition 11.4.** The *general linear Lie algebra over  $\mathbb{F}$*  is

$$\mathfrak{gl}(n, \mathbb{F}) = \{A \in M_n(\mathbb{F})\}.$$

More generally, we may consider

$$\mathfrak{gl}(V) = \{A \in \text{End}(V) \mid V \text{ any vector space over } \mathbb{F}\}.$$

Clearly, if  $V$  is finite-dimensional, then

$$\mathfrak{gl}(V) \cong \mathfrak{gl}(n, \mathbb{F}) \text{ where } n = \dim_{\mathbb{F}} V.$$

**Example 11.5.** Any vector space  $V$  can be regarded as a Lie algebra by defining  $[x, y] = 0$  for all  $x, y \in V$ .

**Definition 11.6.** A Lie algebra  $\mathfrak{g}$  is *abelian* if  $[x, y] = 0$  for all  $x, y \in \mathfrak{g}$ .

**Example 11.7.** On  $\mathbb{R}^3$  we may take  $[u, v] = u \times v$ . This is a Lie algebra. In fact, it is isomorphic to the Lie algebra  $\mathfrak{so}(3)$  of all skew-symmetric  $3 \times 3$ -matrices. An isomorphism is given by

$$e_1 \mapsto J_x, \quad e_2 \mapsto J_y, \quad e_3 \mapsto J_z$$

where  $J_{x,y,z}$  are the natural basis elements of  $\mathfrak{so}(3)$ . This shows that the next natural Euclidean space where we can define the cross product isn't  $\mathbb{R}^4$  but rather  $\mathbb{R}^6$  because  $\dim \mathfrak{so}(4) = \frac{4 \cdot 3}{2} = 6$ . (This is just one perspective; there are several distinct ways to generalize the cross product on  $\mathbb{R}^3$ .)

## 11.2 Subalgebras and Ideals

**Definition 11.8.** Let  $\mathfrak{g}$  be a Lie algebra over  $\mathbb{F}$ . A linear subspace  $\mathfrak{h} \subseteq \mathfrak{g}$  is a

- (*Lie*) *subalgebra* if  $[x, y] \in \mathfrak{h}$  for all  $x, y \in \mathfrak{h}$ .
- (*Lie*) *ideal* if  $[x, y] \in \mathfrak{h}$  for all  $x \in \mathfrak{g}, y \in \mathfrak{h}$ .

**Exercise 11.9.** If  $\mathfrak{h} \subseteq \mathfrak{g}$  is an ideal then  $\mathfrak{g}/\mathfrak{h}$  is a Lie algebra with operation

$$[x + \mathfrak{h}, y + \mathfrak{h}] = [x, y] + \mathfrak{h}$$

for all  $x, y \in \mathfrak{g}$ .

**Exercise 11.10.** (First Isomorphism Theorem) If  $\varphi : \mathfrak{g} \rightarrow \mathfrak{h}$  is a Lie algebra homomorphism, then  $\ker \varphi$  is an ideal of  $\mathfrak{g}$ ,  $\text{im} \varphi$  is a subalgebra of  $\mathfrak{h}$ , and

$$\mathfrak{g} / \ker \varphi \cong \text{im} \varphi.$$

**Example 11.11.** Let  $\varphi : \mathfrak{gl}(n, \mathbb{C}) \rightarrow \mathfrak{gl}(n, \mathbb{C})$  be given by  $\varphi(A) = \text{Tr}(A) \cdot I_n$ . This is a Lie algebra homomorphism. The kernel is

$$\ker \varphi = \{A \in \mathfrak{gl}(n, \mathbb{C}) \mid \text{Tr}(A) = 0\}.$$

By the first isomorphism theorem  $\mathfrak{gl}(n, \mathbb{C}) / \ker \varphi \cong \mathfrak{gl}(1, \mathbb{C})$ .

**Definition 11.12.** Let  $\mathfrak{g}$  be a Lie algebra. The *center* of  $\mathfrak{g}$  is

$$Z(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y] = 0 \text{ } \forall y \in \mathfrak{g}\}.$$

**Exercise 11.13.**  $Z(\mathfrak{g})$  is an ideal of  $\mathfrak{g}$ .

**Exercise 11.14.** The center of  $\mathfrak{gl}(n, \mathbb{C})$  is  $\mathbb{C} \cdot I_n$  (the set of scalar multiples of the identity matrix).

### 11.3 Products of Lie Algebras

If  $\mathfrak{g}_1, \mathfrak{g}_2$  are Lie algebras, the cartesian product  $\mathfrak{g}_1 \times \mathfrak{g}_2$  is Lie algebra with bracket

$$[(x_1, x_2), (y_1, y_2)] = ([x_1, y_1], [x_2, y_2]).$$

The projections  $\pi_i : \mathfrak{g}_1 \times \mathfrak{g}_2 \rightarrow \mathfrak{g}_i$  are Lie algebra homomorphisms and the triple  $(\mathfrak{g}_1 \times \mathfrak{g}_2, \pi_1, \pi_2)$  satisfies the universal property: Given any Lie algebra  $\mathfrak{h}$  and homomorphisms  $\psi_i : \mathfrak{g}_i \rightarrow \mathfrak{h}$  there is a unique homomorphism  $\varphi : \mathfrak{g}_1 \times \mathfrak{g}_2 \rightarrow \mathfrak{h}$  such that  $\psi_i = \pi_i \circ \varphi$ .

Even though there are Lie algebra homomorphisms  $\iota_i : \mathfrak{g}_i \rightarrow \mathfrak{g}_1 \times \mathfrak{g}_2$  sending  $x$  to  $(x, 0)$  and  $(0, x)$  respectively, the triple  $(\mathfrak{g}_1 \times \mathfrak{g}_2, \iota_1, \iota_2)$  does NOT (in general) satisfy the dual universal property:

**Exercise 11.15.** Show that if  $\mathfrak{g}$  is non-abelian, then  $(\mathfrak{g} \times \mathfrak{g}, \iota_1, \iota_2)$  is not a coproduct in the category of Lie algebras.

Conclusion: The category of Lie algebras has finite products but does not have finite coproducts. Nevertheless it is common to see the notation  $\mathfrak{g}_1 \oplus \mathfrak{g}_2$  instead of  $\mathfrak{g}_1 \times \mathfrak{g}_2$  in the literature.

## 12 Further Reading

### 12.1 Lie Subgroups and Lie Subalgebras

**Theorem 12.1** (Thm 3.22). *Let  $G$  be a Lie group and  $\mathfrak{g} = \text{Lie}(G)$ .*

- (1) *If  $H$  is a Lie subgroup of  $G$  then  $T_1 H$  is a Lie subalgebra in  $\mathfrak{g}$ .*
- (2) *If  $H$  is a normal closed subgroup of  $G$  then  $\mathfrak{h} = T_1 H$  is a Lie ideal of  $\mathfrak{g}$  and  $\text{Lie}(G/H) = \mathfrak{g} / \mathfrak{h}$ .*

*Conversely, if  $H$  is a closed Lie subgroup in  $G$ , such that  $H$  and  $G$  are connected and  $\mathfrak{h} = T_1 H$  is an ideal in  $\mathfrak{g}$ , then  $H$  is normal.*

*Proof.* We only prove part (1) here. If  $x \in T_1H$  then the one parameter subgroup  $\gamma: \mathbb{R} \rightarrow H$  with  $\gamma(0) = x$  is also a one parameter subgroup of  $G$  and hence  $\gamma(t) = \exp tX$  by uniqueness. So  $\exp tX \in H$   $\forall t \in \mathbb{R}$ . In particular for  $x, y \in T_1H$ :

$$\log(\exp x \exp y \exp -x \exp -y)$$

belongs to  $\mathfrak{h}$  hence by commutator formula,  $[x, y] \in T_1H$ . Similarly for part (2). (Read in book!)  $\square$

**Theorem 12.2** (Thm 3.35 in Kirillov). *Let  $G$  be a connected Lie group. Then its center*

$$Z(G) = \{g \in G \mid gh = hg \text{ for all } h \in G\}$$

*is a closed Lie subgroup with Lie algebra  $\mathfrak{z}(\mathfrak{g})$ .*

## 12.2 Connection Between Lie Groups and Lie Algebras

Let  $G$  be a Lie group. Then  $\mathfrak{g} = \text{Lie } G = T_1G$  with the bracket discussed previous lecture is a Lie algebra. Every morphism  $\varphi: G_1 \rightarrow G_2$  of Lie groups gives a homomorphism of Lie algebras  $\varphi_*: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ . Moreover,  $\varphi_* \circ d\varphi = d\varphi \circ \varphi_*$ ,  $\text{Id}_* = \text{Id}$ . In other words,  $\text{Lie}$  is a functor from the category of Lie groups to the category of Lie algebras.

**Theorem 12.3** (Thm 3.40). *For any Lie group  $G$  there is a bijection between connected Lie subgroups  $H \in G$  and Lie subalgebras  $\mathfrak{h} \in \mathfrak{g}$  given by  $H \mapsto \text{Lie } H = T_1H$ .*

*Sketch of proof.* An inclusion  $\varphi: H \rightarrow G$  of a subgroup yields an inclusion  $\varphi_*: \mathfrak{h} \rightarrow \mathfrak{g}$  of Lie algebras. For the converse, we have seen a special case: If  $\mathfrak{h}$  is one-dimensional, say  $\mathfrak{h} = \mathbb{R}x$  then we may take  $H$  to be the image of the exponential map  $t \mapsto \exp tX$ . This is a connected subgroup of  $G$  (being the image of the connected set  $\mathbb{R}$  under a continuous map). The general case is not proved in the book and is more difficult. It relies on *Frobenius integrability criterion* which is a generalization of the “integrability of vector fields” theorem from differential equations.  $\square$

**Theorem 12.4.** *Let  $G_1$  and  $G_2$  be Lie groups and  $\mathfrak{g}_i = \text{Lie } G_i$ .*

- (1) *If  $G_1$  is connected this functor is faithful, that is,  $\text{Hom}_{\mathfrak{g}_1, \mathfrak{g}_2} \cong \text{Hom}_{G_1, G_2}$  is injective.*
- (2) *(Thm 3.41) If  $G_1$  is connected and simply connected then the functor is fully faithful, that is,  $\text{Hom}_{\mathfrak{g}_1, \mathfrak{g}_2} \cong \text{Hom}_{G_1, G_2}$ .*

*Proof.* See Section 3.8 of the book.  $\square$

**Theorem 12.5** (Thm 3.42, Lie’s 3rd Thm). *Any finite dimensional real Lie algebra is isomorphic to the Lie algebra of a Lie group.*

*Idea of proof.* Show that every Lie algebra is isomorphic to a subalgebra of  $\mathfrak{gl}(n, \mathbb{R})$ . (Ado’s Theorem<sup>vi</sup>). Then use theorem 12.3.  $\square$

**Corollary 12.6** (Cor 3.43). *For any finite dimensional Lie algebra  $\mathfrak{g}$  over  $\mathbb{R}$ , there is a unique up to isomorphism connected simply-connected Lie group  $G$  with  $\text{Lie } G = \mathfrak{g}$ . Furthermore, any other connected Lie group  $G^1$  with Lie algebra  $\mathfrak{g}$  is of the form  $G^1/Z$  for some discrete central subgroup  $Z \in G$ .*

Since the functor from connected simply connected Lie groups to real Lie algebras is fully faithful (Theorem 12.4 and essentially surjective on objects (Theorem 12.6), we obtain:

**Corollary 12.7** (Cor 3.44). *The category of finite dimensional Lie algebras over  $\mathbb{R}$  is equivalent to the category of connected simply-connected Lie groups.*

<sup>vi</sup>If  $\mathfrak{z}(\mathfrak{g}) = 0$ , then  $x \mapsto \text{ad } x$  is injective.

### 12.3 The Baker-Campbell-Hausdorff Formula

Recall:  $\frac{1}{2}[x, y]$  is the quadratic term of the Taylor expansion of  $\log(\exp x \exp y)$  at 0.

**Question 12.8.** Do higher order terms give more information? Or does the bracket completely determine the multiplication in  $G$ ?

**Theorem 12.9** (Baker-Campbell-Hausdorff). For small enough  $x, y \in \mathfrak{g}$  we have,

$$\exp x \exp y = \exp \sum_{n=0}^{\infty} \mu_n(x, y)$$

where

$$\begin{aligned} \mu_0(x, y) &= 0, \\ \mu_1(x, y) &= x + y, \\ \mu_2(x, y) &= \frac{1}{2}[x, y], \\ \mu_3(x, y) &= \frac{1}{12}([x, [x, y]] + [y, [y, x]]), \\ &\vdots \end{aligned}$$

In general, for  $n \neq 0$   $\mu_n$  is a universal (=independent of  $\mathfrak{g}$ ) expression, in a linear combination of commutators of degree  $n$ .

**Corollary 12.10.** The group operation in a connected Lie group can be recovered from its Lie algebra.

### 12.4 Complex and Real Forms

**Definition 12.11.** The *complexification* of a real Lie algebra  $\mathfrak{g}$  is

$$\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$$

with bracket determined by

$$[x \otimes \lambda, y \otimes \mu] = [x, y] \otimes \lambda \mu \quad @x, y \in \mathfrak{g}, \lambda, \mu \in \mathbb{C}.$$

$\mathfrak{g}$  is a *real form* of  $\mathfrak{g}_{\mathbb{C}}$ .

Under the isomorphism  $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g} \oplus i\mathfrak{g}$ , we can also write it

$$[x + iy, z + iw] = [x, z] + [y, w] + i([y, z] - [x, w]) \quad @x, y, z, w \in \mathfrak{g}.$$

**Example 12.12.**  $\mathfrak{g} = \mathfrak{u}(n) \cong \mathfrak{so}(n, \mathbb{C})$ . With

$$X = \frac{1}{2}(X + iY) \quad X + iY = \frac{1}{2}(X - iY) \quad X - iY$$

with the first piece in  $\mathfrak{u}(n)$  and the second piece in  $i\mathfrak{u}(n)$ . This is clear because  $X$  is hermitian  $\hat{O}$   $iX$  skew-hermitian.

**Example 12.13.**  $\mathfrak{su}(n, \mathbb{C})$  and  $\mathfrak{sl}(n, \mathbb{R})$  are two (different) real forms of  $\mathfrak{sl}(n, \mathbb{C})$ , because

$$\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{sl}(n, \mathbb{R}) \oplus i\mathfrak{sl}(n, \mathbb{R}) = \mathfrak{su}(n, \mathbb{C}) \oplus i\mathfrak{su}(n, \mathbb{C}).$$

**Remark 12.14.** Let  $G$  be a connected complex Lie group,  $\mathfrak{g} = \text{Lie}(\mathfrak{p}G\mathfrak{q})$ . Let  $K \in G$  be a closed real Lie subgroup in  $G$  such that  $\mathfrak{k} = \text{Lie}(\mathfrak{p}K\mathfrak{q})$  is a real form of  $\mathfrak{g}$  (i.e.  $\mathfrak{k} \cong_{\mathbb{R}} \mathbb{C} \otimes \mathfrak{g}$ ). Then  $K$  is said to be a *real form* of  $G$ .

So, the above example shows that  $SU(n, \mathbb{C})$  and  $SL(n, \mathbb{R})$  are two real forms of  $SL(n, \mathbb{C})$ .

Since unitary matrices preserve the standard Hermitian inner product on  $\mathbb{C}^n$ , it is not hard to see that  $SU(n, \mathbb{C})$  is a (closed) and bounded set, hence  $SU(n, \mathbb{C})$  is a so called *compact real form* of  $SL(n, \mathbb{C})$ .

## 13 Lecture 11: Solvable Lie Algebras and Lie's Theorem

### 13.1 The Derived Subalgebra

The following lemma is easy to check.

**Lemma 13.1.**  $I, J \in \mathfrak{g}$  ideals. Then

$$\begin{aligned} I + J &= \{x + y \mid x \in I, y \in J\} \\ I \times J &= \{x + iy \mid x \in I, y \in J\} \\ \langle I, J \rangle &= \text{Span} \{x, y \mid x \in I, y \in J\} \end{aligned}$$

are ideals.

**Definition 13.2.**  $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$  is the *derived subalgebra* (or *commutant*) of  $\mathfrak{g}$ .

Note that, somewhat confusingly, the derived subalgebra is actually an ideal of  $\mathfrak{g}$ .

**Lemma 13.3.**

- (i)  $\mathfrak{g}^1$  is abelian
- (ii) If  $I \in \mathfrak{g}$  is an ideal such that  $\mathfrak{g}^1 \subset I$  is abelian, then  $\mathfrak{g}^1 \in I$ .

**Example 13.4.** Let  $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ , then  $\mathfrak{g}^1 = \mathfrak{sl}(n, \mathbb{C})$

$\mathfrak{p} \in \mathfrak{q}$ : This is clear as  $\text{Tr}(xy - yx) = 0$  for every  $x, y \in \mathfrak{gl}(n, \mathbb{C})$ .

$\mathfrak{p} \perp \mathfrak{q}$ : For  $i \neq j$ :  $E_{ii} - E_{jj} \in \mathfrak{g}^1$  and  $E_{ij} = \frac{1}{2}(E_{ii} - E_{jj}) + E_{ij} \in \mathfrak{g}^1$

### 13.2 The Derived Series

We construct the following sequence of subspaces of a Lie algebra  $\mathfrak{g}$ :

$$\begin{aligned} D^0 \mathfrak{g} &= \mathfrak{g} \\ D^1 \mathfrak{g} &= [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}^1 \\ D^2 \mathfrak{g} &= [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] = [\mathfrak{g}^1, \mathfrak{g}^1] \\ &\vdots \\ D^i \mathfrak{g} &= [D^{i-1} \mathfrak{g}, D^{i-1} \mathfrak{g}] \\ &\vdots \end{aligned}$$

Notice that  $\mathfrak{g} = D^0 \mathfrak{g} \supset D^1 \mathfrak{g} \supset D^2 \mathfrak{g} \supset \dots$ . This sequence is known as the *derived series* of  $\mathfrak{g}$ .

**Exercise 13.5.** Show that  $D^i \mathfrak{g}$  is an ideal of  $\mathfrak{g}$  for every  $i \neq 0$ .

**Definition 13.6.**  $\mathfrak{g}$  is *solvable* if there exists  $n \neq 0$ :  $D^n \mathfrak{g} = 0$ .

**Proposition 13.7.** Let  $\mathfrak{g}$  be a Lie algebra, then TFAE:

- (i)  $\mathfrak{g}$  is solvable;
- (ii) There is a sequence of subspaces

$$\mathfrak{g} = \mathfrak{a}^0 \supset \mathfrak{a}^1 \supset \mathfrak{a}^2 \supset \dots \supset \mathfrak{a}^k = 0$$

such that  $\mathfrak{a}^i, \mathfrak{a}^{i-1} \in \mathfrak{a}^{i-1}$ ,  $i = 0, \dots, k-1$  and  $\mathfrak{a}^i \setminus \mathfrak{a}^{i-1}$  is one-dimensional;

- (iii) There is a sequence of subspaces

$$\mathfrak{g} = \mathfrak{a}^0 \supset \mathfrak{a}^1 \supset \mathfrak{a}^2 \supset \dots \supset \mathfrak{a}^k = 0$$

such that  $\mathfrak{a}^i, \mathfrak{a}^{i-1} \in \mathfrak{a}^{i-1}$ ,  $i = 0, \dots, k-1$  and  $\mathfrak{a}^i \setminus \mathfrak{a}^{i-1}$  is abelian.

*Proof.* Observe the condition  $\mathfrak{a}^i, \mathfrak{a}^{i-1} \in \mathfrak{a}^{i-1}$  for  $i = 0, 1, \dots, k-1$  implies that each  $\mathfrak{a}^i$  is a subalgebra of  $\mathfrak{g}$  and is an ideal of  $\mathfrak{a}^i$ .

$\text{p}iiq \text{ } \bar{r} \text{ } \text{p}iiq$ : Let  $\mathfrak{b}^1$  be any subspace in  $\mathfrak{g} \setminus \mathfrak{g}^1$  of codimension one and let  $\mathfrak{a}^1$  be the inverse image of  $\mathfrak{b}^1$  in  $\mathfrak{g}$  under the canonical map  $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{g}^1$ . Then  $\mathfrak{a}^1, \mathfrak{a}^0 \in \mathfrak{g}^1 \in \mathfrak{a}^1$ . In particular  $\mathfrak{a}^1$  is a (codimension one) subalgebra of  $\mathfrak{g}$  hence is itself solvable. By induction on  $\dim \mathfrak{g}$ , the subalgebra  $\mathfrak{a}^1$  has such a chain of subspaces.

$\text{p}iiq \text{ } \bar{r} \text{ } \text{p}iiq$ : Trivial.

$\text{p}iiiq \text{ } \bar{r} \text{ } \text{p}iiq$ : Since  $\mathfrak{a}^i \setminus \mathfrak{a}^{i-1}$  is abelian, Lemma 13.3(ii) implies that  $\mathfrak{a}^i, \mathfrak{a}^s \in \mathfrak{a}^{i-1}$  for all  $i = 0, 1, \dots, k-1$ . We show by induction on  $i$  that  $D^i \mathfrak{g} \in \mathfrak{a}^i$  for all  $i = 0, 1, \dots, k$ . It is trivial for  $i = 0$ . For  $i \geq 1$  we have  $D^i \mathfrak{g} = [D^{i-1} \mathfrak{g}, D^{i-1} \mathfrak{g}] \in \mathfrak{a}^{i-1}, \mathfrak{a}^{i-1} \in \mathfrak{a}^i$ . Therefore  $D^k \mathfrak{g} = 0$  and  $\mathfrak{g}$  is solvable.  $\square$

### 13.3 Lie's Theorem

The following is the key result that is needed to prove Lie's Theorem.

**Theorem 13.8.** Assume that  $\mathbb{F}$  is algebraically closed and  $\text{char } \mathbb{F} = 0$ . Let  $V \neq 0$  be a finite-dimensional vector space over  $\mathbb{F}$ . Let  $\mathfrak{g}$  be a solvable Lie subalgebra of  $\mathfrak{gl}(V, \mathbb{F})$ . Then there exists a nonzero vector  $v \in V$  which is a common eigenvector for all elements of  $\mathfrak{g}$ .

*Proof.* By Proposition 13.7(i)  $\bar{r}$  (ii), there exists an ideal  $\mathfrak{a} \in \mathfrak{g}$  of codimension one. By induction on  $\dim \mathfrak{g}$  there exists a nonzero vector  $v^1 \in V$  and a linear functional  $\lambda : \mathfrak{a} \rightarrow \mathbb{F}$  such that  $av^1 = \lambda(a)v^1$  for all  $a \in \mathfrak{a}$ . Consider the subspace

$$W = \{w \in V \mid aw = \lambda(a)w \text{ } \forall a \in \mathfrak{a}\}.$$

Since  $v^1 \in W$ ,  $W \neq 0$ . Pick  $x \in \mathfrak{g} \setminus \mathfrak{a}$ ; then  $\mathfrak{g} = \mathfrak{a} \oplus \langle x \rangle$ . We claim that  $xW \in W$ . For any  $a \in \mathfrak{a}$  and  $w \in W$  we have

$$axw = xaw = \lambda(a, x)sw = \lambda(a)qxw = \lambda(x)a, xsqw$$

(we can write this because  $\lambda(a, x) \in \mathbb{F}$ ). So the claim will follow if we show that  $\lambda(x)a, xsq = 0$  for all  $a \in \mathfrak{a}$ .

Pick any nonzero  $w \in W$  and let  $W_i = \text{Span}\{w, xw, x^2w, \dots, x^{i-1}w\}$ . Since  $V$  is finite-dimensional, the ascending sequence  $W_1 \subset W_2 \subset \dots$  must stabilize. Let  $n$  be minimal such that  $W_n = W_{n-1}$ . Then  $\dim W_i = i$  for  $1 \leq i \leq n$ .



We show that any  $a \in \mathfrak{a}$  preserves  $W_n$  and in the basis  $w, xw, \dots, x^{n-1}w$ ,  $a$  is an upper triangular matrix with all diagonal elements equal to  $\lambda \rho a \mathfrak{q}$ . That is, we show

$$ax^i w \in \lambda \rho a \mathfrak{q} x^i w + W_{i-1} \quad \forall a \in \mathfrak{a}, i \geq 0.$$

For  $i = 0$  this holds since  $aw = \lambda \rho a \mathfrak{q} w$ . For  $i \geq 1$  we have  $ax^i w = xax^{i-1}w + [a, x]x^{i-1}w$  hence, by induction,

$$ax^i w \in x \lambda \rho a \mathfrak{q} x^{i-1} w + W_{i-1} + \lambda \rho [a, x] x^{i-1} w + W_{i-1} \in \lambda \rho a \mathfrak{q} x^i w + W_{i-1}$$

since  $xW_{i-1} \in W_i$ . Thus the trace of  $a$  on  $W_n$  is  $n\lambda \rho a \mathfrak{q}$ , for all  $a \in \mathfrak{a}$ . Since both  $x$  and  $a \in \mathfrak{a}$  preserve  $W_n$ , we have  $n\lambda \rho [a, x] \mathfrak{q} = \text{Tr}_{W_n} \rho [a, x] \mathfrak{q} = 0$ . Since  $\text{char } \mathbb{F} \nmid n$  this implies that  $\lambda \rho [a, x] \mathfrak{q} = 0$  for all  $a \in \mathfrak{a}$ . This proves the claim that  $xW \in W$ . Since  $\mathbb{F}$  is algebraically closed, there exists an eigenvector  $v$  for  $x$  in  $W$ . Since  $\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{g}$ ,  $v$  is a common eigenvector for all elements of  $\mathfrak{g}$ .  $\square$

Lie's Theorem is frequently stated in terms of representations.

1. A *representation* of a Lie algebra  $\mathfrak{g}$  is a vector space  $V$  together with a Lie algebra homomorphism  $\rho_V : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ . We say that the representation is *complex* (resp. *finite-dimensional*, etc.) if the vector space  $V$  is.
2. A *subrepresentation* of  $V$  is a subspace  $U$  such that  $\rho_V(\mathfrak{g})U \subseteq U$ . (Then  $U$  becomes a representation by  $\rho_U(\mathfrak{g}) = \rho_V(\mathfrak{g})|_U$ .)
3. If  $U$  is a subrepresentation of a representation  $V$  then the quotient space  $V/U$  is a representation by  $\rho_{V/U}(\mathfrak{g}) = \rho_V(\mathfrak{g}) + U$ .

When no confusion can arise we frequently use *module notation*:

$$x.v = \rho_V(x)v, \quad x \in \mathfrak{g}, v \in V.$$

Note that with this notation we have

$$\rho_V([x, y].v) = [x.y.v - y.x.v]$$

for all  $x, y \in \mathfrak{g}$  and  $v \in V$ .

**Corollary 13.9** (Lie's Theorem). *Any complex finite dimensional representation  $V$  of a solvable Lie algebra  $\mathfrak{g}$  has a flag (sequence of subspaces)*

$$0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq \dots \subseteq V_n = V$$

which is

$$\begin{aligned} \mathfrak{g}\text{-stable} &: x.V_i \subseteq V_i \quad \forall x \in \mathfrak{g} \\ \text{and complete} &: \dim V_i = i \end{aligned}$$

**Remark 13.10.** Consequently, choosing a basis for  $V_1$  and extending to  $V_2$ , then extending to  $V_3$ , and so on, produces a basis for  $V$  in which the matrix  $\rho_V(x)$  is upper-triangular for every  $x \in \mathfrak{g}$ . Thus, Lie's theorem is a generalization of the linear algebra result that commuting complex matrices can be simultaneously upper-triangularized.

*Proof.* Since  $\mathfrak{g}$  is solvable, the image in  $\mathfrak{gl}(V)$  under the representation map  $\rho_V$  is a solvable Lie subalgebra of  $\mathfrak{gl}(V)$ . By Theorem 13.8, there is a vector  $v \in V$  such that  $\langle v \rangle$  is a subrepresentation of  $V$ . By induction on  $\dim V$ , the representation  $W = \langle v \rangle$  has a flag

$$0 = W_0 \subset W_1 \subset \dots \subset W_n = W,$$

such that  $x.W_i \in W_i$  for  $x \in \mathfrak{g}$  and  $\dim W_i = i$ . Define  $V_{i-1} = \{u \in V \mid u \in W_i\}$  be the inverse image of  $W_i$  in  $V$  and let  $V_0 = 0$ . Then

$$0 = V_0 \subset V_1 \subset \dots \subset V_{n-1} \subset V$$

is a  $\mathfrak{g}$ -stable complete flag for  $V$ . □

## 14 Lecture 12: Nilpotent Lie Algebras and Engel's Theorem

### 14.1 Lower Central Series

$D_0 \mathfrak{g} = \mathfrak{g}$ ,  $D_i \mathfrak{g} = [\mathfrak{g}, D_{i-1} \mathfrak{g}]$  for  $i \geq 1$ .

**Definition 14.1.**  $\mathfrak{g}$  is *nilpotent* if there exists  $n \geq 0$ :  $D_n \mathfrak{g} = 0$ .

**Proposition 14.2.** Let  $\mathfrak{g}$  be a Lie algebra, then TFAE:

- (i)  $\mathfrak{g}$  is nilpotent
- (ii) There is a sequence of subspaces

$$\mathfrak{g} = \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \dots \supset \mathfrak{a}_k = 0$$

such that  $[\mathfrak{g}, \mathfrak{a}_i] \subset \mathfrak{a}_{i+1}$ ,  $i = 0, \dots, k-1$ .

By induction it can be shown that  $D^i \mathfrak{g} \in D_i \mathfrak{g}$ , so nilpotent implies solvable.

**Example 14.3.**  $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ . Consider,

$$\begin{aligned} \mathfrak{b} &= \text{upper triangular matrices} = \text{span} \{E_{ij} \mid i \leq j\} \\ \mathfrak{n} &= \text{strictly upper triangular matrices} = \text{span} \{E_{ij} \mid i < j\} \end{aligned}$$

We claim that  $\mathfrak{b}$  is solvable and  $\mathfrak{n}$  is nilpotent.

*Proof of claim.* We will instead prove a more general statement. Let  $V$  be a vector space and let

$$F = \{0 \in V_1 \subset V_2 \subset \dots \subset V_k = V\}$$

be a flag (not necessarily complete). The *standard*  $\mathfrak{ag}$  in  $\mathfrak{gl}(V)$  is given by  $V_i$ :  $\text{span} \{e_1, \dots, e_i\}$ . Put

$$\begin{aligned} \mathfrak{b}_F &= \{x \in \mathfrak{gl}(V) \mid xV_i \subset V_i \text{ for all } i\} \\ \mathfrak{n}_F &= \{x \in \mathfrak{gl}(V) \mid xV_i \subset V_{i-1} \text{ for all } i\} \\ \mathfrak{a}_k &= \{x \in \mathfrak{gl}(V) \mid xV_i \subset V_{i-k} \text{ for all } i\} \end{aligned}$$

with  $V_k = 0$  for  $k > 0$ . Note that  $\mathfrak{b}_F = \mathfrak{b}$ ,  $\mathfrak{n}_F = \mathfrak{n}$ . Abbreviate  $\mathfrak{a}_k = \mathfrak{a}_k \cap \mathfrak{b}_F$ . Obviously,  $\mathfrak{a}_k \subset \mathfrak{a}_{k-1}$  and  $[\mathfrak{a}_k, \mathfrak{a}_k] \subset \mathfrak{a}_k$ , hence  $\mathfrak{n}_F$  is nilpotent. Since diagonal entries of  $xy$  and  $yx$  coincide for  $x, y \in \mathfrak{b}_F$  (check!). We have  $D^1 \mathfrak{b}_F \subset \mathfrak{a}_1$ , and by induction,  $D^i \mathfrak{b}_F \subset \mathfrak{a}_{2i}$ . This implies that  $\mathfrak{b}_F$  is solvable. □

<sup>vii</sup>This called the standard Borel subalgebra of  $\mathfrak{gl}(n, \mathbb{C})$

## 14.2 Engel's Theorem

The *adjoint representation* of a Lie algebra  $\mathfrak{g}$  is  $\mathfrak{g}$  itself with

$$\rho_{\mathfrak{g}} = \text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}), \quad \text{ad } x(qy) = [x, y].$$

That  $\text{ad } x, \text{ad } y = \text{ad}[x, y]$  is equivalent to the Jacobi identity (assuming  $[x, y] = [y, x]$ ). The map  $\text{ad}$  is also called the *adjoint action* of  $\mathfrak{g}$  (on itself).

**Theorem 14.4** (Key Result for Engel's Theorem). *Let  $V$  be an arbitrary field. Let  $V \neq 0$  be a finite dimensional vector space. Let  $\mathfrak{g} \in \mathfrak{gl}(V)$  be a Lie subalgebra consisting of nilpotent transformations. Then there exists a nonzero vector  $v \in V$  such that  $xv = 0$  for all  $x \in \mathfrak{g}$ .*

*Proof.* We use induction on  $n = \dim \mathfrak{g}$ . The conclusion is trivial for  $n = 0, 1$  so suppose  $n \geq 1$ . Let  $I \subseteq \mathfrak{g}$  be a maximal subalgebra. Consider  $\mathfrak{g}$  as a representation of  $I$  via restriction of the adjoint action. Then  $I$  is a subrepresentation, hence  $\mathfrak{g}/I$  is a representation of  $I$ . Explicitly, we have a Lie algebra homomorphism  $\rho : I \rightarrow \mathfrak{gl}(\mathfrak{g}/I)$  given by  $\rho(a)(x+I) = [a, x] + I$  for all  $a \in I, x \in \mathfrak{g}$ . Since  $a$  is nilpotent,  $x \in \mathfrak{g}$   $\Rightarrow [a, x] = a \cdot x - x \cdot a = \rho(a)x - x\rho(a)$  is nilpotent (binomial theorem implies  $\rho(a)^k = \rho(a)^{2k-1} = 0$  if  $a^k = 0$ ). Thus, by the induction hypothesis applied to  $\rho(I)$ , there exists a nonzero  $z \in \mathfrak{g}/I$  such that  $\rho(I)z = 0$ . That is,  $[I, z] \subseteq I$ . That implies that the vector space  $I \cdot z$  is a subalgebra of  $\mathfrak{g}$  strictly containing  $I$ . By maximality of  $I$ ,  $\mathfrak{g} = I \cdot z$ . Let

$$W = \{v \in V \mid av = 0 \ \forall a \in I\}.$$

By the induction hypothesis applied to  $I$ ,  $W \neq 0$ . Furthermore,  $W$  is  $z$ -invariant: if  $v \in W$  and  $a \in I$  then  $azv \in zav = [a, zv] = 0$  (because  $[a, z] \in I$ ). Since  $z$  is nilpotent,  $z|_W$  is nilpotent, so there exists  $v \in W$  such that  $zv = 0$ . Then  $\mathfrak{g}v = 0$ .  $\square$

**Corollary 14.5.** *Let  $V$  be an arbitrary field. Let  $V \neq 0$  be a finite dimensional vector space. Let  $\mathfrak{g} \in \mathfrak{gl}(V)$  be a Lie subalgebra consisting of nilpotent transformations. Then there exists a complete flag  $0 = V_0 \subset V_1 \subset \dots \subset V_n = V$  such that  $xV_i \subseteq V_{i-1}$  for all  $x \in \mathfrak{g}$ .*

**Remark 14.6.** The conclusion is equivalent to there being a basis for  $V$  in which all  $x \in \mathfrak{g}$  are strictly upper-triangular.

**Corollary 14.7.** *If  $\mathfrak{g} \in \mathfrak{gl}(V)$  consists of nilpotent transformations, then  $\mathfrak{g}$  is nilpotent.*

*Proof.* Pick a basis in which all  $x \in \mathfrak{g}$  are strictly upper-triangular. Then  $\mathfrak{g}$  is contained in a nilpotent Lie algebra hence is nilpotent.  $\square$

**Theorem 14.8.** *If  $\mathfrak{g}$  is a finite dimensional Lie algebra, then  $\mathfrak{g}$  is nilpotent iff  $\text{ad } x \in \mathfrak{g}$ , the map  $\text{ad } x : \mathfrak{g} \rightarrow \mathfrak{g}, y \mapsto [x, y]$ , is nilpotent.*

*Proof.* If  $\mathfrak{g}$  is nilpotent, then  $\text{ad } x^n = 0$  for  $n \geq 0$ . Conversely, if  $\text{ad } x$  is nilpotent for all  $x \in \mathfrak{g}$ , then  $\text{ad } \mathfrak{g} = \mathfrak{g}$  is nilpotent by Corollary 14.7. That implies  $\mathfrak{g}$  is nilpotent.  $\square$

## 15 Lecture 13: The Radical; semisimple Lie Algebras; Semi-Direct Products and Levi's Theorem

### 15.1 The Radical of a Lie Algebra

**Definition 15.1.** A Lie algebra  $\mathfrak{g}$  is called *semisimple* if  $\mathfrak{r}(\mathfrak{g})$  is the only solvable ideal in  $\mathfrak{g}$ .

**Definition 15.2.** A Lie algebra  $\mathfrak{g}$  is called *simple* if it is non-abelian and  $\{0\}$  and  $\mathfrak{g}$  are the only ideals.

**Lemma 15.3.**  $\mathfrak{g}$  simple  $\iff \mathfrak{g}$  semisimple.

*Proof.* Let  $\mathfrak{h} = I \in \mathfrak{g}$  be a solvable ideal.  $\mathfrak{g}$  simple  $\iff I = \mathfrak{g}$   $\iff$  the ideal  $\mathfrak{g}/\mathfrak{h}$  is proper  $\iff \mathfrak{g}/\mathfrak{h} = 0$   $\iff \mathfrak{g}$  abelian, which is a contradiction.  $\square$

**Example 15.4.**  $\mathfrak{sl}(2, \mathbb{C})$  is simple. Use the map  $\text{ad } h$ .

**Proposition 15.5.** Let  $\mathfrak{g}$  be a finite-dimensional Lie algebra. Then the set of all solvable ideals of  $\mathfrak{g}$  has a unique maximal element.

**Definition 15.6.** The unique maximal solvable Lie algebra of  $\mathfrak{g}$  is called the *radical* of  $\mathfrak{g}$  and is denoted  $\text{rad } \mathfrak{g}$ .

*Proof.* Let  $\text{rad } \mathfrak{g}$  be the sum of all solvable ideals of  $\mathfrak{g}$ . We need to show  $\text{rad } \mathfrak{g}$  is itself solvable. If  $I$  and  $J$  are solvable ideals of  $\mathfrak{g}$ , then  $\frac{I+J}{I} = \frac{J}{I \times J}$ . Since  $J$  is solvable,  $\frac{J}{I \times J}$  is solvable. Since  $I$  and  $\frac{I+J}{I}$  are solvable  $\iff I+J$  solvable. By induction, any finite sum is solvable.  $\mathfrak{g}$  finite dimensional implies there exists  $N \geq 0$  such that  $D^N I = 0$  for all solvable ideals of  $\mathfrak{g}$  ( $N = \dim \mathfrak{g}$ ). So  $D^N \text{rad } \mathfrak{g} = 0$ .  $\square$

**Exercise 15.7.**  $\text{rad } \mathfrak{g} / \{\text{rad } \mathfrak{g}\} = 0$ , i.e.  $\mathfrak{g} / \{\text{rad } \mathfrak{g}\}$  is semisimple. (Note  $\mathfrak{g}$  is semisimple iff  $\text{rad } \mathfrak{g} = 0$ )

**Definition 15.8.** A finite-dimensional Lie algebra  $\mathfrak{g}$  is *reductive* if  $\text{rad } \mathfrak{g} = \mathfrak{z}(\mathfrak{g})$ .

## 16 Lecture 14: Jordan Decomposition and Cartan's First Criterion

Recall that any matrix  $x \in M_n(\mathbb{C})$  is similar to a matrix in *Jordan canonical form*:

$$x \sim \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & \vdots & \ddots & \\ & & & \lambda_1 & & \\ & & & & \ddots & \\ & & & & & \lambda_k & 1 \\ & & & & & \vdots & \vdots \\ & & & & & \vdots & 1 \\ & & & & & & \lambda_k \end{pmatrix}$$



by

$$\# \begin{matrix} fpTq & pcApTqq & \tilde{N} & pfpTq & I_1, \dots, fpTq & I_kq. \end{matrix}$$

So the system  $\begin{matrix} PpTq & a_i \text{ mod } I_i & @i \\ PpTq & 0 \text{ mod } T \end{matrix}$  has to be a solution  $\text{mod } pcApTqq$ . Put  $QpTq = T - PpTq$  and  $A_S: PpAq, A_n = QpAq$  correspondingly,

$$V = \bigoplus_{i=1}^k V_i$$

where  $V_i = \{v \in V \mid pA - a_i q^{m_i} v = 0\}$ . Then  $A_S|_{V_i} = a_i \text{Id}_{V_i} @i$  i.e.  $pA - a_i q \text{Id}_{V_i} = 0$ . This implies  $A_S$  is diagonal on each  $V_i$ , thus  $A_S$  is semisimple.

Then  $A_n = A - A_S$  is nilpotent on each  $V_i$

$$pA_n|_{V_i} q^{m_i} = pA|_{V_i} - A_S|_{V_i} q^{m_i} = pA - a_i q^{m_i} = 0.$$

Hence  $A = A_S + A_n$  by construction  $(A_S, A_n) = (PpAq, QpAq) = 0$ . To show uniqueness, suppose  $A = S + N$  also. Then  $A = A_S + A_n = S + N \tilde{N} = A_S + S + N - A_n$  is both semisimple and nilpotent  $\tilde{N}$  they are 0.

c)  $\text{ad } A = \text{ad } pA_S q + \text{ad } pA_n q$  as in part c) we get  $\text{ad } pA_S q$  is semisimple,  $\text{ad } pA_n q$  nilpotent, and they commute. So by uniqueness  $\text{ad } pA_S q = \text{ad } pA_S q$  and  $\text{ad } pA_n q = \text{ad } pA_n q$ .  $\square$

**Note 16.3.** If  $\psi: \text{End } pV q \rightarrow \text{End } pW q$  is linear and satisfies  $A$  is semisimple (nilpotent) implies  $\psi pA q$  is semisimple (nilpotent), and  $(A, B) = 0 \tilde{N} (A, B) = 0$  then:

$$\begin{aligned} \psi pA_S q &= \psi pA_S q, \\ \psi pA_n q &= \psi pA_n q. \end{aligned}$$

## 16.1 Cartan's First Criterion

**Proposition 16.4.** Let  $\mathbb{F}$  be an algebraically closed field of characteristic zero, and  $V$  be a finite-dimensional vector space over  $\mathbb{F}$ . Let  $\mathfrak{g}$  be a Lie subalgebra of  $\mathfrak{gl}(V)$ . Then

- (a) If  $\mathfrak{g}$  is solvable, then  $\text{Tr } pxy q = 0$  for all  $x \in \mathfrak{g}, y \in \mathfrak{g}^1$  (recall  $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ );
- (b) If  $\text{Tr } pxy q = 0$  for all  $x, y \in \mathfrak{g}$ , then  $\mathfrak{g}$  is solvable.

*Proof.* .... (to be typed up) ....  $\square$

**Theorem 16.5** (Cartan's First Criterion - algebraically closed field case). Let  $\mathbb{F}$  be an algebraically closed field of characteristic zero, and  $V$  be a finite-dimensional vector space over  $\mathbb{F}$ . Then  $\mathfrak{g}$  is solvable if and only if  $\text{Tr } pax q = \text{Tr } pay q = 0$  for all  $x \in \mathfrak{g}$  and  $y \in \mathfrak{g}^1$ .

*Proof.* If  $\mathfrak{g}$  is solvable, then  $\text{ad } \mathfrak{g} \in \mathfrak{gl}(\mathfrak{g})$  is solvable hence the conclusion follows from part (a) of Proposition 16.4.

Conversely, if  $\text{Tr } pax q = \text{Tr } pay q = 0$  for all  $x \in \mathfrak{g}, y \in \mathfrak{g}^1$ , then  $\text{ad } \mathfrak{g}^1$  is solvable by part (b) of Proposition 16.4. Since  $\text{ad } \mathfrak{g}^1 \in \mathfrak{gl}(\mathfrak{g}^1)$  and  $\mathfrak{g}^1$  is abelian hence solvable, it follows that  $\mathfrak{g}^1$  is solvable. Say  $D^n \mathfrak{g}^1 = 0$ . But then  $D^n \mathfrak{g} = D^n \mathfrak{g}^1 = 0$  so  $\mathfrak{g}$  is solvable.  $\square$

## 17 Lecture 15: Cartan's Second Criterion

This lecture will focus on a method for checking whether or not a given Lie algebra is semisimple or solvable based on the so called Killing<sup>viii</sup> form.

### 17.1 Invariant Symmetric Bilinear Forms

A symmetric bilinear form  $b: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$  is a function satisfying  $b(\lambda x + \mu y, z) = \lambda b(x, z) + \mu b(y, z)$  and  $b(x, y) = b(y, x)$  for all  $x, y, z \in \mathfrak{g}$  and  $\lambda, \mu \in \mathbb{C}$ .

**Definition 17.1.** A symmetric bilinear form  $b: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$  is

(i) *invariant* if

$$b(\rho x, \rho y) = b(x, y) \quad \forall x, y, z \in \mathfrak{g}, \quad (17.1)$$

(ii) *non-degenerate* if  $\forall x \in \mathfrak{g} \exists y \in \mathfrak{g} : b(x, y) \neq 0$ .

**Remark 17.2.** The reason for the name “invariant” is as follows. First, a vector  $v$  in a representation of  $\mathfrak{g}$  is called *invariant* if  $x.v = 0$  for all  $x \in \mathfrak{g}$  (recall we write  $x.v$  for  $\rho_V(x)v$ ). Now if  $V$  is a representation of  $\mathfrak{g}$  then one can verify that the space of all bilinear forms on  $V$  is a representation of  $\mathfrak{g}$  with action  $\rho_V(x).b(v, w) = b(\rho_V(x)v, w) - b(v, \rho_V(x)w)$ . When  $V = \mathfrak{g}$  is the adjoint representation, this is equivalent to  $\rho_V(x).b(y, z) = b(\rho(x)y, z) - b(y, \rho(x)z)$ . Then (17.1) is equivalent to  $x.b = 0$  (check!), i.e.  $b$  is an invariant vector of the representation  $\rho_V \circ \mathfrak{b} : \mathfrak{g} \rightarrow \mathfrak{gl}(V \otimes V)$ .

**Definition 17.3.**

(i) Given a finite-dimensional representation  $V$  of  $\mathfrak{g}$  the *trace form* on  $\mathfrak{g}$  (with respect to  $V$ ) is  $b_V: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ ,

$$b_V(x, y) = \text{Tr}(\rho_V(x)\rho_V(y)), \quad \forall x, y \in \mathfrak{g}.$$

(ii) The trace form on  $\mathfrak{g}$  with respect to the adjoint representation is called the *Killing form* and is denoted by  $\kappa$  or  $\kappa^{\mathfrak{g}}$ . Explicitly:

$$\kappa(x, y) = \text{Tr}(\text{ad}_x \text{ad}_y), \quad \forall x, y \in \mathfrak{g}.$$

**Exercise 17.4.** Show that the trace form (with respect to any finite-dimensional representation  $V$ ) is an invariant symmetric bilinear form on  $\mathfrak{g}$ .

**Lemma 17.5.** If  $b: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$  is an invariant symmetric bilinear form on  $\mathfrak{g}$  and  $I \in \mathfrak{g}$  is an ideal, then

$$I^{\perp} = \{x \in \mathfrak{g} \mid b(x, z) = 0 \quad \forall z \in I\}$$

is an ideal in  $\mathfrak{g}$ .

*Proof.*  $x \in I^{\perp}$ , we show that  $\rho(x)y \in I^{\perp}$  for all  $y \in \mathfrak{g}$ . Indeed, for all  $z \in I$  we have

$$b(\rho(x)y, z) = b(\rho(x), \rho(y)z) = 0$$

hence,  $\rho(x)y \in I^{\perp}$ . □

**Corollary 17.6.**  $I^{\perp} = \ker \rho_{\mathfrak{g}}$  is an ideal in  $\mathfrak{g}$ .

<sup>viii</sup>Wilhelm Killing (1847 – 1923)

**Example 17.7.** Let  $\mathfrak{g} = \langle a, b \rangle$  be a two-dimensional vector space with bracket uniquely determined by the condition  $[a, b] = a$ . This is the unique non-abelian Lie algebra of dimension 2 up to isomorphism (exercise). In the ordered basis  $\{a, b\}$  the matrices for  $\text{ad } a$  and  $\text{ad } b$  are  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  and

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  respectively. Thus it is easy to check that  $\kappa(a, a) = \kappa(a, b) = 0$  and  $\kappa(b, b) = 1$ .

**Example 17.8.** a) The space of column vectors  $\mathbb{R}^n$  is a representation of  $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R})$ , called the *vector representation* (or the *tautological representation*). The trace form is just  $b(x, y) = \text{Tr}(xy)$ . When  $n = 2$ , put

$$\begin{array}{cc} e & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & f & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ h & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & I & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

with respect to the ordered basis  $\{e, f, h, I\}$ , the Gram matrix of the trace form on  $\mathfrak{gl}(2, \mathbb{R})$  is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

For example,  $b(e, e) = \text{Tr} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0$ .

b) By a similar but much longer calculation one can see that the Gram matrix of the Killing form  $\kappa$  on  $\mathfrak{gl}(2, \mathbb{R})$  is

$$\begin{pmatrix} 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

## 17.2 Further reading: reductive Lie algebras

**Theorem 17.9.** *If there exists a representation  $V$  of  $\mathfrak{g}$  such that the trace form  $b_V$  is non-degenerate, then  $\mathfrak{g}$  is reductive i.e.  $\text{rad } \mathfrak{p} \subseteq \mathfrak{z}$ .*

*Proof.*  $\text{rad } \mathfrak{p} \subseteq \mathfrak{z}$  is always true, so all that remains is to show the reverse containment. That is, we must show  $\text{rad } \mathfrak{p} \subseteq \mathfrak{z}$ .

(1)  $x \in \text{rad } \mathfrak{p}$  acts by 0 on any irreducible  $W$  (the proof of this is omitted). This implies that  $x \in \ker \rho_W$ .

(2) For

$$0 = \rho_W(x) = \rho_W([X, Y]) = [\rho_W(X), \rho_W(Y)] = [X_W, Y_W]$$

we have that  $b_V(x, x) = b_W(x, x)$  by

$$\text{Tr} \begin{pmatrix} X_W & & & \\ & X_W & & \\ & & Y_W & \\ & & & Y_W \end{pmatrix} = \text{Tr} X_W Y_W = \text{Tr} X_W Y_W$$

(3) By induction on  $\dim V$  we show that  $x \in \ker \rho_V$  implies  $x = 0$ . □

**Theorem 17.10.** *Each classical Lie algebra  $\mathfrak{g} \in \{\mathfrak{sl}(n), \mathfrak{so}(n), \mathfrak{sp}(n)\}$  is reductive.*

*Proof.*  $b_{\mathfrak{g}}$  is non-degenerate  $\Rightarrow \mathfrak{g}$  reductive. □



### 17.3 Cartan's Criteria

**Lemma 17.11.** *Let  $I$  be an ideal of a finite-dimensional Lie algebra  $\mathfrak{g}$  with Killing form  $\kappa^{\mathfrak{g}}$ . Let  $\kappa^I$  be the Killing form on  $I$  regarded as a Lie algebra in its own right. Then  $\kappa^I(x, y) = \kappa^{\mathfrak{g}}(x, y)$  for all  $x, y \in I$ .*

*Proof.* Choose a linear complement  $V$  to  $I$  in  $\mathfrak{g}$ . Thus  $\mathfrak{g} = I \oplus V$  as vector spaces. Consider a basis for  $\mathfrak{g}$  which is the union of bases for  $I$  and  $V$ . Let  $x, y \in I$ . In the chosen basis,

$$\text{ad } x = \begin{pmatrix} A_x & B_x \\ 0 & 0 \end{pmatrix}, \quad \text{ad } y = \begin{pmatrix} A_y & B_y \\ 0 & 0 \end{pmatrix},$$

for some matrices  $A_x, B_x, A_y, B_y$  of appropriate size. Note that  $A_x$  and  $A_y$  are the matrices for  $\text{ad}^I x : I \rightarrow I$  and  $\text{ad}^I y : I \rightarrow I$ , where  $\text{ad}^I$  denotes the adjoint representation of  $I$ . We have

$$\text{pad } x \text{ pad } y = \begin{pmatrix} A_x A_y & A_x B_y \\ 0 & 0 \end{pmatrix}$$

so that

$$\kappa^{\mathfrak{g}}(x, y) = \text{Tr}(\text{pad } x \text{ pad } y) = \text{Tr}(A_x A_y) = \text{Tr}(\text{ad}^I x \text{ ad}^I y) = \kappa^I(x, y).$$

□

**Lemma 17.12.** *Let  $\mathfrak{g}$  be a finite-dimensional Lie algebra with nonzero radical. Then there exists a nonzero abelian ideal of  $\mathfrak{g}$ .*

*Proof.* Let  $I$  be the radical of  $\mathfrak{g}$ . Since  $I$  is an ideal of  $\mathfrak{g}$ , so is  $\mathfrak{r}I, I^2$  (by the Leibniz Rule form of Jacobi identity:  $\mathfrak{r}x, \mathfrak{r}y, z^2 = \mathfrak{r}x, y^2, z^2 - \mathfrak{r}y, \mathfrak{r}x, z^2$ ). Repeating this argument we see that each term of the derived series for  $I$  is an ideal of  $\mathfrak{g}$ . In particular the last nonzero term of the series is an abelian ideal of  $\mathfrak{g}$ . □

**Theorem 17.13** (Cartan's Criteria). *Let  $\mathbb{F}$  be any field of characteristic zero. Then:*

- a)  $\mathfrak{g}$  solvable  $\iff \kappa^{\mathfrak{g}}(\mathfrak{r}\mathfrak{g}, \mathfrak{g}) = 0$ ;
- b)  $\mathfrak{g}$  semisimple  $\iff \kappa$  non-degenerate.

*Proof.*

a) We proved this in the previous lecture in the case when  $\mathbb{F}$  is algebraically closed. In general, let  $\bar{\mathbb{F}} = \overline{\mathbb{F}}$  be the algebraic closure of  $\mathbb{F}$ . Let  $\bar{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{F}} \bar{\mathbb{F}}$  be the Lie algebra over the algebraic closure  $\bar{\mathbb{F}}$  of  $\mathbb{F}$ . (This process is called *extension of scalars*. The Lie bracket on  $\bar{\mathfrak{g}}$  is  $\mathfrak{r}\lambda \otimes x, \mu \otimes y = \lambda \mu \otimes \mathfrak{r}x, y$  for all  $x, y \in \mathfrak{g}$  and  $\lambda, \mu \in \bar{\mathbb{F}}$ .) Then

$$\begin{aligned} \mathfrak{g} \text{ solvable} &\iff \bar{\mathfrak{g}} \text{ solvable} \\ \iff \kappa^{\bar{\mathfrak{g}}}(\bar{\mathfrak{r}}\bar{\mathfrak{g}}, \bar{\mathfrak{g}}) &= 0 \quad \text{by last time} \\ \iff \kappa^{\mathfrak{g}}(\mathfrak{r}\mathfrak{g}, \mathfrak{g}) &= 0. \end{aligned}$$

b) We prove the contrapositive. Suppose  $\mathfrak{g}$  is not semisimple. By Lemma 17.12, there exists a nonzero abelian ideal  $\mathfrak{a}$  of  $\mathfrak{g}$ . Now for any  $x \in \mathfrak{g}$  and  $a \in \mathfrak{a}$ , the composition  $\text{pad } x \text{ pad } a$  maps  $\mathfrak{g}$  into  $\mathfrak{a}$ , since  $\mathfrak{a}$  is an ideal. Since  $\mathfrak{a}$  is abelian,  $\text{pad } a \text{ pad } x \text{ pad } a$  is identically zero. Thus  $\text{pad } x \text{ pad } a$  is nilpotent, hence has trace zero. Thus  $\kappa^{\mathfrak{g}}(x, a) = 0$  for all  $x \in \mathfrak{g}$  and  $a \in \mathfrak{a}$ . Since  $\mathfrak{a} \neq 0$ , this shows  $\kappa$  is degenerate.

For the reverse implication, suppose  $\mathfrak{g}$  is semisimple. Consider the space  $I = \{x \in \mathfrak{g} \mid \kappa(x, y) = 0 \forall y \in \mathfrak{g}\}$  (this is called the *radical* or *kernel* of the form  $\kappa$ ). We claim that  $I$  is solvable hence equal to zero. First,  $I$  is an ideal by associativity of  $\kappa$ . By Lemma 17.11,  $\kappa^I(\mathfrak{r}I, I) = \kappa^{\mathfrak{g}}(\mathfrak{r}I, I) \in \kappa^{\mathfrak{g}}(I, \mathfrak{g}) = 0$ . By part a),  $I$  is solvable. Since  $\mathfrak{g}$  is semisimple,  $I = 0$ . Thus,  $\kappa$  is non-degenerate. □

**Proposition 17.14.** *If  $\mathfrak{g}$  is semisimple, and  $I \in \mathfrak{g}$  is an ideal, then  $I^{\mathbb{K}} = \{x \in \mathfrak{g} \mid \kappa(x, y) = 0 \forall y \in \mathfrak{g}\}$  is an ideal of  $\mathfrak{g}$ , and  $\mathfrak{g} = I \oplus I^{\mathbb{K}}$ .*

*Proof.*  $I^{\mathbb{K}}$  is an ideal by Lemma 17.5. We claim that  $I \times I^{\mathbb{K}} = \{0\}$ . Let  $x, y, z \in I \times I^{\mathbb{K}}$ . Then  $\kappa(x, ry, zs) = 0$  simply because  $x \in I^{\mathbb{K}}$  and  $ry, zs \in I$ . By Lemma 17.11,  $\kappa(x, ry, zs) = 0$  too. By Cartan's First Criterion, applied to  $I \times I^{\mathbb{K}}$ , we conclude that  $I \times I^{\mathbb{K}}$  is solvable. Since  $\mathfrak{g}$  is semisimple,  $I \times I^{\mathbb{K}} = \{0\}$ .

Next we show that  $I = I^{\mathbb{K}}$ . Let  $\{x_i\}_{i=1}^m$  be a basis for  $I$  and extend it to a basis  $\{x_i\}_{i=1}^n$  for  $\mathfrak{g}$ . By Cartan's Second Criterion,  $\kappa$  is non-degenerate. Therefore there exists a corresponding dual basis  $\{x^j\}_{j=1}^n$  ( $\langle x^j, x_i \rangle = \delta_{ij}$ ,  $x^j \in \mathfrak{g}$ ).  $\kappa(x, y) = \sum_{j=1}^n \langle x^j, x \rangle \langle x^j, y \rangle$  is an isomorphism, so to the linear functional  $\xi^j \in \mathfrak{g}^*$ ,  $\xi^j(x) = \delta_{ij}$ , there corresponds a  $x^j \in \mathfrak{g}$  satisfying  $\kappa(x_i, x^j) = \delta_{ij}$ . Then, clearly,  $x^j \in I^{\mathbb{K}}$  for  $j = m+1, m+2, \dots, n$ . This shows that  $\dim I^{\mathbb{K}} = n - m$ . Since  $I \times I^{\mathbb{K}} = \{0\}$  we have  $\dim I + \dim I^{\mathbb{K}} = \dim \mathfrak{g} = n$ . Thus  $\mathfrak{g} = I \oplus I^{\mathbb{K}}$ . □

We can now prove the following theorem, which motivates the name "semisimple".

**Theorem 17.15.** *Let  $\mathbb{F}$  be a field of characteristic zero and  $\mathfrak{g}$  be a finite-dimensional Lie algebra over  $\mathbb{F}$ . Then  $\mathfrak{g}$  is semisimple if and only if  $\mathfrak{g} = I_1 \oplus I_2 \oplus \dots \oplus I_k$ , where  $I_j$  are simple (as Lie algebras) ideals. Moreover, in this case every ideal of  $\mathfrak{g}$  is equal to  $\sum_{j \in S} I_j$  for some subset  $S \subseteq \{1, 2, \dots, k\}$ .*

*Proof.* Suppose  $\mathfrak{g}$  is semisimple. Let  $I_1$  be a minimal nonzero ideal of  $\mathfrak{g}$ . If  $I_1 = \mathfrak{g}$  we are done. Otherwise, by Proposition 17.14,  $\mathfrak{g} = I_1 \oplus I_1^{\mathbb{K}}$ . Since any solvable ideal  $I_1^{\mathbb{K}}$  would be a solvable ideal of  $\mathfrak{g}$ ,  $I_1^{\mathbb{K}}$  is also semisimple. By induction on  $\dim \mathfrak{g}$  we are done.

The converse is an exercise (see below).

For the last claim, first suppose that  $I$  is a simple ideal of  $\mathfrak{g}$ . Then  $\mathfrak{g} = \sum_{j=1}^k \mathfrak{r}_j I_j$ . By simplicity of  $I$ , all but one summand is zero. Say  $\mathfrak{r}_j I_j = 0$ . Then  $\mathfrak{r}_j I_j = I \cap I_j$  by simplicity of  $I$  and  $I_j$ . Now suppose  $I$  is any ideal of  $\mathfrak{g}$ . Then  $\mathfrak{g} = I \oplus I^{\mathbb{K}}$  and any ideal of  $I$  is an ideal of  $\mathfrak{g}$ . In particular,  $I$  is semisimple. Say  $I = J_1 \oplus \dots \oplus J_r$  for some simple ideals  $J_i$  of  $I$ . But then  $J_i$  are simple ideals of  $\mathfrak{g}$  hence each  $J_i$  is equal to one of the  $I_j$ 's. This proves the claim. □

**Exercise 17.16.** Let  $\mathbb{F}$  be a field of characteristic zero and  $\mathfrak{g}$  be a finite-dimensional Lie algebra over  $\mathbb{F}$ . Suppose that  $\mathfrak{g} = I_1 \oplus I_2 \oplus \dots \oplus I_k$ , where  $I_j$  are simple (as Lie algebras) ideals. Prove that  $\mathfrak{g}$  is semisimple.

## 18 Lecture 16: Semisimple Lie Algebras; the Casimir Operator

Before moving on we point out two important corollaries of previous lecture.

**Corollary 18.1.**  $\mathfrak{g} = \mathfrak{g}^{\mathbb{K}}$  if  $\mathfrak{g}$  is semisimple.

*Proof.*

$$\begin{aligned} \mathfrak{g}^{\mathbb{K}} &= \mathfrak{r} \left( \sum_{j=1}^k I_j \right)^{\mathbb{K}} \\ &= \sum_{j=1}^k \mathfrak{r}_j \left( I_j \right)^{\mathbb{K}} \quad \text{since for } j \neq k \quad \mathfrak{r}_j I_j, I_k \in I_j \times I_k = \{0\} \\ &= \sum_{j=1}^k \mathfrak{r}_j I_j \\ &= \mathfrak{g}. \end{aligned}$$

□

**Corollary 18.2.** *If  $\mathfrak{g}$  is semisimple, then  $I$  and  $\mathfrak{g}\{I$  are semisimple for any ideal  $I$  of  $\mathfrak{g}$ .*

*Proof.* If  $I$  is any ideal of  $\mathfrak{g}$  then  $\mathfrak{g} = I \oplus I^\perp$  with respect to the Killing form. Then for any ideal  $J$  of  $I$  we have  $\mathfrak{r}J \subseteq \mathfrak{r}I, J \subseteq \mathfrak{r}I^\perp, J \subseteq J$  since  $\mathfrak{r}I^\perp, J \subseteq I^\perp \times I = 0$ . Furthermore,  $\mathfrak{g}\{I = I^\perp$  which is an ideal of  $\mathfrak{g}$  hence semisimple by the previous part.  $\square$

We also prove the following uniqueness theorem for the Killing form on a simple Lie algebra. Here we do need algebraically closed field to ensure the existence of at least one eigenvalue.

**Proposition 18.3.** *Let  $\mathfrak{g}$  be a finite-dimensional Lie algebra over an algebraically closed field of characteristic zero. Then there exists a unique (up to nonzero scalar multiple) invariant nondegenerate symmetric bilinear form on  $\mathfrak{g}$ .*

*Proof.* We know the Killing form  $\kappa$  is one such form. Suppose  $\beta : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{F}$  is another. Then we have two vector space isomorphisms  $\kappa_1 : \mathfrak{g} \rightarrow \mathfrak{g}^*$ ,  $x \mapsto \kappa(x, \cdot)$  and similarly  $\beta_1 : \mathfrak{g} \rightarrow \mathfrak{g}^*$ . Let  $x_0 \in \mathfrak{g}$  be an eigenvector of the non-singular transformation  $\beta_1^{-1}\kappa_1$  of  $\mathfrak{g}$ . Thus  $\kappa_1(x_0, y) = \lambda\beta_1(x_0, y)$  for some nonzero  $\lambda \in \mathbb{F}$ . This means  $\kappa(x_0, y) = \lambda\beta(x_0, y)$  for all  $y \in \mathfrak{g}$ . Let

$$I = \{x \in \mathfrak{g} \mid \kappa(x, y) = \lambda\beta(x, y) \text{ for all } y \in \mathfrak{g}\}.$$

We claim  $I$  is an ideal of  $\mathfrak{g}$ . For any  $x \in I$  and  $y, z \in \mathfrak{g}$  we have

$$\kappa([x, y], z) = \kappa(x, [y, z]) = \lambda\beta(x, [y, z]) = \lambda\beta([x, y], z).$$

Thus  $[x, y] \in I$  for all  $x \in I, y \in \mathfrak{g}$ . So  $I$  is an ideal of  $\mathfrak{g}$ . Since  $\mathfrak{g}$  is simple and  $I$  is nonzero (containing  $x_0$ ), we have  $I = \mathfrak{g}$ . This shows that  $\kappa(x, y) = \lambda\beta(x, y)$  for all  $x, y \in \mathfrak{g}$ .  $\square$

## 18.1 The Casimir Operator

Let  $\mathfrak{g}$  be a finite-dimensional Lie algebra, and suppose  $\chi, \psi$  is a non-degenerate invariant symmetric bilinear form on  $\mathfrak{g}$ . For example, if  $\mathfrak{g}$  is semi-simple we could use  $\chi(x, y) = \kappa(x, y)$  (the Killing form) and if  $\mathfrak{g} = \mathfrak{gl}_n$  we can use  $\chi(x, y) = \text{Tr}(xy)$ . (One can show that the existence of such a form is equivalent to that  $\text{rad } \mathfrak{pg} = \{0\}$  i.e.  $\mathfrak{g}$  is reductive.)

Let  $V$  be a representation of  $\mathfrak{g}$ . The *Casimir operator* (on  $V$  with respect to  $\chi, \psi$ ), denoted  $C_V \in \text{End}(V)$ , is defined by

$$C_V = \sum_{i=1}^n \rho_V(x_i) \rho_V(x^i),$$

where  $\{x_i\}_{i=1}^n$  is a basis for  $\mathfrak{g}$  and  $\{x^i\}_{i=1}^n$  is the corresponding dual basis with respect to  $\chi, \psi$ .

**Proposition 18.4** (Properties of the Casimir operator). (a) *The Casimir operator  $C_V$  is independent of the choice of basis  $\{x_i\}_{i=1}^n$  for  $\mathfrak{g}$ ;*

(b)  *$C_V \rho_V(x) = \rho_V(x) C_V$  for all  $x \in \mathfrak{g}$ . In other words,  $C_V$  is an intertwining operator from  $V$  to  $V$ .*

Before the proof, to put the second property in context, we give the following definition.

**Definition 18.5.** Let  $\mathfrak{g}$  be a Lie algebra and  $V, W$  be representations of  $\mathfrak{g}$ . An *intertwining operator*  $T : V \rightarrow W$  is a linear map such that

$$T(\rho_V(x)v) = \rho_W(x)T(v), \quad \forall x \in \mathfrak{g}, v \in V.$$

Another term for intertwining operator is *morphism of representations*. The *category of representations of  $\mathfrak{g}$* , denoted  $\text{Rep}_{\mathfrak{g}}$ , is the category whose objects are representations  $V$  of  $\mathfrak{g}$ , and morphisms  $V \rightarrow W$  are the intertwining operators. Thus the Casimir operators are canonical intertwining operators from any representation to itself. They play a critical role in the study of semisimple (and more generally, reductive) Lie algebras.

*Proof of Proposition 18.4.* (a) Let  $\{y_i\}_1^n$  be another basis for  $\mathfrak{g}$ , and  $\{y^i\}_1^n$  the corresponding dual basis with respect to  $\beta_V$ . Then

$$y_i = \sum_k a_{ik} x_k \quad y^j = \sum_k b_{jk} x^k$$

and furthermore

$$\delta_{ij} = \sum_{k,l} \chi(y_i, y^l y^j) = \sum_{k,l} a_{ik} b_{jl} \chi(x_k, x^l y^j) = \sum_k a_{ik} b_{jk}$$

which is to say  $AB^T = I$  where  $A = (a_{ij})$  and  $B = (b_{ij})$ . By matrix theory, this implies  $B^T A = I$  as well, which can be written

$$\sum_i b_{ik} a_{il} = \delta_{kl}. \tag{18.1}$$

The Casimir operator with respect to the  $y$ -bases equals

$$\sum_i \rho_V \rho_{y_i} \rho_V \rho_{y^i} = \sum_{i;k,l} a_{ik} b_{il} \rho_V \rho_{x_k} \rho_V \rho_{x^l} \stackrel{(18.1)}{=} \sum_{kl} \delta_{kl} \rho_V \rho_{x_k} \rho_V \rho_{x^l} = C_V.$$

(b) Let  $x \in \mathfrak{g}$ . We will use that for any  $x \in \mathfrak{g}$  we have

$$\begin{aligned} \chi(x, x_j y^i) &= \chi(x, x_j) y^i, \\ \chi(x, x^i y_j) &= \chi(x, x^i) y_j. \end{aligned} \tag{18.2}$$

The first equality follows from writing  $x = \sum_j c_j x^j$  and then calculating that  $\chi(x, x_j) = c_j$ . The second one is proved similarly. Put  $\rho = \rho_V$  for brevity. We have:

$$\begin{aligned} \rho C_V \rho &= \sum_i \rho \rho_{x_i} \rho \rho_{x^i} \quad \text{by Leibniz' Rule} \\ &= \sum_{i,j} \rho \chi(x_i, x^j) \rho_{x_j} \rho_{x^i} - \sum_{i,j} \rho_{x_i} \rho \chi(x^i, x_j) \rho_{x^j} \\ &= \sum_{i,j} \chi(x_i, x^j) \rho_{x_j} \rho_{x^i} - \sum_{i,j} \chi(x^j, x_i) \rho_{x_j} \rho_{x^i} = 0 \end{aligned}$$

where, in the last equality, we used that  $\chi(x^j, x_i) = \chi(x_i, x^j)$ ,  $\chi(x_i, x^j) = \chi(x_j, x^i)$  by symmetry and invariance of the form, and by anti-commutativity of the bracket.  $\square$

## 19 Lecture 17: Chevalley-Eilenberg Cohomology

Let  $\mathfrak{g}$  be a Lie algebra and  $M$  be a representation of  $\mathfrak{g}$ . We use module notation:  $x.v = \rho_V \rho_x v$  for  $x \in \mathfrak{g}$  and  $v \in M$ .

## 19.1 Cochains

**Definition 19.1.** Let  $i$  be a non-negative integer. An  $i$ -dimensional cochain for  $\mathfrak{g}$  with values in  $M$ , or  $i$ -cochain for short, is a linear map

$$f : \overset{\text{TM}}{i}\mathfrak{g} \tilde{N} M.$$

In more concrete terms:

- a 0-cochain is an element  $v \in M$  (because  $\overset{\text{TM}}{0}\mathfrak{g} = \mathbb{R}$  and a linear map  $\mathbb{R} \rightarrow M$  is determined by the image of  $1_{\mathbb{R}}$ ),
- a 1-cochain is a linear map  $f : \mathfrak{g} \rightarrow M$ ,
- a 2-cochain is a bilinear map  $f : \mathfrak{g} \times \mathfrak{g} \rightarrow M$  such that  $f(x, y) = -f(y, x)$  for all  $x, y \in \mathfrak{g}$ .

In general, for  $i \geq 1$  an  $i$ -cochain can be viewed as a multilinear map  $f : \mathfrak{g}^i \rightarrow M$  which is alternating i.e. switching any two arguments results in the negative.

The vector space of all  $i$ -cochains is denoted by

$$C^i(\mathfrak{g}, M) = \text{Hom}_{\mathbb{R}}(\overset{\text{TM}}{i}\mathfrak{g}, M).$$

## 19.2 The coboundary map

For each  $i$ -cochain  $f$  we define an  $(i+1)$ -cochain  $df$  (or  $d^i f$  when  $i$  needs to be specified) as follows:

- If  $v \in M$  is a 0-cochain, we define  $dv \in C^1(\mathfrak{g}, M) = \text{Hom}(\mathfrak{g}, M)$  by

$$(dv)(x) = x.v$$

- If  $f : \mathfrak{g} \rightarrow M$  is a 1-cochain, we define  $df \in C^2(\mathfrak{g}, M)$  by

$$(df)(x_1, x_2) = x_1.f(x_2) - x_2.f(x_1) = f(x_1, x_2)$$

- If  $f : \mathfrak{g} \times \mathfrak{g} \rightarrow M$  is a 2-cochain, we define  $df \in C^3(\mathfrak{g}, M)$  by

$$(df)(x_1, x_2, x_3) = x_1.f(x_2, x_3) - x_2.f(x_1, x_3) + x_3.f(x_1, x_2) \\ - f(x_1, x_2, x_3) + f(x_1, x_3, x_2) - f(x_2, x_3, x_1).$$

Here is the general definition:

**Definition 19.2.** Let  $i$  be a non-negative integer. The  $(i+1)$ -th coboundary map is the map

$$d^{i+1} : C^i(\mathfrak{g}, M) \rightarrow C^{i+1}(\mathfrak{g}, M)$$

given by

$$(d^{i+1}f)(x_1, \dots, x_{i+1}) = \sum_{r=1}^{i+1} (-1)^{r-1} x_r.f(x_1, \dots, \hat{x}_r, \dots, x_{i+1}) \\ + \sum_{1 \leq r < s \leq i} (-1)^{r+s} f(x_r, x_s, x_1, \dots, \hat{x}_r, \dots, \hat{x}_s, \dots, x_{i+1}),$$

for all  $f \in C^i(\mathfrak{g}, M)$ . A hat means the variable should be omitted from the list of arguments.

The fundamental property of the coboundary map is that applying it twice gives you zero:

**Lemma 19.3.** For any non-negative integer  $i$ , we have

$$d^{i+1} \circ d^i = 0.$$

*Proof.* We only prove this for  $i = 0$  and  $i = 1$  which are the only cases that we will need anyway.

For  $i = 0$ , let  $v \in M$  be a 0-cochain. Then we have

$$d^1 d^0 v = \rho(x_1, x_2)v - x_1 \cdot \rho(x_2, v) - x_2 \cdot \rho(x_1, v) - \rho(x_1, x_2) \cdot v = 0$$

precisely because  $M$  is a representation of  $\mathfrak{g}$ :  $x_1 \cdot x_2 \cdot v - x_2 \cdot x_1 \cdot v - \rho_V(\rho(x_1, v) - \rho_V \rho(x_2, v) - \rho_V \rho(x_1, v) - \rho_V \rho(x_1, x_2) \cdot v) = 0$ .

For  $i = 1$ , let  $f : \mathfrak{g} \rightarrow M$  be a 1-cochain. Then

$$\begin{aligned} d^2 d^1 f(x_1, x_2, x_3) &= x_1 \cdot x_2 \cdot f(x_3) - x_3 \cdot f(x_2) - f(x_2, x_3) \\ &\quad - x_2 \cdot x_1 \cdot f(x_3) - x_3 \cdot f(x_1) - f(x_1, x_3) \\ &\quad - x_3 \cdot x_1 \cdot f(x_2) - x_2 \cdot f(x_1) - f(x_1, x_2) \\ &\quad - \rho(x_1, x_2) \cdot f(x_3) - x_3 \cdot \rho(x_1, x_2) - \rho(\rho(x_1, x_2), x_3) \\ &\quad - \rho(x_1, x_3) \cdot f(x_2) - x_2 \cdot \rho(x_1, x_3) - \rho(\rho(x_1, x_3), x_2) \\ &\quad - \rho(x_2, x_3) \cdot f(x_1) - x_1 \cdot \rho(x_2, x_3) - \rho(\rho(x_2, x_3), x_1) \\ &= 0. \end{aligned}$$

□

### 19.3 Cocycles, Coboundaries, and Cohomology

For convenience, if  $i$  is a negative integer we put  $C^i(\mathfrak{g}, M) = 0$  and  $d^i = 0$ .

**Definition 19.4.** Let  $i$  be a non-negative integer.

- An  $i$ -cochain  $f \in C^i(\mathfrak{g}, M)$  is called an  $i$ -cocycle if  $df = 0$ . The space of  $i$ -cocycles is denoted

$$Z^i(\mathfrak{g}, M) = \ker d^i$$

- An  $i$ -cochain  $f \in C^i(\mathfrak{g}, M)$  is called an  $i$ -coboundary if  $f = dg$  for some  $g \in C^{i-1}(\mathfrak{g}, M)$ . The space of  $i$ -coboundaries is denoted

$$B^i(\mathfrak{g}, M) = \text{im } d^{i-1}$$

Note that, since  $d^i \circ d^{i-1} = 0$ , every  $i$ -coboundary is an  $i$ -cocycle. Thus the following definition makes sense.

**Definition 19.5.** The  $i$ -th cohomology group of  $\mathfrak{g}$  with values in  $M$  is defined as the quotient vector space

$$H^i(\mathfrak{g}, M) = \frac{Z^i(\mathfrak{g}, M)}{B^i(\mathfrak{g}, M)}.$$

The zeroth cohomology group is already somewhat interesting. Since  $B^0(\mathfrak{g}, M) = \text{im } d^{-1} = 0$ , we have  $H^0(\mathfrak{g}, M) = \ker d^0 = \{v \in M \mid xv = 0 \text{ } \forall x \in \mathfrak{g}\}$ . This is the space of all  $\mathfrak{g}$ -invariants in  $M$  and is usually denoted by  $M^{\mathfrak{g}}$ .

Of particular interest to us will be the first and second cohomology groups:

$$H^1(\mathfrak{g}, M) = \frac{\ker d^1}{\text{im } d^0}, \quad H^2(\mathfrak{g}, M) = \frac{\ker d^2}{\text{im } d^1}.$$

**Exercise 19.6.** Show that for the adjoint representation  $M = \mathfrak{g}$ ,  $Z^1(\mathfrak{g}, \mathfrak{g}) = \text{Der } \mathfrak{g}$  and  $B^1(\mathfrak{g}, \mathfrak{g}) = \text{ad } \mathfrak{g}$  (the space of *inner derivations*). Conclude that  $H^1(\mathfrak{g}, \mathfrak{g}) = \text{Der } \mathfrak{g} / \text{ad } \mathfrak{g}$ , sometimes called the space of *outer derivations*.

## 20 Lecture 18: Whitehead's First Lemma

In general, cohomology can be thought of as measuring the obstruction to carrying out certain constructions. Thus, vanishing of cohomology means there are no obstructions; a desirable property. Whitehead's First and Second Lemma are two fundamental results about the vanishing of cohomology for semisimple Lie algebras.

**Theorem 20.1** (Whitehead's First Lemma). *Let  $\mathfrak{g}$  be a semisimple Lie algebra over a field of characteristic zero, and let  $M$  be a finite-dimensional representation of  $\mathfrak{g}$ . Then*

$$H^1(\mathfrak{g}, M) = 0.$$

In this section we will prove this theorem.

We need two lemmas. The first one is a variant of the Jordan decomposition over not necessarily algebraically closed fields.

**Lemma 20.2** (Fitting's Lemma). *Let  $T$  be a linear transformation of a finite-dimensional vector space  $V$  over a field of characteristic zero. Then there exists a decomposition*

$$V = V_0 \dot{\cup} V_1$$

*into two subspaces  $V_i$  such that  $T|_{V_i} \in \text{GL}(V_i)$  for  $i = 1, 2$ ;  $T|_{V_0}$  is nilpotent and  $T|_{V_1}$  is invertible.*

*Proof.* Let  $p(x)$  be the minimal polynomial of  $T$ . Write  $p(x) = x^d q(x)$  where  $d$  is as large as possible. By the Remainder Theorem, the natural map  $\mathbb{R}[x] / (p(x)) \cong \mathbb{R}[x] / (x^d) \oplus \mathbb{R}[x] / (q(x))$  is a ring isomorphism. For  $i = 0, 1$ , let  $e_i(x) \in \mathbb{R}[x]$  such that

$$\begin{aligned} \# \quad e_0(x) &\equiv 1 \pmod{x^d} & \# \quad e_1(x) &\equiv 0 \pmod{x^d} \\ e_0(x) &\equiv 0 \pmod{q(x)} & e_1(x) &\equiv 1 \pmod{q(x)} \end{aligned}$$

and define  $V_i = e_i(T)V$ . We have  $e_i(T)^2 = e_i(T)$ ,  $e_0(T)e_1(T) = 0$  and  $e_0(T)e_1(T) = 0$  (since when  $T$  is replaced by  $x$  the corresponding congruences hold modulo  $p(x)$ ). Hence  $V = V_0 \dot{\cup} V_1$ . Since  $e_1(T)$  is a polynomial in  $T$  hence commutes with  $T$ , it is clear that  $T|_{V_i} \in \text{GL}(V_i)$  for  $i = 0, 1$ . Since  $e_0(x) = q(x)r(x)$  for some  $r(x) \in \mathbb{R}[x]$  we have  $T^d V_0 = p(T)r(T)V = 0$  so  $T|_{V_0}$  is nilpotent. Since  $q(x)$  is relatively prime to  $x$  we can write  $b(x)q(x) + c(x)x = 1$  for some  $b(x), c(x) \in \mathbb{R}[x]$  (Bezout's identity). Thus for  $v = e_1(T)w \in V_1$  we have  $v = b(T)q(T)e_1(T)w + c(T)T v = c(T)T v$  since  $x^d$  divides  $e_1(x)$  and  $q(x)x^d = p(x)$ . Thus  $T|_{V_1}$  is invertible.  $\square$

The second result we need is a property of the trace of the Casimir operator. A representation  $V$  is called *faithful* if  $\rho_V$  is injective.

**Lemma 20.3.** *Let  $\mathfrak{g}$  be a semisimple Lie algebra, and  $V$  a faithful representation of  $\mathfrak{g}$ , and  $\langle x, y \rangle = \text{Tr } \rho_V(x)\rho_V(y)$  be the trace form (it is non-degenerate, invariant, symmetric, bilinear.) Let  $C_V$  be the corresponding Casimir operator. Then  $\text{Tr } C_V = \dim \mathfrak{g}$ . In particular, if  $\mathfrak{g} \neq 0$ , then  $C_V$  is not nilpotent.*

*Proof.*  $\text{Tr } C_V = \sum_i \rho_M(x_i)\rho_M(x_i) = \sum_i \langle x_i, x_i \rangle = \dim \mathfrak{g}$ . □

We are now ready to prove Theorem 20.1.

*Proof of Whitehead's First Lemma.* Let  $f : \mathfrak{g} \rightarrow M$  be a 1-cocycle. Thus  $f$  is a linear map satisfying

$$f([x, y]) = x.f(y) - y.f(x) \quad \forall x, y \in \mathfrak{g}. \quad (20.1)$$

We must show that  $f$  is a 1-coboundary. That is, we must find  $v \in M$  such that

$$f(x) = x.v \quad (20.2)$$

for all  $x \in \mathfrak{g}$ .

Let  $I = \ker \rho_M$ . Then, in module notation,  $x.v = 0$  for all  $x \in I$  and all  $v \in M$ . By Proposition 17.14 and Corollary 18.2, there exists a semisimple ideal  $\mathfrak{h}$  of  $\mathfrak{g}$  such that  $\mathfrak{g} = I \oplus \mathfrak{h}$  as vector spaces. It suffices to find  $v \in M$  such that (20.2) holds for all  $x \in \mathfrak{h}$  because  $I = \mathfrak{r}I, I\mathfrak{s}$  (by Corollary 18.1) hence, by (20.1),  $f(\mathfrak{r}I) = f(\mathfrak{r}I, I\mathfrak{s}) = I.f(\mathfrak{r}I) = I.f(I) = 0$ . So (20.2) would then hold for all  $x \in \mathfrak{g}$  by linearity.

First, suppose that  $I = \mathfrak{g}$ , i.e.  $\mathfrak{h} = 0$ . In this case we may simply take  $v = 0$ , since then (20.2) holds trivially for all  $x \in \mathfrak{h}$ .

So we may assume  $I \neq \mathfrak{g}$ . Note that  $M$  is a faithful representation of  $\mathfrak{h}$  with representation map  $\rho = \rho_M|_{\mathfrak{h}} : \mathfrak{h} \rightarrow \text{gl } M$ . Let  $\{x_i\}_{i=1}^m$  be a basis for  $\mathfrak{h}$  and  $\{x_i^j\}_{i=1}^m$  be the corresponding dual basis with respect to the trace form  $\langle x, y \rangle = \text{Tr } \rho(x)\rho(y)$ ,  $x, y \in \mathfrak{h}$ . By Lemma 20.3, the Casimir operator  $C_M = \sum_i \rho(x_i)\rho(x_i^j)$  associated to  $\mathfrak{h}$ , is not nilpotent.

Thus, if  $M = M_0 \oplus M_1$  is the Fitting decomposition of  $M$  relative to  $C_M$  (Lemma 20.2), we have  $M_1 = 0$ . Write  $f(x) = f_0(x) + f_1(x)$  where  $f_i(x) \in M_i$ . It is easy to check that  $f_i$  both satisfy (20.1). If  $\dim M_i = \dim M$  for both  $i = 0, 1$  it therefore follows by induction on  $\dim M$  that there exist  $v_i \in M_i$  such that  $f_i(x) = x.v_i$  for all  $x \in \mathfrak{h}$ . Taking  $v = v_0 + v_1$  we have (20.2) for all  $x \in \mathfrak{h}$ .

So the case that remains is that  $M_1 = M$ . That is,  $C_M$  is invertible. Consider the following element of  $M$ :

$$w = \sum_i x_i.f(x_i^j).$$



For any  $x \in \mathfrak{h}$  we have

$$\begin{aligned}
 x.w & \stackrel{\circ}{=} \sum_i \langle x, x_i \rangle \langle f, \rho x^i \rangle = \sum_i x_i \cdot \langle f, \rho x^i \rangle \quad \text{since } M \text{ is a representation of } \mathfrak{h} \\
 & \stackrel{\circ}{=} \sum_{i,j} \langle x, x_i \rangle \langle x^j, y_j \rangle \langle f, \rho x^i \rangle = \sum_i x_i \cdot \langle f, \rho x^i \rangle \quad \text{by dual basis property (18.2)} \\
 & \stackrel{\circ}{=} \sum_{i,j} x_j \cdot \langle f, \rho x_i \rangle = \sum_i x_i \cdot \langle f, \rho x^i \rangle \quad \text{by linearity of } f \text{ and invariance of } \langle \cdot, \cdot \rangle \\
 & \stackrel{\circ}{=} \sum_j x_j \cdot \langle f, \rho x^j \rangle = \sum_j x_j \cdot \langle f, \rho x^j \rangle \quad \text{by dual bases property (18.2) and bracket anti-commutativity} \\
 & \stackrel{\circ}{=} \sum_j x_j \cdot \langle f, \rho x^j \rangle \quad \text{since } f \text{ is a 1-cocycle} \\
 & = C_M \langle f, \rho x \rangle \quad \text{by definition of } C_M.
 \end{aligned}$$

Thus, define

$$v = C_M^{-1} \rho w.$$

Since  $C_M$  commutes with all  $\rho x$ ,  $x \in \mathfrak{h}$ , the same is true of the inverse of  $C_M$ . Consequently, for all  $x \in \mathfrak{h}$ ,

$$x.v = \rho x C_M^{-1} w = C_M^{-1} \rho x w = C_M^{-1} \rho x.w = \rho x.v.$$

□

## 21 Lecture 19: Abstract Jordan Decomposition

In this lecture we assume that  $\mathbb{F}$  is algebraically closed.

Recall that in this case, the words “semisimple” and “diagonalizable” when talking about linear operators on a finite-dimensional vector space.

**Lemma 21.1.** *Assume  $\mathbb{F}$  is algebraically closed. If  $D \in \text{Der}(\mathfrak{g})$  and  $D = D_s + D_n$  is the Jordan decomposition of  $D$ , then  $D_s$  and  $D_n$  belong to  $\text{Der}(\mathfrak{g})$ .*

*Proof.* Write  $\mathfrak{g} = \bigoplus_{\lambda \in \mathbb{F}} \mathfrak{g}_\lambda$ , where  $\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid \rho D = \lambda x\}$ ,  $\lambda \in \mathbb{F}$ . Then  $D_s x = \lambda x$  for all  $x \in \mathfrak{g}_\lambda$ . For  $x, y \in \mathfrak{g}$  we have the identity

$$D(\rho \lambda \langle x, y \rangle) = \sum_{k=0}^N \binom{N}{k} \rho \lambda^k \langle \rho D^k x, \rho D^k y \rangle$$

which can be proved by induction on  $N$ . This proves that  $\rho \lambda \langle x, y \rangle \in \mathfrak{g}$ .

So if  $x \in \mathfrak{g}_\lambda$  and  $y \in \mathfrak{g}_\mu$  then  $D_s \langle x, y \rangle = \rho \lambda \langle \rho x, \rho y \rangle = \rho \langle \lambda x, \mu y \rangle = \rho \langle D_s \rho x, \rho y \rangle = \rho \langle D_s \rho x, \rho y \rangle$ . By bilinearity,  $D_s \in \text{Der}(\mathfrak{g})$ . Then  $D_n = D - D_s \in \text{Der}(\mathfrak{g})$  as well. □

**Exercise 21.2.** Prove the above identity.

The following proposition shows that in a semisimple Lie algebra over  $\mathbb{F}$ , any element can be decomposed in a way that is similar to the Jordan decomposition for a linear transformation.

**Proposition 21.3.** *Let  $\mathfrak{g}$  be a semisimple Lie algebra over  $\mathbb{F}$  and let  $x \in \mathfrak{g}$ . Then there exists a unique pair  $\rho x_s, \rho x_n \in \mathfrak{g}^2$  such that  $x = x_s + x_n$ ,  $\rho x_s, \rho x_n \in \mathfrak{g}$ ,  $\text{ad } x_s$  is semisimple and  $\text{ad } x_n$  is nilpotent. Moreover, for any  $y \in \mathfrak{g}$  such that  $\rho y, \rho x_s \in \mathfrak{g}$  we also have  $\rho y, \rho x_n \in \mathfrak{g}$ .*

*Proof.* Let  $D = \text{ad } x$  and let  $D = D_S + D_N$  be the Jordan decomposition of  $D$ . Since  $D \in \text{Der}(\mathfrak{g})$  we have  $D_S, D_N \in \text{Der}(\mathfrak{g})$  by Lemma 21.1. By Whitehead's First Lemma (Theorem 20.1) and Exercise 19.6,  $\text{Der}(\mathfrak{g}) = \text{ad } \mathfrak{g}$ . Thus there exist  $x_S \in \mathfrak{g}$  and  $x_N \in \mathfrak{g}$  such that  $\text{ad } x_S = D_S$  and  $\text{ad } x_N = D_N$ . Now  $\text{ad } x = D_S + D_N = \text{ad } x_S + \text{ad } x_N = \text{ad } (x_S + x_N)$ . Since  $\text{ad}$  is injective (its kernel equals  $\mathfrak{z}(\mathfrak{g})$  which is a solvable ideal of  $\mathfrak{g}$  hence zero since  $\mathfrak{g}$  is semisimple), we obtain  $x = x_S + x_N$ . Similarly,  $0 = [D_S, D_N] = [\text{ad } x_S, \text{ad } x_N] = \text{ad } [x_S, x_N]$  implies  $[x_S, x_N] = 0$ . And  $\text{ad } x_S = D_S$  is semisimple,  $\text{ad } x_N = D_N$  is nilpotent. This shows existence. If  $\mathfrak{x}_S, \mathfrak{x}_N$  is another such pair, then  $\text{ad } \mathfrak{x}_S = \text{ad } \mathfrak{x}_N$  is the Jordan decomposition of  $\text{ad } x$ , hence  $\text{ad } \mathfrak{x}_S = \text{ad } x_S$  and  $\text{ad } \mathfrak{x}_N = \text{ad } x_N$  by uniqueness. Thus  $\mathfrak{x}_S = x_S$  and  $\mathfrak{x}_N = x_N$ . Lastly, if  $[y, x_S] = 0$  then  $0 = \text{ad } [y, x_S] = \text{ad } [y, \text{ad } x_S]$ . So  $\text{ad } y$  commutes with  $\text{ad } x$ . By the usual Jordan decomposition,  $\text{ad } y$  therefore also commutes with the semisimple and nilpotent parts, i.e. with  $\text{ad } x_S$  and  $\text{ad } x_N$ . So  $0 = \text{ad } [y, \text{ad } x_S] = \text{ad } [y, x_S]$  which implies  $[y, x_S] = 0$ . Similarly  $[y, x_N] = 0$ .  $\square$

The abstract Jordan decomposition behaves well under representations:

**Theorem 21.4.** *Let  $\mathfrak{g}$  be a semisimple Lie algebra and  $V$  a representation of  $\mathfrak{g}$ . Then*

$$\rho_V(\text{ad } x_S) = \rho_V(\text{ad } x)_S, \quad \rho_V(\text{ad } x_N) = \rho_V(\text{ad } x)_N, \quad (21.1)$$

where all subscripts refer to the abstract Jordan decomposition.

*Proof.* Put  $\rho = \rho_V$ . Let  $\text{ad}$  denote the adjoint representation of the Lie algebra  $\rho(\mathfrak{g})$ . We have  $\text{ad } \rho(x) = \text{ad } \rho(x_S) + \text{ad } \rho(x_N)$  since  $x = x_S + x_N$ . Let  $y_1, \dots, y_n$  be a basis for  $\mathfrak{g}$  such that  $\text{ad } \rho(x_S)y_j = \lambda_j y_j$ . That is  $[x_S, y_j] = \lambda_j y_j$ . Applying  $\rho$  this gives  $[\rho(x_S), \rho y_j] = \lambda_j \rho y_j$ . Therefore  $\text{ad } \rho(x_S)$  is semisimple. Similarly  $\text{ad } \rho(x_N)$  is nilpotent, and they commute and sum to  $\text{ad } \rho(x)$ . By uniqueness of the abstract Jordan decomposition, we have  $\text{ad } \rho(x_S) = \text{ad } \rho(x)_S$  and  $\text{ad } \rho(x_N) = \text{ad } \rho(x)_N$ . Since the image of  $\rho$  is isomorphic to a quotient of a semisimple Lie algebra, it is semisimple by Corollary 18.2. Therefore  $\text{ad}$  is injective. Thus we conclude that (21.1) holds.  $\square$

Later on we will prove the following result. It actually requires some more work (Weyl's Theorem).

**Theorem 21.5.** *Suppose  $\mathfrak{g} \cong \mathfrak{g}$ . If  $\mathfrak{g} \in \text{gl}(V)$  is a semisimple Lie subalgebra then the usual and abstract Jordan decompositions coincide.*

Once we establish this theorem, it means that representations map the abstract Jordan decomposition to the concrete (linear) Jordan decomposition.

## 22 Lecture 20: Weyl's Theorem on Complete Reducibility. Short Exact Sequences of Lie Algebras.

### 22.1 Complete Reducibility

**Definition 22.1.** A representation  $V$  is *irreducible* if  $V \neq 0$  and the only subrepresentations are  $\{0\}$  and  $V$ . A representation of  $\mathfrak{g}$  is *completely reducible* if

$$V = \sum_{i=1}^n V_i$$

where  $V_i$  are irreducible subrepresentations.

If  $V$  and  $W$  are representations of a Lie algebra  $\mathfrak{g}$  then the space  $\text{Hom}(V, W)$  of all linear maps from  $V$  to  $W$  is also a representation of  $\mathfrak{g}$  with action

$$(\mathfrak{g} \cdot f)(v) = x \cdot (f(v)) - f(x \cdot v), \quad @x \in \mathfrak{g}, f \in \text{Hom}(V, W), v \in V.$$

**Exercise 22.2.** Check that this makes  $\text{Hom}(V, W)$  into a representation of  $\mathfrak{g}$ . (That is, check that  $(x \cdot (y \cdot f)) = (x \cdot y) \cdot f - y \cdot (x \cdot f)$ .)

Note that  $x \cdot f = 0$  for all  $x \in \mathfrak{g}$  is equivalent to that  $f$  is an intertwining operator. The space of intertwining operators from  $V$  to  $W$  is denoted  $\text{Hom}_{\mathfrak{g}}(V, W)$ . And the space of  $\mathfrak{g}$ -invariants of a representation  $V$  is  $V^{\mathfrak{g}} = \{v \in V \mid x \cdot v = 0 @x \in \mathfrak{g}\}$ . Thus we have  $\text{Hom}(V, W)^{\mathfrak{g}} = \text{Hom}_{\mathfrak{g}}(V, W)$ .

**Theorem 22.3** (Weyl's Theorem on Complete Reducibility). *Let  $\mathfrak{g}$  be a semisimple Lie algebra. Then every finite-dimensional representation of  $\mathfrak{g}$  is completely reducible:*

$$V \cong \bigoplus_{i=1}^n V_i$$

where  $V_i$  are irreducible subrepresentations.

*Proof.* Let  $V$  be a finite-dimensional representation of  $\mathfrak{g}$  and let  $U \in V$  be a subrepresentation. By induction on  $\dim V$  it suffices to show that there exists a subrepresentation  $U^1$  of  $V$  such that  $V = U \oplus U^1$ .

Let

$$\begin{aligned} V &= \{f \in \text{Hom}(V, U) \mid f|_U = \text{Id}_U\}, \\ U &= \{f \in \text{Hom}(V, U) \mid f|_U = 0, u \in U\}. \end{aligned}$$

Let  $p : V \rightarrow U$  be any linear projection, i.e.  $p|_U = \text{Id}_U$ . Then  $p \in V$  and in fact the image of  $p$  in the (one-dimensional) quotient space  $V/U$  is a basis.

For  $x \in \mathfrak{g}$  and  $u \in U$  we have  $(x \cdot p)(u) = x \cdot (p(u)) - p(x \cdot u) = x \cdot u - x \cdot u = 0$ . Thus  $x \cdot p \in U$ . Let  $\varphi : \mathfrak{g} \rightarrow V/U$  be given by  $\varphi(x) = x \cdot p$  for  $x \in \mathfrak{g}$ . We show that  $\varphi \in Z^1(\mathfrak{g}, U)$ :

$$\begin{aligned} \varphi([x, y]) &= [x \cdot p, y \cdot p] = (x \cdot y) \cdot p - (y \cdot x) \cdot p \\ &= x \cdot (y \cdot p) - y \cdot (x \cdot p). \end{aligned}$$

By Whitehead's First Lemma (Theorem 20.1), there exists  $T \in U$  such that  $\varphi = d^0 T$ . That is,  $x \cdot p = x \cdot T$  for all  $x \in \mathfrak{g}$ . Let  $\pi = p - T$ . Then  $x \cdot \pi = 0$  for all  $x \in \mathfrak{g}$ . By the comment preceding the statement of the theorem, this means that  $\pi$  is an intertwining operator from  $V$  to  $U$ . Also,  $\pi^2(v) = \pi(p(v)) = \pi(v)$  for all  $v \in V$  which means  $\pi$  is a projection. Let  $U^1 = \ker \pi$ . Since  $\pi$  is an intertwining operator,  $U^1$  is also a subrepresentation of  $V$ . By standard linear algebra arguments,  $V = U \oplus U^1$ . (Any vector  $v \in V$  can be written  $v = \pi(v) + (v - \pi(v)) = \pi(v) + \pi(v)$  and  $\pi(v) = \pi(v)$  so  $v = \pi(v) + \pi(v) \in U^1$ . If  $v \in U \times U^1$  then  $v = \pi(v) = 0$ .)  $\square$

## 22.2 Short Exact Sequences

Let  $I, \mathfrak{g}$  and  $\mathfrak{g}$  be any three Lie algebras. A *short exact sequence (SES)*

$$0 \rightarrow I \rightarrow \mathfrak{g} \rightarrow \mathfrak{g} \rightarrow 0$$

is a sequence of Lie algebras and Lie algebra homomorphisms such that the kernel of each map is the image of the previous map. This means that  $\ker \iota = \text{im} \rho = 0$  so  $\iota$  is injective;  $\text{im} \rho = \ker \pi = \mathfrak{g}$  so  $\rho$  is surjective; and  $\ker \pi = \text{im} \iota$  which means that  $\text{im} \rho$  is an ideal of  $\mathfrak{g}$  and  $\mathfrak{g}/\text{im} \rho = \mathfrak{g}$ .

**Proposition 22.4.** *Given a SES*

$$0 \rightarrow \mathfrak{h} \xrightarrow{I} \mathfrak{g} \xrightarrow{\pi} \mathfrak{g}/\mathfrak{h} \rightarrow 0$$

*the following are equivalent:*

- (i)  $\mathfrak{h} = \mathfrak{g}$  for some  $\alpha$ .
- (ii) There exists a Lie algebra map  $\sigma : \mathfrak{g}/\mathfrak{h} \rightarrow \mathfrak{g}$  called a section such that  $\pi \circ \sigma = \text{Id}_{\mathfrak{g}/\mathfrak{h}}$ .

## 23 Lecture 21: Whitehead's Second Lemma and Levi's Theorem

### 23.1 Whitehead's Second Lemma

**Theorem 23.1** (Whitehead's Second Lemma). *Let  $\mathfrak{g}$  be a semisimple Lie algebra over a field of characteristic zero, and let  $M$  be a finite-dimensional representation of  $\mathfrak{g}$ . Then*

$$H^2(\mathfrak{g}, M) = 0.$$

*Proof.* Let  $I = \ker \rho_M$ . By Proposition 17.14 and Corollary 18.2 we have  $\mathfrak{g} = I \oplus \mathfrak{h}$  for some semisimple subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ . Let  $\{x_i\}_{i=1}^m$  be a basis for  $\mathfrak{h}$  and  $\{x^i\}_{i=1}^m$  be the dual basis for  $\mathfrak{h}$  with respect to the non-degenerate invariant symmetric bilinear form  $\langle x, y \rangle = \text{Tr}(\rho(x)\rho(y))$ ,  $x, y \in \mathfrak{h}$ ,  $\rho = \rho_M|_{\mathfrak{h}}$ . Let  $C_M = \sum_{i=1}^m \rho(x_i)\rho(x^i)$  be the corresponding Casimir operator.

Let  $f \in Z^2(\mathfrak{g}, M)$ . Thus  $f : \mathfrak{g} \times \mathfrak{g} \rightarrow M$  is a bilinear map satisfying  $f(\rho x, y) = f(\rho y, x)$  and

$$\begin{aligned} y_1 \cdot f(\rho y_2, y_3) - y_2 \cdot f(\rho y_1, y_3) - y_3 \cdot f(\rho y_1, y_2) \\ - f(\rho y_1, y_2, y_3) + f(\rho y_1, y_3, y_2) + f(\rho y_2, y_3, y_1) = 0. \end{aligned}$$

This may be written without minus signs in a cyclic permutation way reminiscent of the Jacobi identity:

$$\begin{aligned} y_1 \cdot f(\rho y_2, y_3) + f(y_1, \rho y_2, y_3) \\ + y_2 \cdot f(\rho y_3, y_1) + f(y_2, \rho y_3, y_1) \\ + y_3 \cdot f(\rho y_1, y_2) + f(y_3, \rho y_1, y_2) = 0. \end{aligned}$$

Now choose  $y_3 = x^i$  and act (via the representation) by  $x_i$ , then sum over  $i$  to get:

$$\begin{aligned} 0 &= \sum_i x_i \cdot y_1 \cdot f(\rho y_2, x^i) - x_i \cdot f(y_1, \rho y_2, x^i) \\ &\quad - x_i \cdot y_2 \cdot f(\rho x^i, y_1) - x_i \cdot f(y_2, \rho x^i, x^i) \\ &\quad - x_i \cdot f(x^i, \rho y_1, y_2) - C_M f(\rho y_1, y_2). \end{aligned}$$

Now use the module identity  $x \cdot y \cdot v = y \cdot v \cdot x - \langle x, y \rangle \cdot v$  in the first and second line (ignore the colors for now):

$$\begin{aligned} &= \sum_i y_1 \cdot x_i \cdot f(\rho y_2, x^i) - \langle x_i, y_1 \rangle \cdot f(\rho y_2, x^i) - x_i \cdot f(y_1, \rho y_2, x^i) \\ &\quad - y_2 \cdot x_i \cdot f(\rho x^i, y_1) - \langle x_i, y_2 \rangle \cdot f(\rho x^i, y_1) - x_i \cdot f(y_2, \rho x^i, y_1) \\ &\quad - x_i \cdot f(x^i, \rho y_1, y_2) - C_M f(\rho y_1, y_2). \end{aligned} \tag{23.1}$$

We have, using the dual basis properties (18.2),

$$\sum_i \langle x_i, y_1 \rangle \langle f p y_2, x^i \rangle = \sum_{ij} \langle x_i, y_1 \rangle \langle x^j y_j, f p y_2, x^i \rangle = \sum_j \langle x_j, f p y_2, \langle y_1, x^j \rangle \rangle$$

and

$$\sum_i \langle x_i, y_2 \rangle \langle f p x^i, y_1 \rangle = \sum_{ij} \langle x_i, y_2 \rangle \langle x^j y_j, f p x^i, y_1 \rangle = \sum_j \langle x_j, f \langle y_2, x^j \rangle, y_1 \rangle$$

thus four terms (red and blue) in (23.1) cancel out pairwise, which yields

$$C_M \langle f p y_1, y_2 \rangle = \sum_i \langle y_1, x_i \rangle \langle f p y_2, x^i \rangle - \sum_i \langle y_2, x_i \rangle \langle f p x^i, y_1 \rangle - \sum_i \langle x_i, f \langle x^i, y_1 \rangle, y_2 \rangle = 0. \quad (23.2)$$

First let us consider the case that  $\mathfrak{h} = 0$ . Let  $M = M_0 \dot{+} M_1$  be the Fitting decomposition of  $M$  relative to  $C_M$  (Lemma 20.2). Since  $C_M$  is an intertwining operator, the  $M_i$  are actually  $\mathfrak{g}$ -subrepresentations of  $M$  (the action of  $I$  is zero anyway and  $C_M$  commutes with  $\rho p x q$  for any  $x \in \mathfrak{h}$ ). Since  $\mathfrak{h} = 0$  and  $M$  is a faithful representation of  $\mathfrak{h}$ , the Casimir  $C_M$  is not nilpotent (Lemma 20.3). Thus,  $M_1 = 0$ . Suppose that  $M_0 = 0$  as well. Write  $f p x, y q = f_0 p x, y q + f_1 p x, y q$  for unique bilinear functions  $f_i : \mathfrak{g} \times \mathfrak{g} \rightarrow M_i$ . Since  $f$  is a 2-cocycle it is immediate that each  $f_i$  is a 2-cocycle with values in  $M_i$ . Thus, by induction on  $\dim M$ , there are  $g_i \in C^1 p \mathfrak{g}, M_i q$  such that  $d^1 g_i = f_i$ . Taking  $g = g_0 + g_1$  we obtain that  $d^1 g = f$  and thus  $f \in B^2 p \mathfrak{g}, M q$ . So we may assume that  $M_0 = 0$ . That is, the linear operator  $C_M : M \rightarrow M$  is invertible. In this case we define  $g : \mathfrak{g} \times \mathfrak{g} \rightarrow M$  by

$$g p y q = \sum_{i=1}^m C_M^{-1} \langle x_i, f p x^i, y q \rangle.$$

Then we have

$$\begin{aligned} & p d^1 g q p y_1, y_2 q = \langle y_1, g p y_2 q \rangle - \langle y_2, g p y_1 q \rangle - \langle g \langle y_1, y_2 \rangle, y_3 \rangle \\ & C_M^{-1} \sum_{i=1}^m \langle y_1, x_i \rangle \langle f p x^i, y_2 q \rangle - \langle y_2, x_i \rangle \langle f p x^i, y_1 q \rangle - \langle x_i, f \langle x^i, y_1 \rangle, y_2 \rangle = \langle f p y_1, y_2 q \rangle \end{aligned}$$

by (23.2).

It remains to deal with the possibility that  $\mathfrak{h} \neq 0$ . That is,  $x.v = 0$  for all  $x \in \mathfrak{g}$  and  $v \in M$ . In this case the identity for  $f$  can be written

$$f \langle y_1, y_2 \rangle, y_3 + f \langle y_2, y_3 \rangle, y_1 + f \langle y_3, y_1 \rangle, y_2 = 0 \quad (23.3)$$

Put  $M = \text{Hom} p \mathfrak{g}, M q$ , regarded as a representation of  $\mathfrak{g}$ , and define  $F \in \text{Hom} p \mathfrak{g}, M q \rightarrow C^1 p \mathfrak{g}, M q$  by  $F p x q p y q = f p x, y q$ . Then (23.3) implies that  $F \in Z^1 p \mathfrak{g}, M q$ . Indeed,

$$F \langle y_1, y_2 \rangle, y_3 q = f \langle y_1, y_2 \rangle, y_3 q$$

while

$$\begin{aligned} y_1 \cdot F p y_2 q &= y_2 \cdot F p y_1 q + p y_3 q - \overline{y_1 \cdot f p y_2, y_3 q} + f \langle y_2, y_1 \rangle, y_3 q \\ &= \overline{y_2 \cdot f p y_1, y_3 q} + f \langle y_1, y_2 \rangle, y_3 q \\ &= f \langle y_3, y_1 \rangle, y_2 + f \langle y_2, y_3 \rangle, y_1. \end{aligned}$$

Therefore, by Whitehead's first lemma (Theorem 20.1), there exists  $g \in M$  such that  $d^0 g = F$ . Written out, this is saying  $p x, g q p y q = f p x, y q$ . By the definition of  $\mathfrak{g}$  action on  $M = \text{Hom} p \mathfrak{g}, M q$  this means  $x \cdot g p y q = g \langle x, y \rangle + f p x, y q$ . Since  $\mathfrak{g}$  is acting trivially on  $M$ , this is equivalent to that  $x \cdot g p y q = y \cdot g p x q + g \langle x, y \rangle + f p x, y q$ . That is  $d^1 g = f$  (now viewing  $g \in \text{Hom} p \mathfrak{g}, M q \rightarrow C^1 p \mathfrak{g}, M q$ ). Thus  $f \in B^2 p \mathfrak{g}, M q$ .  $\square$

It is natural to wonder about the third cohomology group. Although we won't discuss it further here, in general it is nonzero, even for a semisimple Lie algebra.

## 23.2 Semi-Direct Products of Lie Algebras

**Definition 23.2.** If a Lie algebra  $\mathfrak{g}$  is a vector space direct sum of a subalgebra  $\mathfrak{a}$  and an ideal  $I$ , then  $\mathfrak{g}$  is said to be the *(internal) semi-direct product* of  $\mathfrak{a}$  and  $I$  and we write

$$\mathfrak{g} = \mathfrak{a} \ltimes I.$$

(or  $I \rtimes \mathfrak{a}$ .)

Note that if  $\mathfrak{g} = \mathfrak{a} \ltimes I$  then the adjoint action on  $\mathfrak{g}$  gives a map

$$\alpha : \mathfrak{a} \rightarrow \mathfrak{N}(\text{Der}(\mathfrak{g})), \quad \alpha(a) = \text{ad}_a|_I$$

and the bracket in  $\mathfrak{g}$  can be expressed as

$$\begin{aligned} [a + x, b + y] &= [a, b] + [a, y] + [x, b] + [x, y] \\ &\quad + \alpha(a)(y) - \alpha(b)(x) \end{aligned}$$

for any  $a, b \in \mathfrak{a}$  and  $x, y \in I$ .

We can use this formula the basis for an *external* semi-direct product of any two Lie algebras (not a priori subalgebras of the same Lie algebra).

**Definition 23.3.** Let  $\mathfrak{a}$  and  $I$  be any two Lie algebras, and  $\alpha : \mathfrak{a} \rightarrow \mathfrak{N}(\text{Der}(I))$  be a Lie algebra homomorphism. Define  $\mathfrak{a} \ltimes I$  to be the Lie algebra with underlying vector space  $\mathfrak{a} \oplus I$  with bracket

$$[a + x, b + y] = [a, b] + \alpha(a)(y) - \alpha(b)(x) + [x, y]$$

for any  $a, b \in \mathfrak{a}$  and  $x, y \in I$ .

**Exercise 23.4.** Let  $\mathfrak{g} = \mathfrak{a} \ltimes I$ . Show that  $\mathfrak{g}$  is indeed a Lie algebra and that  $\mathfrak{a} = \mathfrak{g}^{\text{ss}}$  is a subalgebra of  $\mathfrak{g}$  and  $I = \mathfrak{g}^{\text{sol}}$  is an ideal of  $\mathfrak{g}$ . Conclude that  $\mathfrak{g}$  is the internal semi-direct product of  $\mathfrak{a}$  and  $I$ .

Thus we can go back and forth between internal and external semi-direct products.

## 23.3 Levi's Theorem

We define

$$\mathfrak{g}_{\text{ss}} = \mathfrak{g} / \text{rad}(\mathfrak{g})$$

This is the largest semisimple quotient of  $\mathfrak{g}$ .

**Theorem 23.5** (Levi's Theorem). *Suppose  $\text{char}(\mathbb{F}) \neq 0$ . Any finite-dimensional Lie algebra  $\mathfrak{g}$  is the semi-direct product of a semisimple and a solvable Lie algebra. More precisely, there exists a semisimple subalgebra  $\mathfrak{g}^{\text{ss}}$  of  $\mathfrak{g}$  such that  $\mathfrak{g} = \mathfrak{g}^{\text{ss}} \ltimes \text{rad}(\mathfrak{g})$  as vector spaces. Equivalently, there is a Lie algebra homomorphism  $\alpha : \mathfrak{g}_{\text{ss}} \rightarrow \mathfrak{N}(\text{Der}(\text{rad}(\mathfrak{g})))$  such that*

$$\mathfrak{g} = \mathfrak{g}_{\text{ss}} \ltimes \text{rad}(\mathfrak{g}).$$

*Proof.*

□

## 24 Lecture 22: Cartan Subalgebras and the Root Space Decomposition

### 24.1 Toral and Cartan Subalgebras

Let  $\mathfrak{g}$  be a semisimple Lie algebra over  $\mathbb{C}$  and assume  $\mathbb{C}$  is algebraically closed.

Then we have the abstract Jordan decomposition available for any element  $x \in \mathfrak{g}$ .

**Definition 24.1.** Call  $x \in \mathfrak{g}$  *semisimple* if  $\text{ad } x$  is semisimple, and *nilpotent* if  $\text{ad } x$  is nilpotent.

**Definition 24.2.** A subalgebra  $\mathfrak{h} \in \mathfrak{g}$  is *toral* if

- (i)  $\mathfrak{h}$  is abelian, and
- (ii)  $\mathfrak{h}$  consists of semisimple elements of  $\mathfrak{g}$ .

Note that, when  $\mathfrak{g} \neq 0$ , there is always at least one nonzero toral subalgebra: If every  $x \in \mathfrak{g}$  is nilpotent then  $\mathfrak{g}$  is nilpotent by Engel's theorem, hence solvable hence  $\mathfrak{g} = \text{rad } \mathfrak{pg} = 0$  which is a contradiction. Therefore there exists  $x \in \mathfrak{g}$  with  $x \neq 0$ . Then  $\langle x \rangle$  is a toral subalgebra of  $\mathfrak{g}$ .

**Definition 24.3.** A subalgebra  $\mathfrak{h} \in \mathfrak{g}$  is a *Cartan subalgebra* if  $\mathfrak{h}$  is toral and not properly contained in another toral subalgebra.

In other words, a Cartan subalgebra is a maximal element of the family of all toral subalgebras of  $\mathfrak{g}$ .

### 24.2 The Root Space Decomposition

Suppose we fix a toral subalgebra  $\mathfrak{h} \in \mathfrak{g}$ . Eventually we will only be interested in the case of a Cartan subalgebra but the contents of this section does not depend on any assumption about maximality.

Let  $h_1, h_2, \dots, h_r$  be a basis for  $\mathfrak{h}$ . Then  $\{\text{ad } h_i\}_{i=1}^r$  is a family of commuting diagonalizable linear operators on  $\mathfrak{g}$ . So there exists a basis for  $\mathfrak{g}$  consisting of vectors that are common eigenvectors for all the operators  $\text{ad } h_i, i = 1, 2, \dots, r$ .

Let  $x$  be such a common eigenvector. That means that there exist  $\alpha_1, \alpha_2, \dots, \alpha_r \in \mathbb{C}$  so that

$$[h_i, x] = \alpha_i x, \quad i = 1, 2, \dots, r.$$

We wish to express this in a basis independent way. Notice that if  $h \in \mathfrak{h}$  and we write  $h = \sum_{i=1}^r c_i h_i$ , then

$$[h, x] = \sum_{i=1}^r c_i [h_i, x] = \sum_{i=1}^r c_i \alpha_i x$$

Thus if we define a linear functional  $\alpha \in \mathfrak{h}^* = \text{Hom}(\mathfrak{h}, \mathbb{C})$  by

$$\alpha(h_i) = \alpha_i$$

then we may express the property that  $x$  has as follows:

$$[h, x] = \alpha(h)x \quad \forall h \in \mathfrak{h}. \tag{24.1}$$

This is the basis independent form.

For any  $\alpha \in \mathfrak{h}^*$  we put

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \quad \forall h \in \mathfrak{h}\}. \tag{24.2}$$

The case  $\alpha = 0$  plays a special role, because  $\mathfrak{g}_0$  coincides with the *centralizer of  $\mathfrak{h}$  in  $\mathfrak{g}$* :

$$\mathfrak{g}_0 = C_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \mid [h, x] = 0 \quad \forall h \in \mathfrak{h}\}.$$

**Definition 24.4.** Let  $\mathfrak{g}$  be a semisimple Lie algebra over  $\mathbb{C}$  and  $\mathfrak{h} \in \mathfrak{g}$  a toral subalgebra.

1. A nonzero linear functional  $\alpha \in \mathfrak{h}^*$  is a *root* (of  $\mathfrak{g}$  with respect to  $\mathfrak{h}$ ) if  $\mathfrak{g}_\alpha \neq 0$ .
2. The set of roots of  $\mathfrak{g}$  is denoted by  $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$  and is called the *root system* of  $\mathfrak{g}$ .
3.  $x \in \mathfrak{g}$  is called a *root vector* if  $x \in \mathfrak{g}_\alpha$  for some  $\alpha \in \Phi$ .
4. When  $\alpha$  is a root, the vector space  $\mathfrak{g}_\alpha$  is called the *root space* associated to  $\alpha$ .

**Example 24.5.** If  $\mathfrak{h}$  is one-dimensional, say  $\mathfrak{h} = \mathbb{C}h$ , then a root of  $\mathfrak{g}$  with respect to  $\mathfrak{h}$  is essentially just an eigenvalue of  $\text{ad } h$  acting on  $\mathfrak{g}$ . Similarly in this case a root vector is just an eigenvector of  $\text{ad } h$ , and root spaces are eigenspaces of  $\text{ad } h$ .

As discussed above, in this new terminology we may say that  $\mathfrak{g}$  has a basis consisting of root vectors. That can also be expressed as follows:

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

Since  $\mathfrak{g}_\alpha = 0$  only for  $\alpha \in \{0\} \cup \Phi$  we obtain the so called *root space decomposition* of  $\mathfrak{g}$  with respect to the toral subalgebra  $\mathfrak{h}$ :

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha \quad (24.3)$$

**Example 24.6.** Let  $\mathfrak{g} = \mathfrak{sl}_3$  and  $\mathfrak{h}$  be the set of diagonal matrices in  $\mathfrak{g}$ . Then  $\Phi = \{\pm \varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq 3, i \neq j\}$  where  $\varepsilon_i : \mathfrak{h} \rightarrow \mathbb{C}$  is defined by  $\varepsilon_i(\text{diag}(a_1, a_2, a_3)) = a_i$ . To see this it suffices to note that  $E_{ij} \in \mathfrak{g}_{\varepsilon_i - \varepsilon_j}$  which follows from the computation

$$[h, E_{ij}] = (a_i - a_j)E_{ij}$$

for any  $h = \sum_{i=1}^3 c_i E_{ii} \in \mathfrak{h}$ .

This decomposition will play a key role in the classification of semisimple Lie algebras. Our first steps towards that goal is the following proposition.

**Proposition 24.7.** Let  $\mathfrak{g}$  be a semisimple Lie algebra over  $\mathbb{C}$  and  $\mathfrak{h} \in \mathfrak{g}$  a toral subalgebra.

(a) For all  $\alpha, \beta \in \mathfrak{h}^*$  we have

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta} \cup \mathfrak{g}_{\alpha-\beta} \cup \{0\} \quad (24.4)$$

(b) If  $\alpha \in \mathfrak{h}^*$  and  $\alpha \neq 0$  then every  $x \in \mathfrak{g}_\alpha$  is nilpotent.

(c) If  $\alpha, \beta \in \mathfrak{h}^*$  and  $\alpha - \beta \neq 0$  then the spaces  $\mathfrak{g}_\alpha$  and  $\mathfrak{g}_\beta$  are orthogonal with respect to the Killing form:

$$\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0 \quad (24.5)$$

*Proof.* (a) Let  $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$  and  $h \in \mathfrak{h}$  then by the Leibniz rule form of the Jacobi identity:

$$[h, [x, y]] - [[h, x], y] - [x, [h, y]] = \alpha(h)[x, y] - \beta(h)[x, y] = (\alpha - \beta)(h)[x, y]$$

(b) Since  $\mathfrak{g}$  is finite-dimensional,  $\Phi$  is a finite set. Let  $\alpha \in \mathfrak{h}^*$  be nonzero. Choose  $n$  to be a large enough positive integer such that  $n\alpha \in \mathfrak{h}^*$ . Then  $\mathfrak{g}_{n\alpha} = \{0\}$ .

(c) Let  $h \in \mathfrak{h}$  be such that  $(\alpha - \beta)(h) \neq 0$ . Then for every  $x \in \mathfrak{g}_\alpha$  and  $y \in \mathfrak{g}_\beta$ :

$$\kappa(x, y) = \kappa([h, x], y) - \kappa(x, [h, y]) = (\alpha - \beta)(h)\kappa(x, y)$$

This implies that  $\kappa(x, y) = 0$ . □

**Corollary 24.8.** The restriction of the Killing form  $\kappa$  to  $\mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$  is non-degenerate.



## 25 Lecture 23: Properties of the Root Space Decomposition

**Example 25.1.** Let  $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ , then  $\mathfrak{h}$  of diagonal matrices of trace 0 is a Cartan subalgebra. Indeed,  $\mathfrak{h}$  is commutative, and if  $h \in \mathfrak{h}$ ,

$$\text{ad } h: \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}$$

$$x \mapsto [h, x] \quad @x \in \mathfrak{g}$$

is diagonalizable  $h$

$$\begin{pmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{pmatrix} \in \mathfrak{h} \quad \text{with } h_j E_{jj}. \text{ Then}$$

$$\text{ad } h \text{ on } E_{ij} \text{ is } [h, E_{ij}] = (h_i - h_j) E_{ij}.$$

So  $\mathfrak{h}$  is toral. Also, if  $x \in C_{\mathfrak{g}}(\mathfrak{h})$ , then  $[h, x] = 0 \quad @h \in \mathfrak{h}$ . Pick  $h$  with distinct eigenvalues. This implies any eigenvector for  $h$  is an eigenvector for  $x$ . Which implies  $x$  is diagonal and thus in  $\mathfrak{h}$ .

**Lemma 25.2.**  $C_{\mathfrak{g}}(\mathfrak{h})$  is a reductive Lie algebra.

*Proof.* If  $I$  is any ideal in  $C_{\mathfrak{g}}(\mathfrak{h})$  then so is  $I^{\mathbb{K}}$  (with respect to the Killing form on  $\mathfrak{g}$ ) and  $C_{\mathfrak{g}}(\mathfrak{h}) \cap I^{\mathbb{K}} = I^{\mathbb{K}}$ . Repeating this argument (the Killing form on  $\mathfrak{g}$  will be non-degenerate on both  $I$  and  $I^{\mathbb{K}}$ ) we can write  $C_{\mathfrak{g}}(\mathfrak{h})$  as a sum of ideals which are either simple or one dimensional. The sum of the one dimensional ideals make up the center. Thus  $C_{\mathfrak{g}}(\mathfrak{h})$  is the sum of a semisimple Lie algebra and a central ideal.  $\square$

**Theorem 25.3.** If  $\mathfrak{h}$  is a Cartan subalgebra then  $C_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$ .

*Proof.* Let  $h \in \mathfrak{g}$  be a Cartan subalgebra.

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_{\alpha} \quad \text{with } \mathfrak{g}_{\alpha} = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \quad @h \in \mathfrak{h}\}.$$

Note,  $C_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{g}_0$ . We claim that  $\mathfrak{g}_0$  is toral. (Then, since  $\mathfrak{h} \in \mathfrak{g}_0$  and  $\mathfrak{h}$  maximal among toral subalgebras,  $\mathfrak{h} = \mathfrak{g}_0$ .) Let  $x \in \mathfrak{g}_0$ . Then  $\text{ad } x|_{\mathfrak{g}_0}$  is nilpotent. Otherwise, it has a nonzero eigenvalue and  $\text{ad } x|_{\mathfrak{g}_0} = 0$ . This implies  $x \in \mathfrak{h}$ , and  $\mathfrak{h} \subsetneq \mathfrak{g}_0$  is a toral subalgebra that strictly contains  $\mathfrak{h}$  which is a contradiction. By Engel's Theorem (Theorem 14.8),  $\mathfrak{g}_0$  is nilpotent. By Lemma 25.2, therefore  $\mathfrak{g}_0$  is abelian.

It remains to show that  $\mathfrak{g}_0$  consists of semisimple elements. Let  $x \in \mathfrak{g}_0$ . We want to show that  $x$  is semisimple.  $\text{ad } x$  is nilpotent and  $\mathfrak{g}_0$  is commutative; therefore,  $\text{ad } x \text{ on } \mathfrak{g}_0$  is nilpotent for every  $y \in \mathfrak{g}_0$ . Thus  $\text{Tr } \text{ad } x \text{ on } \mathfrak{g}_0 = 0 \quad @y \in \mathfrak{g}_0$ . Hence,  $x = 0$  since  $\kappa|_{\mathfrak{g}_0}$  is non-degenerate.  $\square$

**Fact:** If  $\mathfrak{h}_1$  and  $\mathfrak{h}_2$  are two Cartan subalgebras of  $\mathfrak{g}$ , then there exists a Lie algebra automorphism  $\varphi$  of  $\mathfrak{g}$  such that  $\varphi(\mathfrak{h}_1) = \mathfrak{h}_2$ .

**Definition 25.4.** The *rank* of a semisimple Lie algebra  $\mathfrak{g}$  is  $\text{rank } \mathfrak{g} = \dim \mathfrak{h}$  where  $\mathfrak{h}$  is any Cartan subalgebra.

**Example 25.5.**  $\text{rank } \mathfrak{sl}(n, \mathbb{C}) = n - 1$

## 25.1 Representations of $\mathfrak{sl}(2; \mathbb{C})$

Recall  $\mathfrak{sl}(2, \mathbb{C})$  has a basis  $e, f, h$

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with  $[e, f] = h, [h, e] = 2e, [h, f] = -2f$ .

**Theorem 25.6.** For each non-negative integer  $\lambda$  there exists a  $(\lambda + 1)$ -dimensional irreducible representation  $V_\lambda$  of  $\mathfrak{sl}(2, \mathbb{C})$ . Furthermore, any finite-dimensional irreducible representation of  $\mathfrak{sl}(2, \mathbb{C})$  is isomorphic to  $V_\lambda$  for a unique non-negative integer  $\lambda$ .

*Proof.* Let  $\lambda \in \mathbb{Z}_{\geq 0}$ , let  $V = \langle x, y \rangle$  with  $\dim V = \lambda + 1$  and define

$$\rho : \mathfrak{sl}(2, \mathbb{C}) \rightarrow \text{gl}(V_\lambda)$$

by

$$\rho(e)x = (\lambda + 1)y, \quad \rho(f)y = x, \quad \rho(h)x = (\lambda + 1)x, \quad \rho(h)y = -(\lambda + 1)y$$

Note that  $\dim V_\lambda = \lambda + 1$ .

**Exercise 25.7.**  $V_\lambda$  is an irreducible representation of  $\mathfrak{sl}(2, \mathbb{C})$ .

Conversely, let  $\rho : \mathfrak{sl}(2, \mathbb{C}) \rightarrow \text{gl}(V)$  be any finite dimensional irreducible representation of  $\mathfrak{sl}(2, \mathbb{C})$ . Put  $\rho(e) = E, \rho(f) = F, \rho(h) = H$  and for  $\mu \in \mathbb{C}$  put  $V_\mu = \{v \in V \mid Hv = \mu v\}$ .

Step 1:

$$EV_\mu \subseteq V_{\mu+2}$$

$$FV_\mu \subseteq V_{\mu-2}$$

$$HV_\mu \subseteq V_\mu$$

hence,  $V = \bigoplus_{\mu \in \mathbb{Z}} V_\mu$  is a subrepresentation of  $V$ . Since  $V$  is finite dimensional and  $\mathbb{C}$  is algebraically closed,  $\exists \mu \in \mathbb{Z}$  such that  $V_\mu = 0 \neq V_{\mu+2} = 0 \neq V_{\mu+4} = \dots = V_{\mu+2n}$  since  $V$  is irreducible.

Step 2:  $\exists \lambda \in \mathbb{Z}$  and  $v \in V_\lambda$  with  $Ev = 0$ . Indeed, pick any nonzero  $w \in V_\mu$  for some  $\mu \in \mathbb{Z}$ . Then  $E^n w \in V_{\mu-2n}$ . Since  $V$  is finite dimensional, eigenvectors are linearly independent,  $\exists n \in \mathbb{N}$  such that  $E^n w = 0$ . Put  $\lambda = \mu - 2n, v = E^n w$ .

Step 3:  $W = \text{span}\{F^n v \mid n \geq 0\}$  is a submodule of  $V$ , hence  $V = W$ . Indeed

$$HF^n v = (\lambda - 2n)F^n v$$

$$FF^n v = F^{n+1} v$$

$$EF^0 v = Ev = 0$$

and for  $n \geq 0$ :

$$EF^n v = (E, F) F^{n-1} v$$

$$HF^{n-1} v = (H, F) F^{n-1} v.$$

Now  $HF^{n-1} v \in W$  and by induction  $EF^{n-1} v \in W$ . Thus  $EF^n v = (E, F) F^{n-1} v \in W$ .

Step 4:  $\lambda \in \mathbb{Z}_{n \neq 0}, V = \bigoplus_{r \geq 0} V_r$  and  $V_r = \{v \in V \mid F^n v = \lambda v\}$ . To see this, let  $n$  be the smallest positive integer such that  $F^n v = 0$ . Then

$$\begin{aligned} 0 &= EF^n v = rE, F^n s v \\ &= \rho H F^{n-1} + F H F^{n-2} + \dots + F^{n-1} H q v \\ &= \rho \lambda + 2\rho n + 1q + \lambda + 2\rho n + 2q + \dots + \lambda q F^{n-1} v \end{aligned}$$

This implies  $\lambda = n + 1 + \dots + n + \lambda - 1$  and  $v, Fv, \dots, F^{n-1}v$  is a basis.

Step 5: Define

$$\Phi: V \rightarrow \mathbb{C}[x, y], \quad F^n v \mapsto \frac{1}{\rho \lambda + n!} x^n y^n.$$

Then

$$\rho \Phi f q = \Phi \rho F^n v q = \Phi \rho F^{n-1} v q = \frac{1}{\rho \lambda + (n-1)!} x^{n-1} y^{n-1}$$

while

$$\rho \rho \Phi f q = \Phi \rho \rho F^n v q = y \rho \times \rho \frac{1}{\rho \lambda + n!} x^n y^n q = \frac{1}{\rho \lambda + n!} x^{n-1} y^{n-1}.$$

Similarly  $\Phi \rho p e q = \rho \Phi p e q$  and  $\Phi \rho p h q = \rho \Phi p h q$ . This implies a bijective intertwining operator.  $\square$

## 25.2 From Semisimple to Simple

$\mathfrak{g}$  is finite dimensional semisimple Lie algebra over  $\mathbb{C}$  and  $\mathfrak{h} \in \mathfrak{g}$  a fixed choice of Cartan subalgebra. The following is easy to check:

**Theorem 25.8** (Theorem 6.39 in Kirillov). Let  $\mathfrak{g} = \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha}$ . Then

- Every Cartan subalgebra of  $\mathfrak{g}$  has the form  $\mathfrak{h} = \bigoplus_{\alpha \in \Phi} \mathfrak{h}_{\alpha}$  where  $\mathfrak{h}_{\alpha}$  is a Cartan subalgebra of  $\mathfrak{g}_{\alpha}$ .
- Then  $\bigcup_{\alpha \in \Phi} \mathfrak{h}_{\alpha}$  is a disjoint union where  $\mathfrak{h}_{\alpha} \in \rho \mathfrak{h}_{\alpha} \cap \mathfrak{h}$  is the root system of  $\mathfrak{g}_{\alpha}$ .

## 25.3 Coroots

**Definition 25.9.** For each  $\alpha \in \Phi$ , there is a corresponding coroot  $\alpha^{\vee} = \frac{2}{(\alpha, \alpha)} \alpha \in \mathfrak{h}$ .

Let  $\rho, \eta$  be an invariant non-degenerate symmetric bilinear form on  $\mathfrak{g}$ . We know that the restriction to  $\mathfrak{h}$  is non-degenerate. This implies that  $\mathfrak{h} \rightarrow \mathfrak{h}$  by  $h \mapsto \rho(h), \eta(h)$ . Let  $H$  be the inverse image of  $\alpha$  under this map. Then

$$\rho(H), \eta(h) = \alpha \rho(h) \quad \forall h \in \mathfrak{h}.$$

Also convenient to define  $\mathfrak{h}$  by  $\rho\alpha, \beta\eta: \rho(H), \eta(h) = \alpha \rho(H), \eta(h)$ .

**Lemma 25.10.**  $\rho\alpha, \alpha\eta = \rho(H), \eta(h) = 0$ .

Then we can define  $h = \frac{2H}{\rho\alpha, \alpha\eta}$ .

**Note 25.11.**

- The 2 is to get an integer later.

ii) If  $\mathfrak{g}$  is simple we know that  $\rho, \eta$  is unique up to scalar i.e.  $\rho, \eta = \xi \rho, \xi \eta$  for  $\xi \in \mathbb{C} \setminus \{0\}$ . Then

$$h^1 = \frac{2H^1}{\rho\alpha, \alpha\eta^1} = \frac{2\xi^{-1}H}{\xi^2 \xi\rho\alpha, \alpha\eta} = h$$

Hence,  $h$  is independent of  $\rho, \eta$ .

## 26 Lecture 24: The Root System of a semisimple Lie Algebra

Today we discuss the connections between finite dimensional semisimple Lie algebras over  $\mathbb{C}$  and Root systems.

finite dimensional semisimple Lie algebra over  $\mathbb{C}$   $\leftrightarrow$  Root systems  $\leftrightarrow$  Dynkin diagrams <sup>ix</sup>

Let  $\mathfrak{g}$  be a finite-dimensional Lie algebra over  $\mathbb{C}$ ,  $\rho, \eta$  is the Killing form,  $\mathfrak{h}$  a Cartan subalgebra of  $\mathfrak{g}$ , and  $R$  root system of  $\mathfrak{g}$ .

**Lemma 26.1** ( $\mathfrak{sl}_2, \rho, \eta$  triples). *Let  $\alpha \in R$ . Pick  $e \in \mathfrak{P}_{\alpha}$  where  $e \neq 0$ . Choose  $f \in \mathfrak{P}_{-\alpha}$  by*

$$\rho(e, f) = \frac{2}{\rho\alpha, \alpha\eta}.$$

(Recall by lemma 25.10 that  $\rho\alpha, \alpha\eta \neq 0$ .) *Define  $\mathfrak{sl}_2, \mathbb{C}q := \mathbb{C}e \oplus \mathbb{C}f \oplus \mathbb{C}h$ , which is a Lie subalgebra of  $\mathfrak{g}$  isomorphic to  $\mathfrak{sl}_2, \mathbb{C}q$ .*

*Proof.* Claim:  $\rho(e, f) = \rho(e, f)H$ . Indeed for all  $h \in \mathfrak{h}$

$$\begin{aligned} \rho(h, \rho(e, f)) &= \rho(\rho(e, f)H, f) \\ &= \rho(h, e) \rho(f, f) \\ &= \rho\alpha \rho h \eta e, f \\ &= \rho(e, f) \rho H, h \eta \\ &= \rho \rho(e, f) \eta H, h \eta \\ &= \rho h, \rho(e, f) \eta H \eta. \end{aligned}$$

Since the form is non-degenerate, the claim holds. Now  $h = \frac{2H}{\rho\alpha, \alpha\eta}$ , so

$$\rho(e, f) = \rho(e, f) \frac{\rho\alpha, \alpha\eta}{2} h = h.$$

Also,

$$\rho(h, e) = \rho\alpha \rho h \eta e = \rho H, h \eta e = \frac{2\rho H, H \eta}{\rho\alpha, \alpha\eta} = 2e.$$

Similarly  $\rho(h, f) = -2f$ . □

**Lemma 26.2** (Lemma 6.43 in Kirillov). *Let  $\alpha \in R$ . Then*

$$V|_{\mathfrak{h}} \cong \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k \in \mathfrak{g}$$

*is an irreducible subrepresentation of  $\mathfrak{g}$  with respect to the adjoint action of  $\mathfrak{sl}_2, \mathbb{C}q$  on  $\mathfrak{g}$ .*

<sup>ix</sup>It turns out these maps are isomorphisms, but we save this for a later time.

*Proof.* We check  $V$  is a subrepresentation of  $\mathfrak{g}$ . We have  $e \cdot g_k = re, g_k \in \mathfrak{g}_{p_k-1q}$ . Note that  $V \cap \mathfrak{h} = 0$  and  $V \cap \mathfrak{h} = 0$  since  $h \cdot x = 2kx$  for all  $x \in \mathfrak{g}_k$ . By Exercise 4.11 in the text,  $V$  is irreducible.  $\square$

**Theorem 26.3** (Structure of semisimple Lie Algebras over  $\mathbb{C}$ ).

- (1)  $R$  spans  $\mathfrak{h}$  as a  $\mathbb{C}$ -vector space and  $\mathfrak{h} \cup \mathfrak{p}_R$  spans  $\mathfrak{g}$  as a  $\mathbb{C}$ -vector space.
- (2)  $\dim \mathfrak{g}_\alpha = 1$  for all  $\alpha \in \mathfrak{p}_R$ .
- (3)  $\alpha, \beta \in \mathfrak{p}_R$  then

$$\beta \rho h_\alpha = \frac{2\rho\beta, \alpha}{\rho\alpha, \alpha} \rho \alpha.$$

- (4) For any  $\alpha \in \mathfrak{p}_R$ , the reflection

$$s : \mathfrak{h} \rightarrow \mathfrak{h} \\ \lambda \mapsto \lambda - \lambda \rho h_\alpha \\ \lambda \mapsto \lambda - 2 \frac{\rho\lambda, \alpha}{\rho\alpha, \alpha} \alpha$$

preserves  $R$ , i.e.  $s \rho \beta \in \mathfrak{p}_R$   $\alpha, \beta \in \mathfrak{p}_R$ . In particular,  $\alpha \mapsto s \rho \alpha \in \mathfrak{p}_R$   $\alpha \in \mathfrak{p}_R$ .

- (5)  $\alpha \in \mathfrak{p}_R$   $\rho \alpha \in X$   $\alpha \in X$
- (6)  $\alpha, \beta \in \mathfrak{p}_R$  such that  $\beta = \alpha$  then  $V \cong_{\mathfrak{sl}(2, \mathbb{C})} \mathfrak{g}_k$  is an irreducible  $\mathfrak{sl}(2, \mathbb{C})$  representation with respect to the adjoint action.
- (7) If  $\alpha, \beta \in \mathfrak{p}_R$  such that  $\alpha = \beta \in \mathfrak{p}_R$ , then  $\rho \alpha, \rho \beta \in \mathfrak{p}_R$ .

*Proof.* (1) Suppose  $h \in \mathfrak{p}_R$  with  $\alpha \rho h = 0$  for all  $\alpha \in \mathfrak{p}_R$ . Then  $\text{ad } h : \mathfrak{g} \rightarrow \mathfrak{g}$  is the zero map. This implies  $h \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{h} = 0$ . Thus  $R$  spans  $\mathfrak{h}$ . The final part comes from  $\mathfrak{h} \cap \mathfrak{h} = 0$  by  $\mathfrak{h} \cap \mathfrak{h} = \alpha$ .

(2) By Lemma 26.2 and representation theory for  $\mathfrak{sl}(2, \mathbb{C})$ ,  $V \cong_{\mathfrak{sl}(2, \mathbb{C})} \mathfrak{g}_k \oplus \mathfrak{g}_k$  has  $\dim 1$  for all  $k, \alpha \in \mathfrak{p}_R$   $k = 1$  for all  $\alpha \in \mathfrak{p}_R$ .

(3)  $\beta \rho h_\alpha$  is the weight of  $x \in \mathfrak{g}$  with  $\mathfrak{sl}(2, \mathbb{C})$ -action  $h \rho x = \beta \rho h_\alpha x$ . By representation theory of  $\mathfrak{sl}(2, \mathbb{C})$ , they are all integers.

(4)  $x \in \mathfrak{g}$  has weight  $n : \beta \rho h_\alpha$  suppose  $n \geq 0$ , Then  $f^n : \mathfrak{g} \rightarrow \mathfrak{g}$   $n$ , so if  $0 \neq v \in \mathfrak{g}$  then  $\mathfrak{g} \cap \mathfrak{h} = 0$ . Which implies  $\beta = n\alpha \in \mathfrak{p}_R$  with  $s \rho \beta = \beta = n\alpha$ . For  $n = 0$  we use  $e^n$  instead. (5)-(7) Read yourselves.  $\square$

**Example 26.4.**  $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ ,  $\mathfrak{h} = \langle h_1, h_2 \rangle$  with

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that

$$[h_1, E_{12}] = 2E_{12} \quad \text{and} \quad [h_1, E_{13}] = E_{13}$$

so  $E_{12} \in \mathfrak{g}$  with  $\alpha \rho h_1 = 2$  and  $\alpha \rho h_2 = 1$ . Similarly  $E_{23} \in \mathfrak{g}$  with  $\beta \rho h_1 = 1$  and  $\beta \rho h_2 = 2$ .

Now  $E_{13} \in \mathfrak{g}$ . So  $R = \langle \alpha, \beta, \rho\alpha, \rho\beta \rangle$  and  $\alpha = \beta$  with  $\rho\alpha = \rho\beta$  with

$\rho\alpha, \rho\beta \in \mathfrak{h}$ ,  $\mathfrak{h} \cap \mathfrak{h} = 2 \cos(120^\circ) = 1$ .

## 27 Lecture 25: (Abstract) Root Systems

**Definition 27.1.** An (abstract) root system is a finite subset of a Euclidean space  $R \in E \setminus \{0\}$  where  $E$  is a Euclidean space, such that

$$(R1) \quad \text{span}_{\mathbb{R}} R = E;$$

$$(R2) \quad \forall \alpha, \beta \in R: n = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z};$$

(R3) Let  $s : E \rightarrow E$  given by

$$s(\lambda) = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \alpha.$$

Then  $s(\beta) \in R \forall \beta \in R$ .

$$(R4) \quad \forall \alpha \in R \quad \exists \rho(\alpha) \in \mathbb{R}^+ \quad \langle \rho(\alpha), \alpha \rangle = 1.$$

**Note 27.2.**  $s$  is the orthogonal reflection in the root hyperplane  $L$ :

$$L = \{ \lambda \in E \mid (\lambda, \alpha) = 0 \}.$$

**Example 27.3.** Let  $\mathfrak{g}$  be a finite dimensional semisimple Lie algebra over  $\mathbb{C}$  and  $\mathfrak{h} \in \mathfrak{g}$  Cartan subalgebra. Then the set of roots of  $\mathfrak{g}$  with respect to  $\mathfrak{h}$  is a root system in  $\mathfrak{h}_{\mathbb{R}}$  where

$$\mathfrak{h}_{\mathbb{R}} = \text{span}_{\mathbb{R}} \mathfrak{h} \quad \forall \alpha \in R \quad \rho(\alpha) = 1$$

a real form of  $\mathfrak{h}$  i.e.  $\mathfrak{h}_{\mathbb{R}} \cong \mathfrak{h} \otimes_{\mathbb{C}} \mathbb{R}$ .  $\mathfrak{h}_{\mathbb{R}}$  has a positive definite inner product (By a lemma  $\rho, \eta$  Killing form restricted to  $\mathfrak{h}_{\mathbb{R}}$  is a positive definite inner product.)

**Notation 27.4.** If  $v \in V, \lambda \in V^*$  we define  $\chi_v, \lambda_y = \lambda(v) = \langle \lambda, v \rangle$ .

**Definition 27.5.** Let  $R \in E$  be an (abstract) root system. The coroot  $\alpha^-$  of  $\alpha \in R$  is defined by  $\alpha^- \in E$  where

$$\chi_{\alpha^-}, \lambda_y = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}.$$

**Note 27.6.**

(1) This is consistent with the Lie algebra definition of coroot:

$$\alpha^- = \frac{2H}{(\alpha, \alpha)}$$

where  $\rho(H), \eta = \alpha \rho \eta \in \mathfrak{h}$ .

(2) Integrality says  $\chi_{\alpha}, \beta^- \in \mathbb{Z} \forall \alpha, \beta \in R$  and  $s(\lambda) = \lambda - \chi_{\lambda, \alpha^-} \alpha$ .

**Example 27.7** (Root system of type  $A_{n-1}$ ). Let  $\{e_i\}_{i=1}^n$  be the orthonormal basis for  $\mathbb{R}^n$ . Let

$$E = \{ \pm \lambda_1, \dots, \pm \lambda_{n-1} \} \quad \lambda_i = e_i - e_{i+1}$$

$$R = \{ \pm e_i - \pm e_j \mid 1 \leq i, j \leq n, i \neq j \}.$$

Then  $R$  spans  $E$  over  $\mathbb{R}$ .  $\rho_{\varepsilon_i} = \varepsilon_j, \varepsilon_i = \varepsilon_j \rho_{\alpha, \beta} \in \mathbb{Z} \otimes_{\mathbb{R}} \mathbb{R} \cong \frac{2\rho_{\alpha, \beta}}{\rho_{\beta, \beta}} \rho_{\alpha, \beta} \in \mathbb{Z}$ .  
 Now,

$$s_i^{-1} \rho_{\lambda_1, \dots, \lambda_i, \dots, \lambda_j, \dots, \lambda_n} = \rho_{\lambda, \varepsilon_i = \varepsilon_j} \rho_{\varepsilon_i = \varepsilon_j} \\
\lambda = \rho_{\lambda_i = \lambda_j} \rho_{\varepsilon_i = \varepsilon_j} \\
\lambda = \lambda_i \varepsilon_i = \lambda_j \varepsilon_j = \lambda_j \varepsilon_i \\
\lambda_1, \dots, \lambda_j, \dots, \lambda_i, \lambda_n \\
\tilde{S}_i^{-1} \rho_{i,j} \in S_n.$$

Clearly  $s_i \rho_{\beta} \in R$ . Lastly,  $R \otimes_{\mathbb{R}} \mathbb{R} \cong \mathbb{R} \otimes_{\mathbb{R}} \mathbb{R}$  is clear. Fact:  $A_{n-1}$  is the root system of  $\mathfrak{sl}(n, \mathbb{C})$ .

**Definition 27.8.** An isomorphism from  $R_1 \in E_1$  to  $R_2 \in E_2$  is a  $\mathbb{R}$ -linear isomorphism  $\varphi: E_1 \xrightarrow{\sim} E_2$  such that

- i)  $\varphi(\rho_{R_1}) = R_2$ ;
- ii)  $n \cdot \rho_{\alpha} = \rho_{\alpha} \cdot n$  for all  $\alpha \in R_1$ .

**Example 27.9.**  $R \in E$  is isomorphic to  $c \cdot R$  a root system by  $\varphi: E \xrightarrow{\sim} E$  by  $\lambda \mapsto c \cdot \lambda$  for all  $c \in \mathbb{R} \setminus \{0\}$ .

**Definition 27.10.** The Weyl group  $W = W(\rho, R)$  of a root system  $R \in E$  is the subgroup of  $GL(\rho, E)$  generated by  $\{s_{\alpha} \mid \alpha \in R\}$ .

**Lemma 27.11.** Let  $W$  be the Weyl group of a root system  $R \in E$ .

- 1)  $W$  is finite;
- 2)  $W \cong O(\rho, E)$  the orthogonal group
- 3) If  $w \in W, \alpha \in R$  then  $ws = w^{-1} s_{w\rho}$ .

*Proof.* 1) We know  $W(\rho, R) \in R$  since  $s_{\rho} \rho \in R$ , so we get a map  $\varphi: W \xrightarrow{\sim} S_R$ . We claim that  $\varphi$  is injective. Suppose  $w \in \ker \varphi$ , then  $w\rho = \alpha$  for all  $\alpha \in R$ . By  $R$  spanning  $E$  we get that  $w = \text{Id}_E$ .

2) Each  $s_{\alpha}$  is an orthogonal transformation (i.e.  $\rho, s_{\alpha}\rho = \rho, \mu$  for all  $\lambda, \mu$ )  $\tilde{R} W \cong O(\rho, E)$

3) Consider the following calculation

$$\rho, ws = w^{-1} \rho, \lambda = \rho, s w^{-1} \rho, \lambda \\
= \rho, w^{-1} \rho, \lambda = \frac{\rho, w^{-1} \rho, \alpha}{\rho, \alpha} \alpha \\
= \lambda = 2 \frac{\rho, w\rho, \alpha}{\rho, w\rho, \alpha} w\rho \\
= s_{w\rho} \rho, \lambda.$$

□

## 28 Lecture 26: More on Root Systems

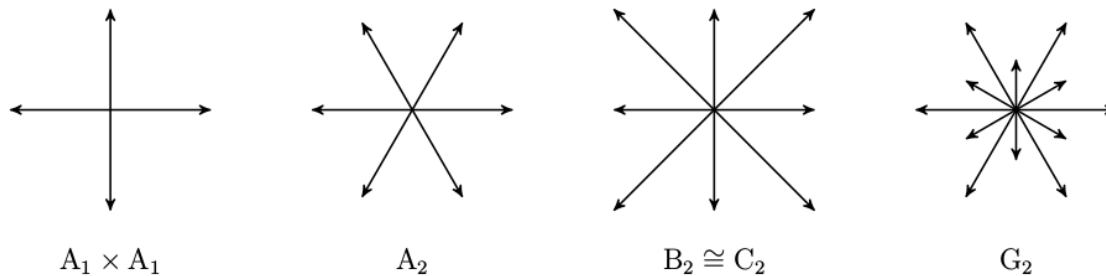
**Theorem 28.1.** Let  $\alpha, \beta$  be non-parallel roots of a root system  $R$ . Let  $\theta \in (0, 2\pi)$  be the angle between them. Then  $\theta \in \left\{ \frac{2\pi}{12}, \frac{2\pi}{8}, \frac{2\pi}{6}, \frac{2\pi}{4}, \frac{2\pi}{3}, \frac{2\pi}{2} \right\}$  and  $n \in \{0, 1, 2, 3\}$ .

*Proof.*

$$\cos \theta = \frac{2(\alpha, \beta)}{|\alpha| |\beta|} = \frac{2(\beta, \alpha)}{|\beta| |\alpha|} = \frac{4|\alpha|^2 |\beta|^2 \cos^2 \theta}{4|\alpha|^2 |\beta|^2} = 4 \cos^2 \theta$$

□

### 28.1 Root Systems of Rank 2



$$\begin{aligned}
 A_1 &= A_1 \text{ or } A_1 \setminus A_1 \cong \mathfrak{sl}(2, \mathbb{C}) \\
 A_2 &\cong \mathfrak{sl}(3, \mathbb{C}) \\
 B_2 &= C_2 \cong \mathfrak{so}(5, \mathbb{C}) \\
 G_2 &\cong \mathfrak{der}(\mathbb{O})
 \end{aligned}$$

Where  $\mathfrak{der}(\mathbb{O})$  is the Lie algebra of derivations of the octonions, which is one of the five exceptional simple Lie algebras.

### 28.2 $E_{\text{reg}}$ , Weyl chambers, and Positive Roots

**Definition 28.2.** Let  $R \in E$  be a root system. The set of *regular vectors* in  $E$  is

$$E_{\text{reg}} = \{ \tau \in E \mid (\alpha, \tau) \neq 0 \text{ for all } \alpha \in R \}$$

The connected components of  $E_{\text{reg}}$  are called the *Weyl chambers* of  $R$ .

Given a Weyl chamber  $C$ , we associate a *polarization* of  $R$ :

$$R = R^+ \cup R^-, \text{ where } R^+ = \{ \alpha \in R \mid (\alpha, \tau) > 0 \text{ for } \tau \in C \}$$

Call  $\alpha \in R^+$  (respectively  $\alpha \in R^-$ ) a *positive* (respectively *negative*) *root*.



**Remark 28.3.** This doesn't depend on the choice of  $\tau \in C$ .  $C$ , being defined as a set of linear inequalities of the form  $\rho\alpha, \tau\eta \geq 0$  (or  $\rho\alpha, \tau\eta < 0$ ), is intersection of (open) half-spaces. Therefore,  $C$  is convex. So if  $\tau, \tau^1 \in C$  then the straight line segment  $[\tau, \tau^1] \subset C$ . By the intermediate value theorem,  $R = R^1$ .

**Remark 28.4.** If  $\tau \in E_{\text{reg}}$  and  $\alpha \in R$ , then  $\langle \beta, \tau \rangle \in R$ :

$$\langle \beta, s^{-1}\tau \rangle = \langle \beta, \tau \rangle \langle \beta, s \rangle \in R$$

Now  $s^{-1}\tau \in R$ , hence  $s^{-1}\langle \beta, \tau \rangle \in E_{\text{reg}}$ . Moreover, if  $\tau, \tau^1$  belong to the same connected component  $C$  of  $E_{\text{reg}}$ . Let  $p = [0, 1]$  be a path from  $\tau$  to  $\tau^1$ . Then  $s^{-1}p$  is a path from  $s^{-1}\tau$  to  $s^{-1}\tau^1$ . So  $W$  acts on  $\pi_0(E_{\text{reg}})$  transitively. Lastly, note that, the Weyl group acts on  $R$  by root system automorphisms. (More generally we have  $\text{OpEq} \cong \text{AutpRq}$ ).

**Theorem 28.5.** The action of  $W$  on  $\pi_0(E_{\text{reg}})$  is transitive.  $\langle C, C^1 \rangle \in \pi_0(E_{\text{reg}})$   $Dw \in W$ :  $w[C] = C^1$ . Thus every root system has a unique polarization, up to a Weyl group automorphism of  $R$ .

*Proof.* Let  $C, C^1$  be any two Weyl chambers. Pick  $\tau \in C, \tau^1 \in C^1$  such that  $\tau = \tau^1 = 0$ . Then the line segment  $[\tau, \tau^1]$  is contained in  $E_{\text{reg}}$  and crosses some of the hyperplanes  $L_1, L_2, \dots, L_k$ . Claim: If  $C_1$  and  $C_2$  are adjacent, i.e.  $L = \text{span}_{\mathbb{R}}(\overline{C_1} \times \overline{C_2})$ , then  $s^{-1}C_1 \subset C_2$ . Pick  $\tau \in C_1$ , the line segment  $[\tau, s^{-1}\tau]$  only intersects  $L$ . Hence,  $\langle \beta, \tau \rangle, \langle \beta, s^{-1}\tau \rangle$  and  $\langle \beta, \tau \rangle, \langle \beta, s^{-1}\tau \rangle$  have the same sign for all  $\alpha \in R$ . Thus  $C^1 = s^{-1}C$ .  $\square$

## 29 Lecture 27: Simple Roots

Last time discussed how any root system  $R$  has a unique (up to Weyl group automorphism) polarization  $R = R^+ \cup R^-$ .

**Definition 29.1.**  $\alpha \in R$  is simple if it is not a sum of two positive roots.

**Lemma 29.2.** Every positive root is a sum of simple roots.

*Proof.* Let  $\alpha \in R$ . If  $\alpha$  is simple we are done. If not,  $\alpha = \beta + \gamma$ , for  $\beta, \gamma \in R$ . Then pick  $\tau \in C = \{ \lambda \in E \mid \rho\lambda, \alpha^1 \geq 0 \}$ . Then  $\rho\alpha, \tau\eta = \rho\beta, \tau\eta + \rho\gamma, \tau\eta$  each of which is strictly greater of zero  $\rho\beta, \tau\eta, \rho\gamma, \tau\eta > \rho\alpha, \tau\eta$ . The set  $\{ \lambda \in R \mid \rho\lambda, \tau\eta > 0 \}$  is finite; therefore, totally ordered. So we proceed by induction (or by contradiction).  $\square$

**Proposition 29.3.** Every root is a unique combination of simple roots with integer coefficients:

$$\alpha = \sum_{i=1}^r n_i \alpha_i, \quad n_i \in \mathbb{Z}$$

Where  $\Pi = \{ \alpha_1, \dots, \alpha_r \}$  is the set of simple roots. Moreover,  $\alpha \in R \iff n_i \neq 0 \forall i$

*Proof.* Without loss of generality  $\alpha \in R^+$ .

Step 1: Linear algebra fact: If  $\{v_i\} \in E$  is a set of vectors, all lying on the same side of some hyperplane, such that  $\langle v_i, v_j \rangle \geq 0 \forall i, j$ , then  $\{v_i\}$  is linearly independent. (Exercise)

Step 2: By Step 1, it suffices to show that  $\langle \alpha_i, \alpha_j \rangle \geq 0 \forall i, j$ . Fix  $i, j$  and let  $R^1 = \{ \lambda \in R \mid \langle \lambda, \alpha_j \rangle \geq 0 \}$ . By Exercise 29.4 below,  $R^1$  is a root system of rank two. Let  $R^1 = R^+ \cup R^-$ . Then  $R^1 = R^+ \setminus R^-$  is a polarization of  $R^1$  and  $\alpha_i, \alpha_j$  are the two simple roots with respect to this polarization. By the description of the root systems of rank two, it now follows that the angle between  $\alpha_i$  and  $\alpha_j$  is obtuse, hence  $\langle \alpha_i, \alpha_j \rangle \geq 0$ .  $\square$

**Exercise 29.4.** If  $\alpha, \beta \in R$ ,  $\alpha \neq \beta$ , then  $\{Z\alpha - Z\beta\} \times R$  is a root system of rank 2.

**Corollary 29.5.**  $\Pi$  is a basis for  $E$  over  $R$ , so  $|\Pi| = \dim E = \text{rank } R$ .

*Proof.*  $\Pi$  linear independent and spans  $R$  over  $Z$ , and  $R$  spans  $E$  over  $R \cap \Pi$  spans  $E$  over  $R$ .  $\square$

## 29.1 Simple Reflections

**Definition 29.6.**  $R \setminus R^+$ ,  $\Pi = \{\alpha_1, \dots, \alpha_r\}$ . Then  $s_i = s_{\alpha_i}$  are called *simple reflections*.

**Proposition 29.7.** Any simple reflection  $s_i$  permutes the positive roots other than  $\alpha_i$ , i.e.  $s_i R^+ \setminus \{\alpha_i\} = R^+ \setminus \{\alpha_i\}$ .

*Proof.* Let  $\beta \in R^+ \setminus \{\alpha_i\}$  with  $\beta = \sum_{j=1}^r n_j \alpha_j$ ,  $n_j \geq 0$ . Then  $s_i \beta = \beta - \langle \beta, \alpha_i \rangle \alpha_i$ . So if  $s_i \beta \in R^+$ , then  $n_j = 0$  for all  $j = i$ . This implies that  $n_j = 0$  for all  $j = i$ . Thus  $\beta = \sum_{j \neq i} n_j \alpha_j \in R^+ \setminus \{\alpha_i\}$ .  $\square$

**Corollary 29.8.** The Weyl vector,  $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$  satisfies:

$$\langle \rho, \alpha_i \rangle = 1 \quad \forall i = 1, \dots, r$$

equivalently  $s_i \rho = \rho - \alpha_i$ .

**Theorem 29.9.**

- i) The simple reflections generate the Weyl group.
- ii)  $\forall \alpha \in R^+, w \in W, \alpha_i \in \Pi: \alpha = w \alpha_i$ .
- iii)  $W$  acts simply transitively on  $R^+ \setminus \{0\}$  i.e. if  $w \alpha = \beta$ , then  $w = 1$ .

## 30 Example: $\mathfrak{sp}(4, \mathbb{C})$

For  $\mathbb{C}^4$  we have symplectic form  $w: \mathbb{C}^4 \times \mathbb{C}^4 \rightarrow \mathbb{C}$ , i.e.  $w$  is

- bilinear
- non-degenerate
- skew-symmetric

$\mathfrak{sp}(4, \mathbb{C}) = \{X \in \mathfrak{gl}(4, \mathbb{C}) \mid w(Xv, u) = w(v, Xu) \quad \forall v, u \in \mathbb{C}^4\}$ .

**Theorem 30.1.** Over  $\mathbb{C}$  all symplectic forms are equivalent (i.e. coincide after a change of basis).

We have matrix for  $w$ :

$$J = (w(e_i, e_j))_{i,j}$$

i.e.,  $w(a, b) = a^T J b$ .

**Example 30.2.** A good choice  $J = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$  or  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$ .



So  $F_{12} \in \mathfrak{g}$  and  $F_{21} \in \mathfrak{g}$ , where  $\alpha \rho h_1 \mathfrak{q} = 1$  and  $\alpha \rho h_2 \mathfrak{q} = 1$ . Similarly,

$$\begin{aligned} \rho h_1, F_{13} \mathfrak{S} &= \rho E_{11} - E_{33}, E_{13} \mathfrak{S} \\ &= 2 F_{13}; \\ \rho h_2, F_{12} \mathfrak{S} &= \rho E_{22} - E_{44}, E_{13} \mathfrak{S} \\ &= 0. \end{aligned}$$

So  $F_{13} \in \mathfrak{g}$  and  $F_{31} \in \mathfrak{g}$ , where  $\beta \rho h_1 \mathfrak{q} = 2$  and  $\beta \rho h_2 \mathfrak{q} = 0$ . By similar calculations we get

$$\begin{aligned} \rho h_1, F_{14} \mathfrak{S} &= \rho E_{11} - E_{23}, E_{14} - E_{23} \mathfrak{S} = F_{14}; \\ \rho h_2, E_{14} \mathfrak{S} &= F_{14}; \end{aligned}$$

with  $F_{14} \in \mathfrak{g}$  and  $F_{41} \in \mathfrak{g}$ , where  $\gamma \rho h_1 \mathfrak{q} = \gamma \rho h_2 \mathfrak{q} = 1$ , and

$$\begin{aligned} \rho h_1, F_{24} \mathfrak{S} &= \rho E_{11} - E_{33}, E_{24} \mathfrak{S} \\ &= 0; \\ \rho h_2, F_{24} \mathfrak{S} &= \rho E_{22} - E_{44}, E_{24} \mathfrak{S} \\ &= 2 F_{24}; \end{aligned}$$

with  $F_{24} \in \mathfrak{g}$  and  $F_{42} \in \mathfrak{g}$ , where  $\delta \rho h_1 \mathfrak{q} = 0$  and  $\delta \rho h_2 \mathfrak{q} = 2$ .

**Note 30.4.** The following relations hold for our roots:  $\alpha = \delta = \gamma, \alpha = \gamma = 2\alpha = \delta = \beta$ .

Thus  $R = \text{tr}(\alpha, \delta, \gamma) = \alpha = \delta, \beta = \delta = 2\alpha$ . Using the trace form on  $\mathfrak{h}_R = \mathbb{R}h_1 \oplus \mathbb{R}h_2$ , we have  $\rho h_i, h_j \mathfrak{q} = \text{Tr}(\rho h_i h_j \mathfrak{q}) = 2\delta_{ij}$ . By rescaling we define  $\rho h_i, h_j \mathfrak{q} = \delta_{ij}$ .

Now we work to find the coroots.  $H = a_1 h_1 - a_2 h_2$ , by the definition of coroots we know that

$$\begin{aligned} 1 &= \alpha \rho h_1 \mathfrak{q} = \rho(H, h_1 \mathfrak{q}) = a_1 \\ 1 &= \alpha \rho h_2 \mathfrak{q} = \rho(H, h_2 \mathfrak{q}) = a_2. \end{aligned}$$

This implies that  $H = h_1 - h_2$ .  $\tilde{\alpha} = \frac{2H}{\rho(H, H \mathfrak{q})} = H = h_1 - h_2$ . Similarly,  $H = 2h_2$

implies  $\tilde{\alpha} = \frac{2H}{\rho(H, H \mathfrak{q})} = \frac{1}{2}H = h_2$ .

Now we compute  $n$  and  $n$ :

$$\begin{aligned} n &= \frac{2\rho\delta, \alpha \mathfrak{q}}{\rho\alpha, \alpha \mathfrak{q}} = \frac{\delta \rho \alpha - \mathfrak{q}}{\delta \rho h_1 - h_2 \mathfrak{q}} = 2 \\ n &= \frac{\alpha \rho \delta - \mathfrak{q}}{\alpha \rho h_2 \mathfrak{q}} = 1. \end{aligned}$$

**Note 30.5.**  $\rho\delta, \delta \mathfrak{q} = 2\rho\alpha, \alpha \mathfrak{q}$ .

So

$$\begin{aligned} 4 \cos^2 \theta &= \frac{n}{n} \\ &= \frac{\rho(1 \mathfrak{q} - 2 \mathfrak{q})}{\rho(1 \mathfrak{q} - 2 \mathfrak{q})} \\ \cos^2 \theta &= \frac{1}{2}. \end{aligned}$$

Now  $\rho\alpha, \delta \mathfrak{q} = 0$  implies that  $\cos \theta = 0$ . Hence,  $\cos \theta = \frac{1}{2}$ .  $\tilde{\alpha} = \theta = 3 \frac{2}{8}$ . A suitable choice for  $R$

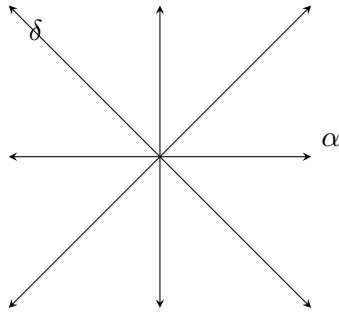


Figure 17: Root System of  $\mathfrak{sp}(4, \mathbb{C})$

would be  $R = \{\pm\alpha, \pm\delta, \pm(\alpha + \delta), \pm 2\alpha\}$  and  $\Pi = \{\alpha, \delta\}$ .

The Weyl group  $W = \langle s_1, s_2 \rangle$  such that  $s_1 s_2 = \rho_{\frac{\pi}{4}}$ , so  $W = D_4$  the dihedral group of order 8.

The Cartan matrix (which will be discussed more next time) is:

$$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

### 31 Lecture 28: Cartan Matrices and Dynkin Diagrams

Let  $R$  root system  $\in E$ ;  $R^+$  choice of positive roots;  $\Pi \in R^+$  simple roots  $\Pi = \{\alpha_1, \dots, \alpha_r\}$ .

**Theorem 31.1.**  $R$  can be recovered from  $\Pi$  and the numbers  $n_i = \langle \alpha_i, \alpha_i \rangle$ .

*Proof.* Recall  $R = W(\Pi)$  and  $W$  is generated by the simple reflections (by Theorem 29.9). So if  $\alpha \in R$ , then  $\alpha = s_{i_1} \dots s_{i_k} \alpha_{j_1}$  for  $i_1, \dots, i_k, j_1 \in \{1, \dots, r\}$ .

**Note 31.2.**  $\langle s_{i_1} \dots s_{i_k} \alpha_{j_1}, \alpha_{j_1} \rangle = \langle \alpha_{j_1}, \alpha_{j_1} \rangle - 2 \langle \alpha_{j_1}, \alpha_{i_1} \rangle + \dots$

By linearity,  $\langle s_{i_1} \dots s_{i_k} \alpha_{j_1}, \alpha_{j_1} \rangle$  can be computed from  $n_{j_1}, n_{j_1 - i_1}, n_{j_1 - i_1 - i_2}$ . By induction  $\alpha$  can be computed using only knowledge of  $\Pi$  and  $n_i$  for  $\alpha, \beta \in \Pi$ .  $\square$

**Definition 31.3.** The *cartan matrix* of  $R$  is defined by:  $A = (a_{ij})_{i,j=1}^r$  where

$$a_{ij} = \frac{2 \langle \alpha_j, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}$$

**Corollary 31.4.**  $R$  is uniquely determined by  $A$ .

**Example 31.5.** We look at the cartan matrices for rank 2 root systems.

$$A_1 \setminus A_1 : \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$A_2 : \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

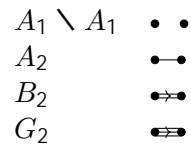
$$B_2 \quad C_2 : \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$$

$$G_2 : \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

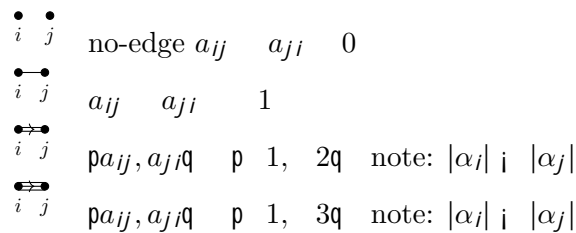
**Note 31.6.**

- i)  $a_{ii} = 2$  for all  $i$
- ii)  $\rho a_{ij}, a_{ji} \in \mathbb{P} \setminus \{0\}, \rho \in \{1, 2, 3\}$  and  $\rho a_{ij} \in \mathbb{P} \setminus \{0\}, \rho \in \{1, 2, 3\}$
- iii)  $|a_{ij}| = \rho |a_{ji}| \iff |\alpha_i| = \rho |\alpha_j|$ .

### 31.1 Rank 2 Dynkin diagrams



**Definition 31.7.** The *Dynkin diagram*  $D$  of  $R$  is a graph with vertex set  $\Pi$  (often identified with  $\{1, \dots, r\}$ ) and edges of the following four kinds:



Main point:  $A$ , hence  $R$ , can be recovered from  $D$ .

**Definition 31.8.** A set  $S \in E$  is the *orthogonal union* of two subsets  $S_1, S_2 \in S$  if

- 1)  $S = S_1 \cup S_2$
- 2)  $u \perp v \iff u \in S_1, v \in S_2$ .

Notation:  $S = S_1 \sqcup S_2$

**Definition 31.9.**  $S \in E$  decomposable if  $S = S_1 \sqcup S_2, S_i \neq \emptyset$ .

## 32 Lecture 29: Classification of Root Systems

**Lemma 32.1.** Let  $R$  be a root system,  $\Pi$  a set of simple roots,  $A$  the cartan matrix, and  $D$  the Dynkin diagram. TFAE:

- 1)  $R$  is irreducible, i.e.  $R = R_1 \sqcup R_2$  with  $R_1 \cap R_2 = \emptyset$  or  $R_2 = \emptyset$ .
- 2)  $\Pi$  is irreducible.
- 3)  $D$  is connected.
- 4)  $A$  is indecomposable, i.e. it cannot be written as a block diagonal matrix  $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$  even after reordering  $\Pi$ .

*Proof.* Bonus Homework Question. □

**Lemma 32.2.** If  $R$  is a reducible subset of  $E$ :  $R = R_1 \sqcup R_2$  with  $R_i \neq \emptyset$ , then  $R_i$  are root systems in  $E_i = \text{span}R_i$ .

*Proof.* Bonus Homework Question. □

**Corollary 32.3.** Without loss of generality, we may assume  $R$  is irreducible.

Thus our goal should be to describe all connected Dynkin diagrams!

### 32.1 Coxeter Graphs

**Definition 32.4.** A *coxeter graph*  $\Gamma = (\Gamma_0, \Gamma_1)$  is an undirected loopless graph such that each edge  $e \in \Gamma_1$  has multiplicity  $m_e \in \mathbb{Z}_{\neq 0}$ . By thinking of no-edge as multiplicity 0, we may think of

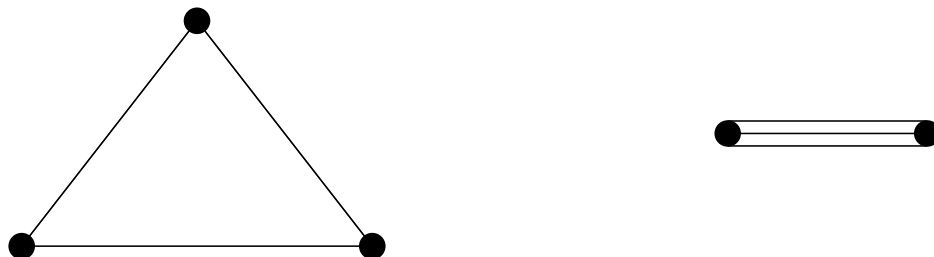
$$\Gamma_1 \in \mathbb{Z}_{\neq 0}^{\Gamma_0 \times \Gamma_0}$$

and the adjacency matrix for  $\Gamma$

$$A = \begin{pmatrix} 0 & a_{ij} \\ \vdots & \vdots \\ a_{ji} & 0 \end{pmatrix}$$

is symmetric with  $a_{ij} \in \mathbb{Z}_{\neq 0}$ .

**Example 32.5.** Below are two examples of coxeter graphs.



**Example 32.6.** Forgetting the direction of a Dynkin diagram gives a coxeter-graph.

## 32.2 Admissible Sets

**Definition 32.7.** An *admissible set*  $U = \{e_1, \dots, e_r\} \subseteq E$  is a set of linearly independent unit vectors such that  $\langle e_i, e_j \rangle = 0$ :

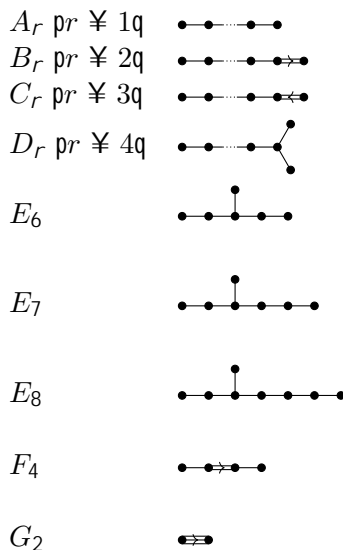
- 1)  $\langle e_i, e_j \rangle = 0$ ;
- 2)  $\langle e_i, e_j \rangle^2 \in \{0, 1, 2, 3\}$ .

**Example 32.8.** The following will be our main example to consider  $U = \{e_i \mid \alpha_i \in \Pi\}$ .

To any admissible set  $U$ , we can attach a coxeter graph  $\Gamma = (\Gamma_0, \Gamma_1)$  where  $\Gamma_0 = U$  and  $e_j, e_i$  are connected by an edge of multiplicity  $\langle e_i, e_j \rangle^2$ . So we have the following commuting diagram:

$$\begin{array}{ccc} \Pi & & U \\ & \div & \\ D & & \Gamma \end{array}$$

**Theorem 32.9** (Classification of Root systems). *Let  $R$  be any root system. Then its Dynkin diagram  $D$  is a union of diagram of the following type:*



Moreover, any two from this list correspond to non-isomorphic root systems.

**Remark 32.10.** Types  $A$  through  $D$  are called *classical types*, and types  $E$  through  $G$  are the five *exceptional types*.

*Proof.* We show that any connected admissible graph is one the above types (with orientation removed). Let  $U = \{e_1, \dots, e_r\}$  be any admissible set, and  $\Gamma$  be its coxeter graph. Then:

- 1) Any subset of an admissible set is admissible. □

*Proof of 1).* Clear. □

- 2)  $c = |\{e_i, e_j \mid \langle e_i, e_j \rangle^2 = 2\}|$ . Then  $c = r$ .

*Proof of 2).* Let  $e = e_i$ . Then

$$0 \leq \langle e, e \rangle = r = \sum_{i,j} \langle e_i, e_j \rangle^2 = r + 2c.$$

Hence,  $c = r$ . □

- 3)  $\Gamma$  has no cycles.



*Proof of 3).* If  $\Gamma^1 \in \Gamma$  is a cycle, by 1)  $\Gamma^1$  is an admissible graph, but this contradicts 2) because in cycles  $c = r$ . □

4) The degree of any vertex (with multiplicity) is  $\leq 3$ .

*Proof of 4).* Suppose  $e \in U$  has edges to  $\eta_1, \eta_2, \dots, \eta_k$ , then  $4p_e, \eta_i q^2 \in \{0, 1, 2, 3\}u$ . By 3)  $p_{\eta_i, \eta_j} q = 0$  if  $i \neq j$  (else  $\Gamma$  would contain a cycle).  $W = \text{span}\{\eta_i u\}_{i=1}^k : \text{proj}_W e = \sum_{i=1}^k p_{e, \eta_i} q \eta_i$ . Now  $e^1 = e$  since  $\{e, \eta_1, \dots, \eta_k u\}$  are linearly independent. Hence,

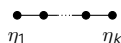
$$4 \sum_{i=1}^k p_{e, \eta_i} q = 4 \sum_{i=1}^k p_{e^1, \eta_i} q = \sum_{i=1}^k 4 p_{e, \eta_i} q^2 = \text{deg } e.$$

□

5) By 4) the only possible diagrams containing a triple edge is



6) If  $\{\eta_1, \dots, \eta_k\} \in U$  with their induced subgraph is



then they can be replaced by a single point, i.e.  $p_{U^1} \{ \eta_1, \dots, \eta_k \} \in U^1$ , where  $\eta = \sum_{i=1}^k \eta_i$  is, is admissible. □

### 33 Lecture 30: Classification Proof Continued

*Proof of Theorem 32.9 Con't.* 6) Linear roots can be deformed to a single point.  $U^1 = p_{U^1} \{ \eta_1, \dots, \eta_k \} \in U^1$  with  $\eta = \sum_{i=1}^k \eta_i$ .

*Proof of 6).* Clearly,  $U^1$  is linearly independent. Now

$$p_{\eta, \eta} q = \sum_{i=1}^k p_{\eta_i, \eta_i} q = \sum_{i=1}^k p_{k-1} q = p_{k-1} q = k-1$$

We want to show that  $4p_{\varepsilon, \eta} q^2 \in \{0, 1, 2, 3\}u$  for all  $\varepsilon \in U^1$ . Now any  $\varepsilon \in U^1$  is connected to at most one of the  $\{\eta_1, \dots, \eta_k\}$  (else there would be a cycle), so  $p_{\varepsilon, \eta} q = 0$  or  $p_{\varepsilon, \eta} q = p_{\varepsilon, \eta_i} q$  for exactly one  $i$ . Hence,  $4p_{\varepsilon, \eta} q^2 = 4p_{\varepsilon, \eta_i} q^2 \in \{0, 1, 2, 3\}u$ . □

7)  $\Gamma$  contain no subgraphs of the form shown in Figure 19.

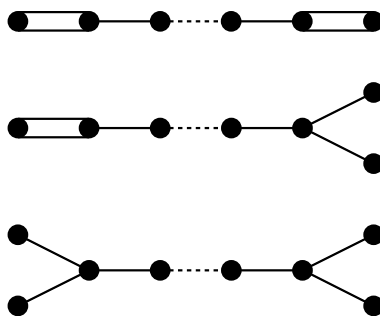


Figure 19: None of these can occur as subgraphs in an admissible graph.

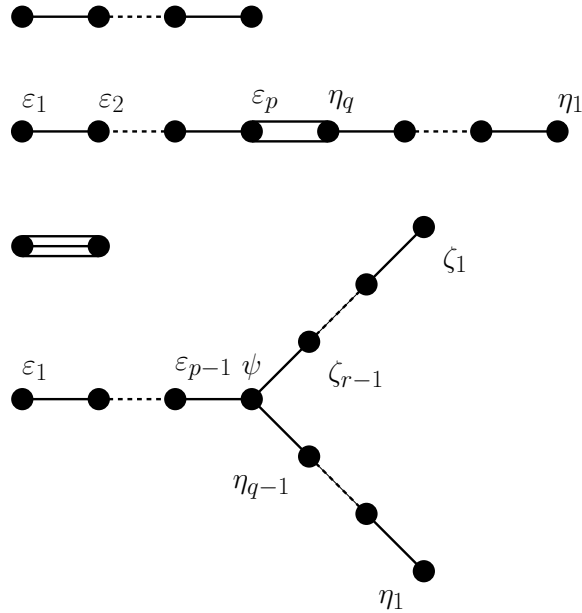
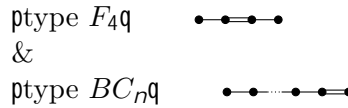


Figure 20: Any connected graph  $\Gamma$  of an admissible set must be one of these.

*Proof of 7).* If we collapse lines to points as in 6), then we would reach a vertex of degree greater than 3 from each of these graphs which is a contradiction to 4).  $\square$

8) Any connected  $\Gamma$  of an admissible set must be of the form shown in Figure 20 9) The only graphs of the second variety are



*Proof of 9).* Define  $\varepsilon : \sum_{i=1}^p i\varepsilon_i$  and  $\eta : \sum_{j=1}^q j\eta_j$ . Then

$$\rho_{\varepsilon, \varepsilon} = \sum_{i=1}^p i^2, \quad \rho_{\eta, \eta} = \sum_{j=1}^q j^2, \quad \rho_{\varepsilon, \eta} = \frac{pq}{2}$$

Similarly,

$$\rho_{\eta, \eta} = \frac{qp}{2}$$

Now

$$\rho_{\varepsilon, \eta}^2 = p^2 q^2 \rho_{\varepsilon, \varepsilon} \rho_{\eta, \eta}$$

By Cauchy-Schwarz:

$$\frac{p^2 q^2}{2} \rho_{\varepsilon, \eta}^2 = \rho_{\varepsilon, \varepsilon} \rho_{\eta, \eta} = \frac{pq}{2} \frac{qp}{2}$$

This implies  $p = q = 2$ , so either  $p = 1$  or  $q = 1$  (Type  $BC_n$ ), or  $p = q = 2$  (Type  $F_4$ ).  $\square$

10) The only  $\Gamma$  of the fourth type is  $D_n$  or  $E_6, E_7, E_8$ .

*Proof of 10).* Define  $\varepsilon : \sum_{i=1}^p i\varepsilon_i$ ,  $\eta : \sum_{j=1}^q j\eta_j$ , and  $\zeta : \sum_{l=1}^r l\zeta_l$ . Then  $\varepsilon$ ,  $\eta$ , and  $\zeta$  are necessarily orthogonal and linearly independent, and  $\psi \in \text{span}\{\varepsilon, \eta, \zeta\}$  as in 4)  $\cos^2 \theta = \cos^2 \theta$

$\cos^2 \theta = 1$  where these are the angles between the projection of  $\text{proj}_{\text{span}\{e_i\}} \psi$  and the respective roots. On the other hand,

$$\cos^2 \theta = \frac{(\rho, \psi)^2}{|\rho|^2} = \frac{(\rho, \psi)^2}{|\rho|^2} = \frac{1}{2} \left( \frac{1}{p} + \frac{1}{q} \right)$$

Similarly for  $\theta$  and  $\theta'$ . This implies that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$$

Without loss of generality,  $p \leq q \leq r$ . If  $r = 1$  we get  $A_n$ , else  $r = 2$ . Then  $2 \leq q \leq 4$ . As such we get  $(p, 2, 2q)$  for type  $D_n$ ,  $(3, 3, 2q)$ ,  $(4, 3, 2q)$ ,  $(5, 3, 2q)$  are  $E_6, E_7, E_8$  respectively.  $\square$

This completes the proof.  $\square$

### 33.1 Serre's Theorem

We have seen that

the set of simple Lie algebras over  $\mathbb{C}$  is not necessarily connected Dynkin diagrams

by

$$\mathfrak{g} \xrightarrow{\text{Choose } C, S, A} \mathfrak{N} \xrightarrow{\text{Choose } \rho \in E_{\text{reg}}} R \xrightarrow{\text{ordering}} \mathfrak{N} \xrightarrow{\text{ordering}} A \xrightarrow{\text{ordering}} D.$$

Serre's Theorem shows that  $D$  is bijective by constructing an explicit inverse  $D \rightarrow \mathfrak{g}$ .

**Theorem 33.1** (Chevalley-Serre). *Let  $\mathfrak{g}$  be as above, and  $D$  its Dynkin Diagram,  $A =$  cartan matrix (with respect to some order on vertices). Then  $\mathfrak{g}$  is generated by a subset  $\{e_i, f_i, h_i\}_{i=1}^r$  satisfying*

Chevalley 1940's	\$	\$\{e_i, f_i\}\$	\$\delta_{ij} h_i\$
	%	\$\{h_i, e_j\}\$	\$a_{ij} e_j\$
	#	\$\{h_i, f_j\}\$	\$a_{ij} f_j\$
Serre 1960's		\$\text{ad}_{e_i}^{-1}\$	\$a_{ij} e_j\$
		\$\text{ad}_{f_i}^{-1}\$	\$a_{ij} f_j\$

**Theorem 33.2** (Serre's Theorem). *Let  $D$  be a Dynkin diagram and  $A$  its cartan matrix. Let  $\mathfrak{g}(D, A)$  be the free Lie algebra on  $r$  symbols  $\{e_i, f_i, h_i\}_{i=1}^r$  module the Chevalley-Serre relations.*

*Then  $\mathfrak{g}(D, A)$  is a finite dimensional semisimple Lie algebra over  $\mathbb{C}$  with Dynkin diagram  $D$ . Moreover,  $\mathfrak{g} \cong \mathfrak{g}(D, A)$  for any  $\mathfrak{g}$  whose Dynkin diagram is  $D$ .*

## 34 Further Reading: Summary of Representation Theory for Lie Groups

### 34.1 Representations of Lie Groups and Lie Algebras

**Definition 34.1.** A representation of a Lie group is a finite dimensional vector space  $V$  together with a morphism  $\rho : G \rightarrow GL(V)$  of Lie groups.

$$\begin{array}{ccc}
 V & \xrightarrow{f} & W \\
 \rho_V \rho_g \downarrow & & \downarrow \rho_W \rho_g \\
 V & \xrightarrow{f} & W
 \end{array}$$

**Definition 34.2.** A *morphism* between two representations  $V$  and  $W$  of  $G$  is a linear map  $f: V \rightarrow W$  which intertwines the action of  $G$ :

$$f \circ \rho_V \rho_g = \rho_W \rho_g \circ f$$

for all  $g \in G$  i.e. commutes.

**Definition 34.3.**  $V$  and  $W$  are *equivalent* (or *isomorphic*) if there exists invertible  $f: V \rightarrow W$ . Denoted  $V \cong W$ .

**Definition 34.4.** A *representation* of a Lie algebra  $\mathfrak{g}$  is a vector space with a morphism  $\rho: \mathfrak{g} \rightarrow \text{gl}(V)$  of Lie algebras.

A morphism of Lie algebra reps  $f: V \rightarrow W$  is defined as for groups. As are  $\text{Hom}_{\mathfrak{g}}(V, W)$  and  $V \cong W$ .

**Note 34.5.** We always assume that  $V$  is a complex vector space. If  $G$  is a real Lie group we regard  $\text{GL}(V, \mathbb{C})$  as a real Lie group, and want  $\rho: G \rightarrow \text{GL}(V, \mathbb{C})$  to be smooth.

**Theorem 34.6.**  $G$  Lie group,  $\mathfrak{g} = \text{Lie}(G)$ .

- (1) Every representation  $\rho: G \rightarrow \text{GL}(V, \mathbb{C})$  gives a representation  $\rho: \mathfrak{g} \rightarrow \text{gl}(V, \mathbb{C})$ . Every morphism of  $G$ -representation is automatically a  $\mathfrak{g}$ -representation. In other words we have a functor

$$\begin{array}{ccc}
 \text{Rep } G & \cong & \text{Rep } \mathfrak{g} \\
 \rho: V, \rho & \mapsto & \rho: V, \rho
 \end{array}$$

- (2) If  $G$  is connected and simply connected, then the above is an equivalence of categories. In particular, any representation of  $\mathfrak{g}$  can be lifted to a representation of  $G$ , and  $\text{Hom}_G(V, W) \cong \text{Hom}_{\mathfrak{g}}(V, W)$ .

**Example 34.7.**

$$\rho: \text{GL}(2, \mathbb{C}) \rightarrow \text{GL}(C^2) \cong \text{GL}(C^2) \quad A \mapsto \rho(A) = \begin{pmatrix} ax & by \\ cx & dy \end{pmatrix}$$

where  $C^2 = \mathbb{C}^2$  and  $\mathbb{A}_{n \times n}^d = \{ \sum_{j=0}^d c_j x^j y^{n-j} \}$ . Also,

$$\rho: \text{gl}(2, \mathbb{C}) \rightarrow \text{gl}(C^2) \quad E_{ij} \mapsto x_i y_j$$

with  $\rho(x_1, x_2) = \rho(x, y)$ .

**Example 34.8.**

$$\text{Ad}: G \rightarrow \text{GL}(\mathfrak{g}) \quad \text{Ad}_g = \text{ad} \circ \rho(g) \quad x \mapsto \rho(g)x = \text{ad}(g)x$$

**Remark 34.9.** If  $\mathfrak{g}$  is a real Lie group then

$$\text{Rep } \mathfrak{g} = \text{Rep } \mathfrak{g}_{\mathbb{C}} \oplus \text{Rep } \mathfrak{g}_{\mathbb{C}}$$

where

$$\rho_{\mathbb{C}}(x + iy) = \rho(x) - i\rho(y)$$

for any representation  $\rho$  of  $\mathfrak{g}$  and

$$\text{Hom}_{\mathfrak{g}_{\mathbb{C}}}(\rho V, W) = \text{Hom}_{\mathfrak{g}}(\rho V, W).$$

**Note 34.10.** The theory of representations in real and/or  $\mathfrak{S}$ -dimensional vector spaces is very different.

### 34.2 Operations on Representations

Let  $G$  be a Lie group,  $\mathfrak{g}$  a Lie algebra.

**Definition 34.11.** Let  $V$  be a representation of  $G$  (respectively  $\mathfrak{g}$ ). A *subrepresentation* of  $V$  is a linear subspace  $W \subseteq V$  such that

$$\rho(x)W \subseteq W$$

for all  $x \in G$  (respectively  $x \in \mathfrak{g}$ ).

**Definition 34.12.** If  $V$  is a representation of  $G$  (or  $\mathfrak{g}$ ), and  $W \subseteq V$  is a subrepresentation then  $V/W$  becomes a representation:

$$\rho_{V/W}(x)v = \rho_V(x)v \pmod{W}$$

for all  $x \in G$  (or  $\mathfrak{g}$ ) and  $v \in V/W$ .

**Definition 34.13.** For representations  $V$  and  $W$  we can define  $V \otimes W$  by

$$\rho_{V \otimes W}(x)(v \otimes w) = \rho_V(x)v \otimes \rho_W(x)w$$

**Definition 34.14.** For a representation  $V$  we can define a representation structure on the dual space  $V^*$ . For Lie group case:

$$\rho_{V^*}(x)(\lambda)(v) = \lambda(\rho_V(x^{-1})v)$$

for all  $v \in V$ ,  $\lambda \in V^*$ ,  $x \in G$ . Lie algebra case:

$$\rho_{V^*}(x)(\lambda)(v) = \lambda(\rho_V(x)v)$$

for  $x \in \mathfrak{g}$ .

**Definition 34.15.** For two representations  $V$  and  $W$  define a representation structure on  $V \otimes W$ . For the Lie group case:

$$\rho_{V \otimes W}(x)(v \otimes w) = \rho_V(x)v \otimes \rho_W(x)w$$

For Lie algebras: We compute  $\rho_{V \otimes W}$  to find the correct definition. Let  $x \in \mathfrak{g}$ . Consider  $\gamma(t) = \exp(tx)$ .

$$\rho_{V \otimes W}(x)(v \otimes w) = \left. \frac{d}{dt} \right|_t \rho_{V \otimes W}(\gamma(t))(v \otimes w)$$

$$= \left. \frac{d}{dt} \right|_t (\rho_V(\gamma(t))v \otimes \rho_W(\gamma(t))w)$$

$$\stackrel{\text{Leibniz}}{=} \rho_V(x)v \otimes \rho_W(x)w + \rho_V(\gamma(0))v \otimes \rho_W(\gamma'(0))w$$

$$= \rho_V(x)v \otimes w + v \otimes \rho_W(x)w.$$

This motivates defining

$$\rho_{V \otimes W}(\rho_V \otimes \rho_W)(x) = \rho_V(x) \otimes \rho_W(x)$$

**Exercise 34.16.** Check that  $\rho_{V \otimes W}$  thus defined is indeed a representation of any Lie algebra  $\mathfrak{g}$ , given representations  $\rho_V, \rho_W$ .

**Corollary 34.17.** If  $V$  is a representation of  $G$  (or  $\mathfrak{g}$ ) then so is  $V^{\otimes k} \otimes \rho_V$ .

**Definition 34.18.** Let  $V$  and  $W$  be representations. Then  $\text{Hom}(V, W) = V \otimes W^*$  by  $\rho_V \otimes \rho_{W^*}$ . This gives  $\text{Hom}(V, W)$  the structure of a representation with

$$(\rho_V \otimes \rho_{W^*})(g)(\varphi)(v) = \rho_V(g)v \otimes \rho_{W^*}(g)\varphi$$

or

$$(\rho_V \otimes \rho_{W^*})(g)(\varphi)(v) = \rho_V(g)v \otimes \rho_{W^*}(g)\varphi$$

### 34.3 Invariants

**Definition 34.19.** A vector  $v$  in a representation  $V$  of  $G$  (or  $\mathfrak{g}$ ) is *invariant* if  $(\rho_V(g)v = v \text{ } \forall g \in G)$ .  $V^G = \{v \in V \mid v \text{ is invariant}\}$ .

**Example 34.20.**  $\rho_{\text{Hom}(V, W)}(g) = \text{Hom}_G(\rho_V(g), \rho_W(g))$  (respectively  $\rho_{\text{Hom}(V, W)}(g) = \text{Hom}_{\mathfrak{g}}(\rho_V(g), \rho_W(g))$ ).

**Example 34.21.**  $B$  be a bilinear form on a representation  $V$ .

$$B: V \otimes V \rightarrow \mathbb{C} \text{ bilinear} \iff B: V \otimes V \rightarrow \mathbb{C} \text{ linear} \\ \iff B \in \rho_V \otimes \rho_V^*$$

So  $G$  (respectively  $\mathfrak{g}$ ) acts on  $\rho_V \otimes \rho_V^*$  via

$$(\rho_V \otimes \rho_V^*)(g)(B)(v, w) = \rho_V(g)v \otimes \rho_V^*(g)B$$

respectively

$$(\rho_V \otimes \rho_V^*)(g)(B)(v, w) = \rho_V(g)v \otimes \rho_V^*(g)B$$

so  $B$  is invariant iff

$$B(\rho_V(g)v, w) = B(v, \rho_V^*(g)w) \text{ } \forall g \in G$$

respectively

$$0 = B(\rho_V(x)v, w) - B(v, \rho_V^*(x)w) \text{ } \forall x \in \mathfrak{g}$$

### 34.4 Irreducible Representations

**Definition 34.22.** A representation  $V \neq 0$  is *irreducible* (or *simple*) if the only subrepresentations of  $V$  are  $0$  and  $V$ . Otherwise  $V$  is *reducible*.

**Example 34.23.** The standard representation  $C^n$  of  $SL(n, C)$  is irreducible. (Exercise) Hint: Use  $E_{ij}, i < j$  to get  $(1, 0, \dots, 0)_q$ .

Suppose  $V \neq 0$  is reducible. Let  $W \in V$  be a proper nonzero subrepresentation. We get a SES

$$0 \rightarrow W \rightarrow V \rightarrow V/W \rightarrow 0$$

**Note 34.24.**  $\dim W$  and  $\dim V \setminus W$  are both strictly less than  $\dim V$ .

(When) does  $\rho|_W$  split? i.e.  $V \setminus W \setminus V \setminus W$ ?

**Definition 34.25.** A representation  $V$  is *completely reducible* (or *semisimple*) if

$$V \cong \bigoplus_{i=1}^N V_i, \quad V_i \text{ irreducible.}$$

Then:

$$V \cong \bigoplus_{i=1}^k n_i V_i \cong \bigoplus_{i=1}^k V_i^{\oplus n_i}$$

where  $V_i \cong V_j$  @  $i = j$ .  $n_i$  is the *multiplicity* of  $V_i$  in  $V$ .

**Example 34.26.**  $G = \mathbb{R}$ ,  $\mathfrak{g} = \text{Lie}(\mathfrak{gl}(n, \mathbb{C})) = \mathbb{R}$ . A representation of  $\mathfrak{g}$  is just a complex finite dimensional vector space  $V$  with an  $\mathbb{R}$ -linear map  $\rho: \mathbb{R} \rightarrow \text{Hom}_{\mathbb{C}}(V, V) = \text{End}_{\mathbb{C}}(V)$ .  $\rho(t) = t\rho(1) = tA$ , where  $A = \rho(1) \in \text{End}_{\mathbb{C}}(V)$ . Conversely any  $A \in \text{End}_{\mathbb{C}}(V)$  gives a representation  $\rho: \mathbb{R} \rightarrow \text{Hom}_{\mathbb{C}}(V, V)$ .  $V \setminus W \iff A|_W = TAT^{-1}$ , some  $T \in \text{GL}(V, \mathbb{C})$ . This implies that Jordan canonical form classifies up to equivalence all representations of the Lie algebra  $\mathbb{R}$ . A representation of  $V$  is completely reducible iff  $A|_W$  is diagonalizable.  $V$  is irreducible iff  $\dim V = 1$ .

Some Goals of Representation Theory

- 1) Given  $G$ , classify irreducible representations of  $G$ .
- 2) Given a reducible representation, how to decompose it into irreducible representations?
- 3) For which  $G$  are all representations completely reducible?

### 34.5 Intertwining Operators (=morphisms of representation)

Suppose  $A: V \rightarrow V$  is a diagonalizable intertwining operator:

$$V \cong \bigoplus_{\lambda \in \mathbb{C}} V_{\lambda}, \quad V_{\lambda} = \{v \in V \mid Av = \lambda v\}$$

Then  $\rho|_G \in \mathfrak{g}$ ,  $\rho|_V \in V$ :

$$A\rho(g)v = \rho(g)Av = \rho(g)\lambda v = \lambda\rho(g)v.$$

This implies  $\rho(g)v \in V_{\lambda}$ , so  $\rho(g)v \in V_{\lambda}$  for all  $g \in G$ . So  $\rho|_V: V_{\lambda} \rightarrow V_{\lambda}$  is a subrepresentation of  $V$ .

**Corollary 34.27.** If  $\lambda \in \mathbb{C}$  such that  $\rho|_{V_{\lambda}}$  is diagonalizable, then  $V \cong \bigoplus_{\lambda \in \mathbb{C}} V_{\lambda}$  where  $V_{\lambda} = \text{eigenspace of } \rho|_{V_{\lambda}}$ .

*Proof.*  $\rho|_{V_{\lambda}} \in \mathfrak{g}$ ,  $\rho|_{V_{\lambda}} \in V_{\lambda}$ ,  $\rho|_{V_{\lambda}} \in \mathfrak{g}$ ,  $\rho|_{V_{\lambda}} \in V_{\lambda}$  @  $g \in G$ . This show that  $\rho|_{V_{\lambda}}$  is an intertwining operator.  $\square$

**Example 34.28.**  $V = \mathbb{C}^n \oplus \mathbb{C}^n$  representations of  $G = \text{GL}(n, \mathbb{C})$ ,  $\rho: v \oplus w \rightarrow w \oplus v$  commutes with  $G$  action implies it is an intertwiner.  $V = V \oplus V$  eigenspace decomposition.

$$V = \text{Span}\{v \oplus w, w \oplus v\}$$

$$V = \text{Span}\{v \oplus w, w \oplus v\}$$

In face,  $V$  are irreducible representations of  $G$ .

### 34.6 Schur's Lemma

Recall all representations are assumed complex and finite dimensional.

**Lemma 34.29** (Schur's Lemma).

1) Let  $V$  be an irreducible representation of  $G$ . Then

$$\text{Hom}_G(\rho V, V) = \mathbb{C} \text{Id}$$

2) If  $V$  and  $W$  are irreducible representations, such that  $V \cong W$ . Then  $\text{Hom}_G(\rho V, W) = \mathbb{C}$ .

*Proof.*  $\Phi: V \rightarrow W$  intertwining operator. Then  $\ker \Phi$  and  $\text{Im} \Phi$  are subrepresentations of  $V$  and  $W$  respectively.  $V$  and  $W$  irreducible implies  $\Phi = 0$  or an isomorphism. Which shows 2). To get 1), pick any eigenvalue  $\lambda$  of  $\Phi$ . Then  $\Phi - \lambda \text{Id}$  is an intertwiner with  $\ker(\Phi - \lambda \text{Id}) = 0$ . By  $V$  irreducible, we have  $V = \ker(\Phi - \lambda \text{Id})$ . Thus  $\Phi = \lambda \text{Id}$ .  $\square$

### 34.7 Unitary representations

Goal: Show that any representation of a compact real Lie group is completely reducible. Steps:

(1) Any unitary representation is completely reducible.

(2) Any representation of a compact real Lie group is unitary.

**Definition 34.30.** A representation  $(\rho, V)$  of  $G$  is *unitary* (or *unitarizable*) if there exists a positive definite Hermitian form on  $V$ ,  $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{C}$  that is  $G$ -invariant, i.e.

$$\langle \rho(g).u, \rho(g).v \rangle = \langle u, v \rangle \quad \forall g \in G, u, v \in V.$$

A representation  $V$  of a Lie algebra  $\mathfrak{g}$  is *unitarizable* if  $\exists \langle \cdot, \cdot \rangle$  positive definite Hermitian form which is  $\mathfrak{g}$ -invariant:

$$\langle \rho(x).v, \rho(x).w \rangle = \langle v, w \rangle \quad \forall x \in \mathfrak{g}, v, w \in V.$$

**Example 34.31.** Let  $G$  be a finite group acting on a set  $X$ .

$$V = \mathbb{C}^X \quad \text{functions } f: X \rightarrow \mathbb{C}.$$

Define  $\rho: G \rightarrow GL(V)$  by  $(\rho(g)f)(x) = f(g^{-1}.x)$ ,  $f \in V$ , and  $g \in G$ . Then  $(\rho, V)$  is a rep of  $G$ . Define  $\langle f, g \rangle = \sum_{x \in X} f(x)\overline{g(x)}$ ,  $f, g \in V$ . Then  $\langle \cdot, \cdot \rangle$  is a  $G$ -invariant positive definite Hermitian form hence  $V$  is unitarizable.

**Theorem 34.32.** Every unitarizable representation is completely reducible.

*Proof.* The proof is by induction on  $\dim V$ .  $W \in V$  nonzero proper subrepresentation. Consider  $W^\perp$  with respect to  $\langle \cdot, \cdot \rangle$  on  $V$ . We have  $W \cap W^\perp = 0$ .

$$\langle \rho(g).v, \rho(g).w \rangle = \langle v, w \rangle = 0.$$

Where  $\langle \cdot, \cdot \rangle$  is by  $G$ -invariance and  $W^\perp$  because it is a subrepresentation. This implies that  $\rho(g).W^\perp \in W^\perp$ , so  $W^\perp$  is also a subrepresentation,  $V = W \oplus W^\perp$ . So  $\dim W, \dim W^\perp < \dim V$ . Proceed by induction. Same idea for Lie algebras.  $\square$

#### 34.7.1 The Haar Measure

Let  $G \in \mathbb{R}^n$  be a real Lie group. Recall  $A \in G$  is *open in  $G$*  (respectively. *closed in  $G$* ) if  $A = B \times G$  for some open (resp. closed) subset  $B \in \mathbb{R}^n$ . Let  $\Sigma \in \mathcal{P}(G)$  be the smallest subset closed under complements and countable  $\cup, \cap$ , containing all open sets in  $G$ .  $\Sigma$

---

$\Sigma$  is called a  $\sigma$ -algebra.



**Definition 34.33.** A *measure* on  $G$  is a map  $\mu: \Sigma \tilde{N} \rightarrow \mathbb{R}_0^+, \mathcal{B}$  such that

- i)  $\mu \left( \bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \mu(A_n)$  ( $\bigcup$  denotes a disjoint union).
- ii)  $\mu(\emptyset) = 0$ .

**Definition 34.34.**

$$\sum_{i=1}^n c_i \chi_{A_i} d\mu \rightarrow \sum_{i=1}^n c_i \mu(A_i).$$

Where  $\chi_{A_i} = \begin{cases} 1 & x \in A_i \\ 0 & \text{otherwise} \end{cases}$ . Through a limit process we can define  $\int_G f d\mu$  *integral of  $f$  over  $G$* .

**Definition 34.35.** A *right Haar measure* on a real Lie group is a Borel measure  $dg$  which is invariant under right action of  $G$  on itself.

Thus if  $dg$  is a Haar measure on  $G$  then for an integrable function  $f: G \rightarrow \mathbb{R}$  (i.e.  $f \in L^1(G, dg)$ )

$$\int_G f(pgh) dq = \int_G f(pg) dq \quad \forall h \in G.$$

**Example 34.36.** Lebesgue measure on  $\mathbb{R}$ .

$$\int_{\mathbb{R}} f(x+y) dx = \int_{\mathbb{R}} f(x) dx \quad \forall y \in \mathbb{R}.$$

**Example 34.37.** The Haar measure on  $U(1)$  is given by  $\frac{dz}{2\pi iz}$ . We have:

$$\int_{U(1)} f(z) \frac{dz}{2\pi iz} = \int_0^{2\pi} f(e^{i\theta}) \frac{e^{i\theta} i d\theta}{2\pi i e^{i\theta}} = \int_0^{2\pi} f(e^{i\theta}) d\theta.$$

**Note 34.38.**  $\forall w = e^{i\theta} \in U(1)$  we have

$$\begin{aligned} \int_{U(1)} f(zw) \frac{dz}{2\pi iz} &= \int_0^{2\pi} f(e^{i(\theta+\phi)}) \frac{e^{i\theta} i d\theta}{2\pi i e^{i\theta}} \\ &= \int_0^{2\pi} f(e^{i\theta}) d\theta \\ &= \int_0^{2\pi} f(e^{i\theta}) d\theta = \int_0^{2\pi} f(e^{i\theta}) d\theta \end{aligned}$$

by periodicity of  $e^{i\theta}$

$$\int_0^{2\pi} f(e^{i\theta}) d\theta = \int_0^{2\pi} f(e^{i\theta}) d\theta.$$

**Theorem 34.39.** Let  $G$  be a compact real Lie group. Then  $G$  has a canonical Borel measure  $dg$  which is invariant under

$$\begin{aligned} g \tilde{N} gh & \quad \forall h \in G \\ g \tilde{N} hg & \quad \forall h \in G \\ g \tilde{N} g^{-1} & \end{aligned}$$

and such that  $\int_G dg = 1$ . This is the Haar measure on  $G$ .

### 34.8 Complete Reducibility

We assume that  $G$  is a compact real Lie group.

**Theorem 34.40.** Any (finite dimensional) representation of  $G$  is unitary, hence completely reducible.

*Proof.* Let  $\rho, \eta$  be any positive definite Hermitian form on a representation  $V$  of  $G$ . Define  $\chi, \psi: V \times V \rightarrow \mathbb{C}$  by

$$\chi(v, w) = \int_G \rho(\rho(g)v, \rho(g)w) dg.$$

Then  $\chi, \psi$  is positive definite and Hermitian. Also  $\chi \in \mathcal{H}(G)$ ,

$$\chi(\rho(h)v, \rho(h)w) = \chi(v, w)$$

by left invariance of Haar measure. □

### 34.9 Characters

**Definition 34.41.** The *character* of a representation  $V$  of  $G$  is  $\chi_V: G \rightarrow \mathbb{C}$ ,  $\chi_V(g) = \text{Tr}(\rho(g))$ .

**Lemma 34.42** (Lemma 4.4 in Kirillov).

- (1)  $\chi_{\text{triv}} = 1$  (Recall:  $\text{triv} = \mathbb{C}$  and  $g \cdot 1 = 1 \cdot g$ ).
- (2)  $\chi_V \cdot \chi_W = \chi_{V \otimes W}$
- (3)  $\chi_{V \oplus W} = \chi_V + \chi_W$
- (4)  $\chi_V(\rho(g)h) = \chi_V(\rho(h)g)$   $\forall g, h \in G$
- (5)  $\chi_V(\rho(g)) = \overline{\chi_V(\rho(g^{-1}))}$   $\forall g \in G$ .

*Proof.* Homework. □

**Theorem 34.43** (Orthonormality of characters).

- (1) If  $V, W$  are non-isomorphic irreducible representations, then

$$\int_G \chi_V(g) \overline{\chi_W(g)} dg = 0.$$

- (2) If  $V$  is any irreducible representation

$$\int_G |\chi_V(g)|^2 dg = 1.$$

**Corollary 34.44.** If  $V = \sum n_i V_i$  where  $V_i$  are nonisomorphic irreducible representations, then  $n_i = \int_G \chi_V(g) \overline{\chi_{V_i}(g)} dg$ .

**Corollary 34.45.** If  $V$  and  $W$  are two representations then  $\int_G \chi_V(g) \overline{\chi_W(g)} dg = \sum n_i \int_G \chi_{V_i}(g) \overline{\chi_{W_i}(g)} dg$

*Proof.* Homework. □

### 34.10 The Hilbert space $L^2(G; dg)$

Let  $G$  be a compact real Lie group. Let

$$L^2(G; dg) = \left\{ f: G \rightarrow \mathbb{C} \mid \int_G |f(g)|^2 dg < \infty \right\}$$

This is a Hilbert space with respect to

$$\langle f_1, f_2 \rangle = \int_G \overline{f_1(g)} f_2(g) dg.$$

Recall: Hilbert Normed metric topological space. Hence we have a notion of closure and denseness.

$G$  acts on the left and right:

$$\begin{aligned} (fg) \cdot f_1 &= f_1 \cdot hg & @g, h \in G \\ (f \cdot g) \cdot f_1 &= f \cdot gh & @f \in L^2(G; dg). \end{aligned}$$

### 34.11 Matrix Coefficients

Let  $V$  be a representation of  $G$ . To  $\rho, \lambda, v \in V \otimes V$  we associate a function  $\rho_V^{\lambda, v}: G \rightarrow \mathbb{C}$  by  $\rho_V^{\lambda, v}(g) = \lambda(\rho_V(g)v)$ .

**Definition 34.46.**  $\rho_V^{\lambda, v}$  is the *matrix coefficient* corresponding to  $V, \lambda, v$ .

**Note 34.47** (Notes).

- (1)  $\rho_V^{\lambda, v} \in L^2(G; dg)$ , because  $\rho_V^{\lambda, v}$  is continuous and  $G$  is compact.
- (2) The right hand side depends bilinearly on  $\rho, \lambda, v$ , therefore we get a map  $V \otimes V \rightarrow L^2(G; dg)$  by  $(\rho, \lambda, v) \mapsto \rho_V^{\lambda, v}$ .
- (3) Under the isomorphism  $V \otimes V \cong \text{End}(V)$  this is equivalent to defining for  $\varphi \in \text{End}(V)$ ,

$$\rho_V^{\lambda, v} = \text{Tr}(\varphi \rho_V)$$

**Example 34.48.** Fix a basis  $\{v_i\}$  for  $V$ , let  $v_i^* \in V^*$  be dual basis:  $v_i^*(v_j) = \delta_{ij}$ . Then  $\rho_V^{v_i^*, v_j}$  is just the  $(i, j)$  entry of the matrix  $\rho_V$  in the basis  $\{v_i\}$ .

$$\rho_V \in \text{Hom}(V, V) \cong \text{GL}(V) \cong \text{GL}(n, \mathbb{C}).$$

Taking  $\mathbb{1} = \sum_i v_i \otimes v_i^*$ , we get

$$\rho_V^{\mathbb{1}} = \text{Tr}(\rho_V) = \chi_V.$$

So matrix coefficients generalize characters.

**Theorem 34.49** (Orthogonality of Matrix Coefficients). *Let  $G$  be a compact real Lie group.*

- 1) Let  $V, W$  be irreducible representations. Then

$$\langle \rho_V, \rho_W \rangle = 0$$

for all  $\varphi \in \text{End}(V), \psi \in \text{End}(W)$ .

2) Let  $V$  be an irreducible representation. Then

$$\rho_{V^1}, \rho_{V^2} \sim \frac{\text{Tr}(\rho_{\varphi_1} - \rho_{\varphi_2})}{\dim V}$$

for all  $\varphi_1, \varphi_2 \in \text{End}(\rho_V)$ .

*Proof.* For all irreducible representations  $V, W$  we have:

$$\begin{aligned} \rho_{V^1}, \rho_{W^1} &\sim \text{Tr}_G \rho_V \rho_W \sim \text{Tr}(\rho_V \rho_W) \\ &\sim \text{Tr}_{V \otimes W} (\rho_V \otimes \rho_W) \\ &\sim \text{Tr}_{V \otimes W} (\rho_V \otimes \rho_W) \\ &\sim \text{Tr}_{V \otimes W} (\rho_V \otimes \rho_W) \\ &\sim \text{Tr}_{V \otimes W} (\rho_V \otimes \rho_W) \\ &\sim \text{Tr}_{V \otimes W} (\rho_V \otimes \rho_W) \end{aligned}$$

where  $\Phi: V \otimes W \rightarrow V \otimes W$  is the value-average of  $\varphi \otimes \psi$ :

$$\Phi = \frac{1}{\dim V \dim W} \sum_{g \in G} \rho_V(g) \otimes \rho_W(g)$$

The image of  $\Phi$  is thus contained in  $\rho_V \otimes \rho_W$ .

Now if  $V = W$  then

$$\rho_V \otimes \rho_W \cong \text{Hom}_G(\rho_W, \rho_V) \oplus 0$$

in which case we get  $\Phi = 0$ . Hence,

$$\rho_{V^1}, \rho_{W^1} \sim \text{Tr} \Phi = 0.$$

On the other hand if  $W \neq V$  then

$$\rho_V \otimes \rho_W \cong \text{Hom}_G(\rho_V, \rho_W) \oplus \mathbb{C} \text{Id}_V.$$

Which means  $\dim \text{range } \Phi = 1$ . So

$$\Phi \text{Id}_V \sim \text{Tr}(\Phi \text{Id}_V) \text{Id}_V \sim \begin{pmatrix} \text{Tr} \Phi \\ 0 \\ \vdots \\ 0 \end{pmatrix} \sim \text{Tr}_V(\Phi \text{Id}_V) \sim \text{Tr}(\Phi) \text{Id}_V.$$

On the other hand,

$$\begin{aligned} \text{Tr}_V(\Phi \text{Id}_V) &\sim \text{Tr}_V \left( \frac{1}{\dim V \dim W} \sum_{g \in G} \rho_V(g) \otimes \rho_W(g) \right) \\ &\sim \text{Tr}_V \left( \frac{1}{\dim V \dim W} \sum_{g \in G} \rho_V(g) \otimes \rho_W(g) \right) \\ &\sim \text{Tr}_V \left( \frac{1}{\dim V \dim W} \sum_{g \in G} \rho_V(g) \otimes \rho_W(g) \right) \\ &\sim \text{Tr}_V \left( \frac{1}{\dim V \dim W} \sum_{g \in G} \rho_V(g) \otimes \rho_W(g) \right) \\ &\sim \text{Tr}_V \left( \frac{1}{\dim V \dim W} \sum_{g \in G} \rho_V(g) \otimes \rho_W(g) \right) \\ &\sim \text{Tr}_V \left( \frac{1}{\dim V \dim W} \sum_{g \in G} \rho_V(g) \otimes \rho_W(g) \right) \end{aligned}$$

This implies that

$$\langle \rho_V, \rho_V \rangle = \text{Tr}_{V \otimes V} \langle \rho, \rho \rangle = \frac{\text{Tr} \langle \rho, \rho \rangle}{\dim V}$$

□

### 34.12 The Peter-Weyl Theorem

Let  $G$  be a compact real Lie group. The Peter-Weyl Theorem

- 1) describes  $L^2(\mathfrak{g}, dg)$  as a  $G$ -bimodule
- 2) says that any  $f \in L^2(\mathfrak{g}, dg)$  can be approximated by a linear combination of matrix coefficients coming from irreducible representations.

Let  $\hat{G}$  be the set of equivalence classes of irreducible representations of  $G$ .

For  $\rho \in \hat{G}$ , define a  $G$ -invariant inner product on  $V \otimes V = \text{End} V$

$$\langle \rho, \psi \rangle = \frac{\text{Tr} \langle \rho, \psi \rangle}{\dim V}$$

Let  $\hat{\rho} \in \hat{G}$  denote the Hilbert space completion with respect to this form.

**Theorem 34.50** (Peter-Weyl). *The map*

$$m: \hat{\rho} \in \hat{G} \rightarrow L^2(\mathfrak{g}, dg) \quad \text{by} \quad \rho \mapsto \rho_X \in \hat{\rho}$$

is an isometric isomorphism of  $G$ -bimodules.

*Proof.* The onto part requires analysis, we skip the proof. □

**Corollary 34.51.** *The set of characters  $\chi_V \mid \rho \in \hat{G}$  is an orthonormal Hilbert space basis for  $L^2(\mathfrak{g}, dg)^G$ , the space of conjugate-invariant functions on  $G$ .*

## 35 Lecture 31: Universal enveloping algebra

Important tool in representation theory.

Recall If  $A$  is an associative  $\mathbb{C}$ -algebra can turn  $A$  into a Lie algebra,  $LA$ , by defining  $[x, y] = xy - yx$ .

Consider a representation  $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ .

**Note 35.1.**  $\mathfrak{gl}(V) \cong \text{End} V$ .

Replacing  $\text{End} V$  by an arbitrary associative, we get the notion of an enveloping algebra.

**Definition 35.2.** An *enveloping algebra* of a Lie algebra  $\mathfrak{g}$  is a pair  $(A, j)$  where  $A$  is an associative algebra and  $j: \mathfrak{g} \rightarrow LA$  is a Lie algebra morphism, i.e.  $j([x, y]) = j(x)j(y) - j(y)j(x)$  and  $j$  is linear.

Any representation  $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  gives an enveloping algebra  $(\text{End} V, \rho)$ .

**Question 35.3.** What is the "most general" enveloping algebra?

**Definition 35.4.** Let  $\mathfrak{g}$  be a Lie algebra over a field  $\mathbb{F}$ . The *universal enveloping algebra* of  $\mathfrak{g}$ , denoted  $U\mathfrak{g}$  (or  $U\mathfrak{pg}$ ), is the associative  $\mathbb{F}$ -algebra with 1 generated by symbols  $ipxq$  for  $x \in \mathfrak{g}$  subject to relations

$$\begin{aligned} ipx - yq &= ipxq - ipyq \quad @x, y \in \mathfrak{g} \\ ipcxq &= cipxq \quad @x \in \mathfrak{g}, c \in \mathbb{F} \end{aligned}$$

and

$$iprx, ysq = ipxqipyq - ipyqipxq \quad @x, y \in \mathfrak{g}.$$

**Remark 35.5.** The map

$$\begin{aligned} \mathfrak{g} &\xrightarrow{\tilde{N}} U\mathfrak{g} \\ x &\xrightarrow{\tilde{N}} ipxq \end{aligned}$$

which we might as well call  $i$ , is a Lie algebra morphism

$$i: \mathfrak{g} \xrightarrow{\tilde{N}} U\mathfrak{g}$$

This is immediate by construction of  $U\mathfrak{g}$ :  $i$  is linear and  $iprx, ysq = ipxqipyq - ipyqipxq$ . Abusing notation, we write  $x$  instead of  $ipxq$ . This is ok since we will show  $\mathfrak{g} \xrightarrow{\tilde{N}} U\mathfrak{g}, x \xrightarrow{\tilde{N}} ipxq$  is injective.

**Remark 35.6.**  $U\mathfrak{g} \cong T\mathfrak{g}/I$  where  $I = \langle px \circ y - y \circ x \mid x, y \in \mathfrak{g} \rangle$  and  $T\mathfrak{g} = \mathbb{F}\langle \mathfrak{g} \rangle$ . The isomorphism sends  $ipx_1q \dots ipx_nq$  to  $x_1 \circ \dots \circ x_n \in I$ .

**Example 35.7.**  $\mathfrak{g}$  abelian Lie algebra with basis  $\{x_i\}_{i=1}^n$ . This implies  $U\mathfrak{g} \cong S\mathfrak{g} = \mathbb{F}\langle x_1, \dots, x_n \rangle$  a polynomial algebra. (Recall:  $S\mathfrak{g} = T\mathfrak{g}/\langle px \circ y - y \circ x \rangle$ )

**Example 35.8.**  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ . Then  $U\mathfrak{g}$  is the associative  $\mathbb{C}$ -algebra generated by  $e, f, h$  subject to the relations  $ef - fe = h, he = eh = 2e, hf = fh = -2f$ .

**Note 35.9.**

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{C})$$

In  $M_2(\mathbb{C})$ ,  $e^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$  but in  $U\mathfrak{g}$ ,  $e^2 \neq 0$  (as we will see).

**Example 35.10.** The quadratic casimir element of  $U\mathfrak{psl}(2, \mathbb{C})$  is  $c = ef - fe - \frac{1}{2}h^2$ . Then  $c \in Z(U\mathfrak{psl}(2, \mathbb{C}))$ . Consider the following calculation

$$\begin{aligned} cf &= ef^2 - fef - \frac{1}{2}h^2f \\ pfe - hqf &= fpfe - hq - \frac{1}{2}hpfh - 2fq \\ fef - fh - 2f &= ffe - fh - \frac{1}{2}pfh - 2fqph - 2q \\ fpef - feq &= fph - 2h - \frac{1}{2}ph - 2q^2q \\ fpef - fe &= \frac{1}{2}h^2q - fc. \end{aligned}$$

$$\begin{array}{ccc}
 \mathfrak{g} & \xrightarrow{i} & U\mathfrak{g} \\
 & \searrow j & \downarrow D!j \\
 & & A
 \end{array}$$

**Theorem 35.11** (Universal property of  $U\mathfrak{g}$ ). *Let  $\rho: \mathfrak{g} \rightarrow A$  be any enveloping algebra of  $\mathfrak{g}$ . Then there exists a unique morphism  $j: U\mathfrak{g} \rightarrow A$  of associative algebras such that  $j \circ i = \rho$ , i.e. the following diagram commutes*

**Corollary 35.12.** *Given any representation*

$$\rho: \mathfrak{g} \rightarrow \text{gl}(V)$$

*there is a unique algebra morphism*

$$j: U\mathfrak{g} \rightarrow \text{End}(V)$$

*such that  $j(\rho(x)) = j(x)$ .*

This makes  $V$  into a  $U\mathfrak{g}$ -module:  $X.v = j(X)v$ . Conversely, given any  $U\mathfrak{g}$ -module  $V$ , we get a representation of  $\mathfrak{g}$  by

$$\begin{aligned}
 \rho: \mathfrak{g} &\rightarrow \text{gl}(V) \\
 \rho(x)v &= j(x)v
 \end{aligned}$$

So  $\text{Rep}(U\mathfrak{g}) \cong \text{Mod } U\mathfrak{g}$  is an equivalence of categories.

## 36 Lecture 32: The Poincaré-Birkhoff-Witt Theorem

**Theorem 36.1** (PBW Theorem).  *$U\mathfrak{g}$  is a filtered algebra and its associated graded algebra is congruent to  $S\mathfrak{g}$ .*

### 36.1 Graded Algebras

**Definition 36.2.** A *gradation*  $G$  on an algebra  $A$  is a collection of subspaces  $G_n A$  (or just  $A_n$  if the gradation is obvious) such that

- i)  $A = \sum_{n \geq 0} A_n$ ;
- ii)  $A_n A_m \subseteq A_{n+m}$ .

**Example 36.3.** The tensor algebra on a vector space is naturally graded

$$\begin{aligned}
 T(V) &= \bigoplus_{n \geq 0} V^{\otimes n} \\
 \text{namely } T(V)_n &= \bigoplus_{i \geq 0} V^{\otimes i} \otimes V^{\otimes n-i}
 \end{aligned}$$

## 36.2 Filtered Algebras

**Definition 36.4.** A filtration  $F$  on an algebra  $A$  is a collection of subspaces  $F_n A$  (or  $A_{p_n q}$  if the filtration is obvious) such that

- i)  $A_{p_0 q} \subseteq A_{p_1 q} \subseteq A_{p_2 q} \subseteq \dots$ ;
- ii)  $\bigcup_{n=0}^{\infty} A_{p_n q} = A$ ;
- iii)  $A_{p_m q} A_{p_n q} \subseteq A_{p_{m+n} q}$ .

Any graded algebra  $A$  is naturally filtered by  $A_{p_n q} = A_0 \oplus \dots \oplus A_n$ , but not conversely.

**Example 36.5.**  $ef \in U_{pslp2, Cqq_{p2q}}$  Note:  $ef = fe + h$ , so it is not graded. This filtration is uniquely determined by requiring  $x \in U_{pgq_{p1q}} @ x \in \mathfrak{g}$ .

## 36.3 Associated Graded Algebra

Think of this as formalizing the idea of a leading term.

**Definition 36.6.** Given a filtered algebra  $A$  the associated graded algebra is as follows

$$\text{gr } A = \bigoplus_{n=0}^{\infty} A_{p_n q} / A_{p_{n-1} q} \quad (A_{p_{-1} q} = 0 \text{ by convention})$$

**Example 36.7.**  $ef \in U_{pslp2, Cqq_{p1q}} = fe \in U_{pslp2, Cqq_{p1q}}$  in  $\text{pgr } U_{pslp2, Cqq_{p1q}}$ .

So the PBW theorem says that  $\text{gr } U_{pgq} = Sp_{pgq}$  as graded algebras. As a word of warning, there is no algebra homomorphism from  $A \rightarrow \text{gr } A$  when  $A$  is filtered.

However there is a function  $f: A \rightarrow \text{gr } A$  defined as follows. Let  $a \in A$  and  $n \in \mathbb{N}$  be the smallest such that  $a \in A_{p_n q}$ . Then  $f(a) = a + A_{p_{n-1} q} \in \text{pgr } A_{p_n q}$ . We do have that  $f(ab) = f(a)f(b)$ , but  $f$  is not linear.

*Proof of PBW thm (sketch).*

Step 1: Define  $\varphi: Sp_{pgq} \rightarrow \text{gr } U_{pgq}$  by

$$\varphi(x_1 \dots x_n) = x_1 \dots x_n + U_{pgq_{p_{n-1} q}}$$

Is  $\varphi$  well-defined?

$$\varphi(x_1 \dots x_i x_{i+1} \dots x_n) - \varphi(x_1 \dots x_i x_{i+1} x_n) - \varphi(x_1 \dots x_i x_{i+1} x_{i+2} \dots x_n) \in U_{pgq_{p_{n-1} q}} = 0$$

So we have that  $\varphi$  is well-defined. To show  $\varphi$  is onto, let  $\{x_i\}_1^n$  be a basis.

$$\varphi \left( \sum_{k \in \mathbb{N}^n} c_k x_1^{k_1} \dots x_n^{k_n} \right) = \sum_{k \in \mathbb{N}^n} c_k x_1^{k_1} \dots x_n^{k_n} + U_{pgq_{p_{n-1} q}}$$

Using commutators we can reorder the terms, where  $\varphi(x_1^{k_1} \dots x_n^{k_n}) = x_1^{k_1} \dots x_n^{k_n} + U_{pgq_{p_{n-1} q}}$  are called *ordered monomials*. We want to show that any element  $U_{pgq}$  can be written as a linear combination of ordered monomials.



It follows by induction in the following way,

$$xy - yx = [x, y]$$

were  $x \leq y$ . As an example

$$h^2e = hpeh - 2eq + heh - 2he.$$

Finally to show that  $\varphi$  is one-to-one, we show that there exists a unique way to reduce the following: for  $x \leq y \leq z$  we want to reorder  $zyx$ . We can either switch  $z \leq y$  or  $y \leq x$ . For the first choice we have

$$zyx - yzx = [z, y]x - yxz = y[z, x] - [z, y]x - xyz = [y, x]z - y[z, x] - [z, y]x, \quad p < q$$

and for the later we have,

$$zyx - zxy = z[y, x] - xzy = [z, x]y - z[y, x] - xyz = x[z, y] - [z, x]y - z[y, x]. \quad p < q$$

We need to show that the  $p < q - p < q = 0$ . We notice that the LHS is zero due to the Jacobi identity.  $\square$

The following corollary is also known as the PBW theorem:

**Corollary 36.8.** *Let  $\mathfrak{g}$  be a Lie algebra and let  $\{x_i\}_{i \in I}$  be an ordered basis for  $\mathfrak{g}$  ( $I$  is some index set). Then the set of ordered monomials*

$$\{x_{i_1} \cdots x_{i_k} \mid i_j \in I; i_1 \leq \cdots \leq i_k\}$$

*is a basis for  $U(\mathfrak{g})$ . When  $\mathfrak{g}$  is finite-dimensional, say  $I = \{1, 2, \dots, n\}$ , then the basis may be written*

$$\{x_1^{a_1} \cdots x_n^{a_n} \mid a_i \in \mathbb{Z}_{\geq 0}\}.$$

### 37 Lecture 33: Highest Weight Theory

Motivation: Recall the representation  $V_n$  of  $\mathfrak{sl}_2, \mathbb{C}$ :

$$\begin{aligned} V_n &= \text{span}\{x^n, x^{n-1}y, \dots, y^n\} \\ \rho(e)y &= xB_y \\ \rho(f)x &= yB_x \\ \rho(h)x &= xB_x - yB_y \end{aligned}$$

$V_n$  contains a special vector  $v_0 = x^n$  with three properties:

1)  $v_0$  is a *weight vector*:

$$h.v_0 = \rho(B_x - yB_y)x^n = nx^n = nv_0.$$

2)  $v_0$  is a *highest weight vector*:

$$e.v_0 = \rho(B_y)x^n = 0.$$

3)  $V_n$  is generated by  $v_0$ :

$$V_n = \text{span}_{\mathbb{C}} \{ a_1 \rho a_2 \rho \dots \rho a_k v_0 \mid k \in \mathbb{N}, a_i \in \mathfrak{sl}(2, \mathbb{C}) \}$$

equivalently, using that

$$a_1 \rho a_2 \rho \dots \rho a_k v_0 = \rho a_1 a_2 \dots a_k v_0$$

where  $\rho a_1 a_2 \dots a_k v_0 \in U(\mathfrak{sl}(2, \mathbb{C})) v_0$ .

**Definition 37.1.** Let  $\mathfrak{g}$  be a finite dimensional semisimple Lie algebra over  $\mathbb{C}$ , choose a Cartan subalgebra  $\mathfrak{h} \in \mathfrak{g}$ , and  $R \subseteq R^+$  a choice of positive roots. A representation of  $\mathfrak{g}$   $(V, \rho)$  is a *highest weight representation* if  $\exists v_0 \in V$  and  $\lambda \in \mathfrak{h}^*$  such that

1.  $v_0$  is a *weight vector* of weight  $\lambda$ :

$$h.v_0 = \lambda(h)v_0 \quad \forall h \in \mathfrak{h}.$$

2.  $v_0$  is a *highest weight vector*:

$$e_{\alpha}.v_0 = 0$$

where  $\alpha \in R^+$  i.e.  $e_{\alpha}.v_0 = 0 \quad \forall \alpha \in R^+$ .

3.  $V$  is generated by  $v_0$

$$V = U(\mathfrak{g}).v_0$$

**Theorem 37.2.** Any finite dimensional irreducible representation of  $\mathfrak{g}$  is a highest weight representation.

*Proof.*  $\{ \rho(h) \mid h \in \mathfrak{h} \}$  is a family of commuting linear operators on  $V$ , hence there is at least one common eigenvector,  $w$ , say. Let  $\mu \in \mathfrak{h}^*$  be defined by  $\rho(h)w = \mu(h)w$ . So  $V^1 = \text{span}_{\mathbb{C}} \{ w \}$  is a non-zero subspace of  $V$ , where  $V^1 = \{ v \in V \mid \rho(h)v = \mu(h)v \}$ . So  $w \in V^1$ . But  $V^1$  is actually a subrepresentation:

$$\mathfrak{g}.V^1 \subseteq V^1.$$

$V$  irreducible implies that  $V = V^1$ .

Let  $\text{Supp} V = \{ \alpha \in R^+ \mid V_{\alpha} \neq 0 \}$  be the support of  $V$ . Choose  $h \in \mathfrak{h}$  such that  $\alpha(h), \beta(h) > 0 \quad \forall \alpha \in R^+$  (e.g.  $h = \sum_{\alpha \in R^+} \alpha$ , where  $\tau$  defines  $R^+$ ). Then let  $\lambda \in \text{Supp} V$  be such that  $\alpha(h), \beta(h)$  is maximal. Then  $\alpha(h) > \beta(h) \quad \forall \alpha \in R^+$ . This implies that  $\alpha \in R^+ \implies \alpha \in \text{Supp} V$ . Hence,  $\mathfrak{g}.V \subseteq V$  for all  $\alpha \in R^+$ .

Let  $v_0$  be any nonzero vector in  $V$ . Then 1) and 2) hold. that  $v_0$  generates  $V$  is obvious since  $V$  is irreducible.  $\square$

**Proposition 37.3.** If  $V$  is a finite dimensional irreducible representation of  $\mathfrak{g}$  of highest weight  $\lambda$ , then  $\lambda(\alpha_{\tau}) \neq 0$  for every simple coroot  $\alpha_{\tau} = -h_{\tau}$ .

*Proof.* Consider the action of  $\mathfrak{sl}(2, \mathbb{C})_{\tau}$  on  $V$ . Let  $v_0 \in V$  be a highest weight vector. Consider

$$V_{\tau} = \text{span}_{\mathbb{C}} \{ f_{\tau}^k v_0 \mid k \in \mathbb{N} \}.$$

This is an  $\mathfrak{sl}(2, \mathbb{C})_{\tau}$ -subrepresentation of  $V$ .  $v_0$  satisfies  $h_{\tau}v_0 = \lambda(h_{\tau})v_0$ ,  $e_{\tau}v_0 = 0$  and  $v_0$  generates a finite dimensional  $\mathfrak{sl}(2, \mathbb{C})_{\tau}$ -representation.

Let  $N$  be minimal such that  $f_{\tau}^N.v_0 = 0$ . Then

$$0 = e_{\tau}.f_{\tau}^N.v_0 = \rho(f_{\tau}^N e_{\tau}).v_0 = \rho(e_{\tau}, f_{\tau}^N).v_0 = \rho(\lambda(h_{\tau}) - N) f_{\tau}^{N-1}.v_0$$

Thus  $\lambda(h_{\tau}) = N - 1$ , so we are done.  $\square$

So we have a map

$$\text{tf.d. irreps of } \mathfrak{gu}_{(iso)} \cong \tilde{N} P$$

where  $P = \{ \lambda \in \mathfrak{h}^* \mid \lambda \rho_{\alpha_i} \neq 0, i = 1, 2, \dots, r \}$  are the *dominant integral weights*.

Goal: Show that we can go back (i.e. there exists an inverse of  $\Phi$ ).

Plan:

- 1) To any  $\lambda \in P$  construct a universal highest weight representation  $M(\lambda)$  of highest weight  $\lambda$ . (Verma Module)
- 2) Each  $M(\lambda)$  has a unique irreducible quotient  $L(\lambda)$ .
- 3) Show that  $L(\lambda)$  is finite dimensional iff  $\lambda \in P$ .
- 4)  $L(\lambda) \otimes L(\mu) \cong L(\lambda + \mu)$ .

### 38 Lecture 34: Verma Modules

We will need the definition of tensor product of modules over a noncommutative ring.

**Definition 38.1.** Let  $R$  be a ring,  $M$  a right  $R$ -module, and  $N$  a left  $R$ -module. Then  $M \otimes_R N$  is the abelian group generated by the symbols  $m \otimes n$  subject to the relations:

- 1)  $\mathbb{Z}$  bilinearity

$$\begin{aligned} (m_1 + m_2) \otimes n &= m_1 \otimes n + m_2 \otimes n \\ m \otimes (n_1 + n_2) &= m \otimes n_1 + m \otimes n_2 \end{aligned}$$

- 2)  $R$ -balanced

$$(m \cdot r) \otimes n = m \otimes (r \cdot n)$$

Moreover, if  $M$  is a  $(S, R)$ -bimodule, then  $M \otimes_R N$  is a left  $S$ -module via  $s \cdot (m \otimes n) = (s \cdot m) \otimes n$ .

Recall that a *highest weight representation* of  $\mathfrak{g}$  is a representation generated by a *highest weight vector*:

- 1)  $h v_0 = \lambda(h) v_0$  for all  $h \in \mathfrak{h}$
- 2)  $e v_0 = 0$  for all  $e \in \mathfrak{p}$  and  $\alpha \in R$
- 3)  $V = U(\mathfrak{p}) v_0$

The following defines a universal highest weight representation associated to any  $\lambda \in \mathfrak{h}^*$ .

**Definition 38.2.** Let  $\lambda \in \mathfrak{h}^*$ . The corresponding *Verma module*  $M(\lambda)$  is defined by

$$M(\lambda) = U(\mathfrak{p}) \otimes_{U(\mathfrak{b})} \mathbb{C} 1$$

where  $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ ,  $\mathfrak{h} \cong \mathfrak{p} \oplus \mathfrak{q}$ , and  $\mathbb{C} 1$  is the 1-dimensional representation of  $\mathfrak{b}$  given by:

$$\begin{aligned} h \cdot 1 &= \lambda(h) 1 & @h \in \mathfrak{h} \\ x \cdot 1 &= 0 & @x \in \mathfrak{n} \end{aligned}$$

and  $U(\mathfrak{p})$  is regarded as a right  $U(\mathfrak{b})$  module.

Note that, since  $M(\rho, \lambda) = U(\rho, \mathfrak{g}) \otimes_{U(\mathfrak{b}, \mathfrak{q})} C_1$  with  $C_1$  is a left  $U(\mathfrak{b}, \mathfrak{q})$ -module and  $U(\rho, \mathfrak{g})$  is a  $\rho U(\rho, \mathfrak{g}), U(\mathfrak{b}, \mathfrak{q})$ -bimodule, the Verma module  $M(\rho, \lambda)$  is a left  $U(\rho, \mathfrak{g})$ -module.

**Example 38.3.**  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}), \mathfrak{b} = \mathbb{C}h \oplus \mathbb{C}e, C_1$  for  $\lambda \in \mathfrak{h}^*$  such that  $h \cdot 1 = \lambda(h)1 = \lambda^2 1$  and  $e \cdot 1 = 0$ . Below we see how we can simplify elements in  $M(\rho, \lambda)$ .

$$\begin{aligned} M(\rho, \lambda) \otimes_{U(\mathfrak{b}, \mathfrak{q})} \mathbb{C}1 &= \mathbb{C}\langle f^k e^{-\ell} \rangle \otimes_{U(\mathfrak{b}, \mathfrak{q})} \mathbb{C}1 \\ &= \mathbb{C}\langle f^k e^{-\ell} \rangle \otimes_{U(\mathfrak{b}, \mathfrak{q})} \mathbb{C}1 \\ &= \mathbb{C}\langle f^k e^{-\ell} \rangle \otimes_{U(\mathfrak{b}, \mathfrak{q})} \mathbb{C}1 \\ &= \mathbb{C}\langle f^k e^{-\ell} \rangle \otimes_{U(\mathfrak{b}, \mathfrak{q})} \mathbb{C}1 \\ &= \mathbb{C}\langle f^k e^{-\ell} \rangle \otimes_{U(\mathfrak{b}, \mathfrak{q})} \mathbb{C}1 \\ &= \mathbb{C}\langle f^k e^{-\ell} \rangle \otimes_{U(\mathfrak{b}, \mathfrak{q})} \mathbb{C}1 \end{aligned}$$

Recall: The PBW theorem (Thm 36.1) says that  $U(\mathfrak{sl}(2, \mathbb{C}))$  has a basis

$$\{f^k h^\ell e^m \mid k, \ell, m \in \mathbb{Z}\}$$

So

$$M(\rho, \lambda) \cong \bigoplus_{k, \ell, m \in \mathbb{Z}} \mathbb{C} f^k h^\ell e^m \otimes_{U(\mathfrak{b}, \mathfrak{q})} \mathbb{C}1 \cong \bigoplus_{k \in \mathbb{Z}} \mathbb{C} f^k \otimes_{U(\mathfrak{b}, \mathfrak{q})} \mathbb{C}1$$

This property holds in general:

**Theorem 38.4.**  $M(\rho, \lambda)$  is a left  $U(\mathfrak{n}, \mathfrak{q})$ -module

*Proof.*  $\varphi: U(\mathfrak{n}, \mathfrak{q}) \otimes_{U(\mathfrak{b}, \mathfrak{q})} M(\rho, \lambda) \rightarrow M(\rho, \lambda)$  by  $x \otimes y \mapsto xy$ .  $\varphi$  is a surjective using the PBW theorem (thm 36.1):

$$\begin{aligned} M(\rho, \lambda) &= U(\rho, \mathfrak{g}) \otimes_{U(\mathfrak{b}, \mathfrak{q})} \mathbb{C}1 \\ &= U(\mathfrak{n}, \mathfrak{q}) \otimes_{U(\mathfrak{b}, \mathfrak{q})} U(\rho, \mathfrak{h}) \otimes_{U(\mathfrak{b}, \mathfrak{q})} \mathbb{C}1 \\ &= U(\mathfrak{n}, \mathfrak{q}) \otimes_{U(\mathfrak{b}, \mathfrak{q})} \mathbb{C}1 \\ &\in \text{im } \varphi. \end{aligned}$$

$\varphi$  is injective: By PBW theorem  $U(\rho, \mathfrak{g})$  is free as a right  $U(\mathfrak{b}, \mathfrak{q})$ -module on a basis for  $U(\mathfrak{n}, \mathfrak{q})$ :  $U(\rho, \mathfrak{g}) \otimes_{U(\mathfrak{b}, \mathfrak{q})} U(\mathfrak{n}, \mathfrak{q}) \otimes_{U(\mathfrak{b}, \mathfrak{q})} \mathbb{C}1$  by properties of tensors one can show that

$$\begin{aligned} U(\rho, \mathfrak{g}) \otimes_{U(\mathfrak{b}, \mathfrak{q})} \mathbb{C}1 &= U(\mathfrak{n}, \mathfrak{q}) \otimes_{U(\mathfrak{b}, \mathfrak{q})} U(\rho, \mathfrak{h}) \otimes_{U(\mathfrak{b}, \mathfrak{q})} \mathbb{C}1 \\ &= U(\mathfrak{n}, \mathfrak{q}) \otimes_{U(\mathfrak{b}, \mathfrak{q})} U(\rho, \mathfrak{h}) \otimes_{U(\mathfrak{b}, \mathfrak{q})} \mathbb{C}1 \\ &= U(\mathfrak{n}, \mathfrak{q}) \otimes_{U(\mathfrak{b}, \mathfrak{q})} \mathbb{C}1 \\ &= U(\mathfrak{n}, \mathfrak{q}). \end{aligned}$$

□

**Corollary 38.5.** The support of  $M(\rho, \lambda)$  is

$$\text{Supp } M(\rho, \lambda) = \{ \lambda - \sum_{i=1}^r k_i \alpha_i \mid k_i \in \mathbb{Z}_{\geq 0} \}$$

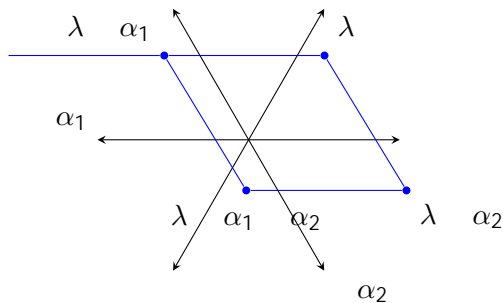
where  $\alpha_1, \dots, \alpha_r \in \Pi$  the set of simple roots.  $Q = \mathbb{Z}R = \sum_{i=1}^r \mathbb{Z}\alpha_i$  and  $Q^+ = \sum_{i=1}^r \mathbb{Z}_{\geq 0}\alpha_i$

*Proof.*  $f_1^{k_1} \dots f_n^{k_n} \mathfrak{b} \neq 1$  where  $R = \{ \beta_1, \dots, \beta_n \}$ . Then

$$\begin{aligned} h \cdot \rho f_1^{k_1} \dots f_n^{k_n} \mathfrak{b} &= \rho f_1^{k_1} \dots f_n^{k_n} h \mathfrak{b} \\ &= \rho f_1^{k_1} \dots f_n^{k_n} (h + \sum_{i=1}^n \beta_i) \mathfrak{b} \\ &= \rho f_1^{k_1} \dots f_n^{k_n} \mathfrak{b} + \sum_{i=1}^n \beta_i \rho f_1^{k_1} \dots f_n^{k_n} \mathfrak{b} \end{aligned}$$

Where  $\rho \mathfrak{b}$  is by  $\beta_i \rho f_i \mathfrak{b} = \beta_i \rho h f_i \mathfrak{b}$  for every  $i$ . □

**Example 38.6.**  $\mathfrak{sl}(3, \mathbb{C})$   $\lambda = k_1 \alpha_1 + k_2 \alpha_2$  where  $w_i \in \mathfrak{h}$  and  $w_i \rho h_j \mathfrak{b} = \delta_{ij}$  which are the *fundamental weights*. Where the blue lattice is  $\text{Supp } M_{\rho, \lambda}$ .



### 39 Lecture 35: Classification of Finite-Dimensional Irreducible Representations

Goal: Classify all finite dimensional irreducible represents of a finite dimensional semisimple Lie algebra over  $\mathbb{C}$   $\mathfrak{g}$ ,  $\rho V$ ,  $\rho \mathfrak{q}$ .

$$M_{\rho, \lambda} = U(\mathfrak{g}) \mathfrak{b}_{\rho, \lambda} \mathbb{C} 1$$

**Proposition 39.1.**

- i)  $M_{\rho, \lambda}$  is a highest weight representation of  $\mathfrak{g}$  of highest weight  $\lambda$ .
- ii) Every highest weight representation of  $\mathfrak{g}$  of highest weight  $\lambda$  is a quotient of  $M_{\rho, \lambda}$ .
- iii)  $M_{\rho, \lambda} \cong \mathbb{C} \oplus \mathfrak{b} \oplus \mathfrak{q}$
- iv)  $\text{Supp } M_{\rho, \lambda} = \lambda + Q = \{ \lambda + \sum_{i=1}^n k_i \alpha_i \mid k_i \in \mathbb{Z}_{\geq 0} \}$
- v)  $M_{\rho, \lambda}$  has a unique maximal (proper) submodule  $N_{\rho, \lambda}$ . Hence  $M_{\rho, \lambda}$  has a unique irreducible quotient  $L_{\rho, \lambda} = M_{\rho, \lambda} / N_{\rho, \lambda}$ .

*Proof.* i) Put  $v = 1 \otimes 1 \in M_{\rho, \lambda}$ ,

$$\begin{aligned} h.v &= h \cdot 1 \otimes 1 = h \cdot 1 \otimes 1 \\ &= h \cdot 1 \otimes 1 \\ &= 1 \otimes h \cdot 1 \\ &= 1 \otimes \lambda \rho(h) \cdot 1 \\ &= \lambda \rho(h) \cdot 1 \otimes 1. \end{aligned}$$

This implies  $v$  is a weight vector of weight  $\lambda$ . Also  $e \in \mathfrak{p} \cap \mathfrak{n}$ ,

$$e.v = e \cdot 1 \otimes 1 = 0 \otimes 1 = 0$$

because  $\mathfrak{n} \subseteq \mathfrak{b}$ . So  $v$  is a highest weight vector.

$$U_{\rho, \lambda} v = U_{\rho, \lambda} \cdot 1 \otimes 1 = U_{\rho, \lambda} \cdot 1 \otimes 1 = M_{\rho, \lambda}.$$

ii) Let  $W$  be any highest weight representation of  $\mathfrak{g}$  of highest weight  $\lambda$ .  $w \in W$  be a (non-zero) highest weight vector of weight  $\lambda$ .

Consider the map

$$\begin{aligned} \psi: U_{\rho, \lambda} \otimes \mathbb{C} \cdot 1 &\rightarrow W \\ \rho(a, \xi) \otimes 1 &\mapsto \xi a \cdot w \end{aligned}$$

Then

- $\psi$  is  $\mathbb{Z}$ -bilinear (biadditive)
- Write  $U_{\rho, \lambda} = U_{\rho, \lambda} \cap U_{\rho, \lambda}$ . We show that  $U_{\rho, \lambda} \subseteq U_{\rho, \lambda} \psi$ . If  $b \in \mathfrak{h} \cap \mathfrak{p}$ , then

$$\begin{aligned} \psi(\rho(a, \xi) \otimes 1) &= \xi a \cdot w \\ &= \xi a \lambda \rho(h) \cdot w \\ &= \lambda \rho(h) \xi a \cdot w \\ &= \psi(\rho(a, \lambda \rho(h) \xi) \otimes 1) \\ &= \psi(\rho(a, h \cdot \xi) \otimes 1). \end{aligned}$$

Similarly for  $b \in \mathfrak{p} \cap \mathfrak{n}$ . So  $\psi$  induces a map from  $M_{\rho, \lambda} = U_{\rho, \lambda} \otimes U_{\rho, \lambda} \subseteq U_{\rho, \lambda} \otimes \mathbb{C} \cdot 1 \rightarrow W$ . Since  $W$  is generated by  $w$ , the map  $\psi$  is surjective. So  $W \cong \frac{M_{\rho, \lambda}}{\ker \psi}$ .

v) Let

$$N_{\rho, \lambda} := \sum_{S \in \widehat{M_{\rho, \lambda}}} S$$

subrepresentations

We want to show that  $N_{\rho, \lambda} \cong M_{\rho, \lambda}$ .

Facts: Any subrepresentation of a weight representation is a weight representation. This implies all  $S$  have a weight decomposition  $S = \sum_{\mu} S_{\mu}$  and so does  $N_{\rho, \lambda}$ :

$$N_{\rho, \lambda} = \sum_{\mu} N_{\rho, \lambda}^{\mu}.$$

Since each  $S \in M_{\rho, \lambda}$   $S \neq 0$ ,  $\rho(S) \in M_{\rho, \lambda} \subseteq C v \otimes \mathfrak{q}$ . Hence  $N_{\rho, \lambda} = \sum_{S \in \widehat{M_{\rho, \lambda}}} S \neq 0$ .

Thus  $N_{\rho, \lambda}$  is proper subrepresentation. □

**Lemma 39.2.** *If  $\lambda, \mu \in \mathfrak{h}$  such that  $L_{\rho\lambda} \cong L_{\rho\mu}$ , then  $\lambda = \mu$ .*

*Proof.* Suppose  $\varphi: L_{\rho\lambda} \xrightarrow{\sim} L_{\rho\mu}$  is an isomorphism. Then  $\varphi(\rho v) = \rho v + \lambda s$ . Therefore  $\lambda = \mu \in Z_{\neq 0} \mathfrak{h}$ , i.e.  $\lambda = \mu$ . Switching roles of  $\lambda$  and  $\mu$  we also have  $\mu = \lambda$ . Thus  $\lambda = \mu$ .  $\square$

To summarize:

- Every finite dimensional irreducible representation is a finite dimensional irreducible highest weight representation.
- Every irreducible highest weight representation is  $L_{\rho\lambda}$  for some unique  $\lambda \in \mathfrak{h}$ .
- Question: When is  $\dim L_{\rho\lambda} = \mathfrak{S}$ ?

So  $\{L_{\rho\lambda} \mid \lambda \in \mathfrak{h}\}$  is a complete set of representatives for isoclasses of irreducible highest weight representations of  $\mathfrak{g}$ .

**Theorem 39.3.**  $\dim L_{\rho\lambda} = \mathfrak{S} \iff \lambda \in P^+$  i.e.  $\lambda = \sum_{i=1}^r \alpha_i \in P^+$ .

*Proof.*  $\Rightarrow$ : Last time using  $\mathfrak{sl}(2, \mathbb{C})$ .

$\Leftarrow$ : Somewhat lengthy.  $\square$

Consequently, we obtain the following theorem, which provides a classification of finite-dimensional irreducible representations of a semisimple Lie algebra  $\mathfrak{g}$ :

**Theorem 39.4.** *The set  $P^+$  of dominant integral weights is in bijection with the set of isomorphism classes of finite-dimensional irreducible representations of  $\mathfrak{g}$ . The bijection is given by  $\lambda \mapsto L_{\rho\lambda}$ , where  $L_{\rho\lambda}$  is the unique irreducible quotient of the Verma module  $M_{\rho\lambda}$ .*

## 40 Lecture 36: Examples

## 41 Lecture 37: Central Characters

**Definition 41.1.** A *character* of an associative algebra  $A$  is an algebra homomorphism  $\chi: A \rightarrow \mathbb{C}$ .

Recall that by  $Z(\mathfrak{U}(\mathfrak{g}))$  we mean the center of the universal enveloping algebra of  $\mathfrak{g}$ . That is,  $Z(\mathfrak{U}(\mathfrak{g})) = \{z \in \mathfrak{U}(\mathfrak{g}) \mid zu = uz \text{ for all } u \in \mathfrak{U}(\mathfrak{g})\}$ .

**Definition 41.2.** A *central character* of a Lie algebra  $\mathfrak{g}$  is a character of  $Z(\mathfrak{U}(\mathfrak{g}))$ .

**Example 41.3.** Let  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ . Choosing the ordered basis  $\{e, h, f\}$ , it is easy to check that the dual basis for  $\mathfrak{g}$  with respect to the trace form is given by  $\{\rho f, \frac{1}{2}h, \rho e\}$ . The  $\frac{1}{2}$  comes from the fact that square of the matrix  $h$  (in  $M_2(\mathbb{C})$ ) has trace 2. Thus the Casimir element is

$$C = ef + fe - \frac{1}{2}h^2.$$

This is an element of  $Z(\mathfrak{U}(\mathfrak{g}))$ . In fact, as we shall see, this element generates the center (as an associative algebra). That is,  $Z(\mathfrak{U}(\mathfrak{g})) = \mathbb{C}[C]$ . Furthermore,  $C$  is algebraically independent over  $\mathbb{C}$  so  $\mathbb{C}[C]$  is a polynomial algebra in one variable.

Thus a central character  $\chi$  for  $\mathfrak{g}$  is determined by the value  $\chi(C)$  at the Casimir.

We will show that each finite-dimensional simple  $\mathfrak{U}(\mathfrak{g})$ -module gives rise to a central character. We need an instance of Schur's Lemma which can be stated as follows in our case.

**Lemma 41.4** (Schur's Lemma). *If  $V$  is a simple  $U\mathfrak{pgq}$ -module, then*

$$\text{End}_{U\mathfrak{pgq}} V \cong \mathbb{C} \cdot \text{Id}_V.$$

*That is, if  $\varphi : V \rightarrow V$  is a  $U\mathfrak{pgq}$ -module endomorphism of  $V$ , then  $\varphi$  must be a scalar multiple of the identity.*

*Proof.* First we show that any nonzero  $U\mathfrak{pgq}$ -module endomorphism  $\psi$  of  $V$  has to be invertible. Indeed, since  $\ker \psi \subseteq V$  and  $V$  is a simple module, we have  $\ker \psi = 0$ . Similarly,  $\text{im } \psi = V$  so  $\text{im } \psi = V$  by simplicity.

Now let  $\varphi : V \rightarrow V$  be a  $U\mathfrak{pgq}$ -module endomorphism and let  $\xi \in \mathbb{C}$  be an eigenvalue of  $\varphi$ . Such a  $\xi$  exists because  $V$  is finite-dimensional and  $\mathbb{C}$  is assumed to be algebraically closed. Then consider  $\psi = \varphi - \xi \text{Id}_V$ . Since  $\xi$  is an eigenvalue for  $\varphi$ , the map  $\psi$  is not injective. Therefore, by the first part,  $\psi$  must be identically zero. Thus  $\varphi = \xi \text{Id}_V$ .  $\square$

Now we can show the following result.

**Proposition 41.5.** *Let  $V$  be a finite-dimensional simple  $U\mathfrak{pgq}$ -module. Then there exists a central character  $\chi$  for  $\mathfrak{g}$  such that*

$$z.v = \chi(z)v \quad \forall z \in \mathfrak{Z}\mathfrak{pgq}, v \in V. \quad (41.1)$$

*Proof.* Let  $z \in \mathfrak{Z}\mathfrak{pgq}$ . Then the map  $v \mapsto z.v$  is a  $U\mathfrak{pgq}$ -module endomorphism of  $V$ :  $u.pz.v = pzu.v = z.pu.v = u.pz.v$ . Thus, by Schur's Lemma, there exists a scalar  $\chi(z)$  such that (41.1) holds. Thus, for any  $v \in V$  and  $z_1, z_2 \in \mathfrak{Z}\mathfrak{pgq}$ ,

$$z_1 z_2.v = z_1.pz_2.v = z_1.\chi(z_2)v = \chi(z_2)z_1.v = \chi(z_1)\chi(z_2)v$$

which, choosing  $v \neq 0$ , implies that  $\chi(z_1 z_2) = \chi(z_1)\chi(z_2)$ . Similarly one shows that  $\chi$  is linear. Thus  $\chi$  is a central character of  $\mathfrak{g}$ .  $\square$

**Notation 41.6.** We know that when  $\mathfrak{g}$  is semisimple, any finite-dimensional simple  $U\mathfrak{pgq}$ -module is isomorphic to  $L(\lambda)$  for some dominant integral weight  $\lambda$ . In this case we denote the associated central character by  $\chi_\lambda$ .

**Example 41.7.** If we go back to the case of  $\mathfrak{g} = \mathfrak{sl}_2$ , let  $V = L(\rho) = \mathbb{C}[x, y]$  be the  $n+1$ -dimensional simple module  $\rho = \frac{1}{2}\alpha$ . Let us describe the corresponding central character  $\chi = \chi_\rho$ . Since  $\mathfrak{Z}\mathfrak{pgq}$  is generated by the Casimir  $C$  it suffices to compute  $\chi(C)$ . Furthermore, since we know  $C$  has to act by a scalar, it suffices to compute the action of  $C$  on any vector we want. We choose the highest weight vector. This motivates us to rewrite  $C$  in the PBW basis coming from the ordered basis  $f, h, e$ , because it is very easy to compute the action on the highest weight vector by a monomial having  $e$  on the right: it is zero. We have

$$\begin{aligned} C &= ef - fe + \frac{1}{2}h^2 \\ &= pfe - hq - fe + \frac{1}{2}h^2 \\ &= 2fe - \frac{1}{2}ph^2 - 2hq \\ &= 2fe - \frac{1}{2}ph - 1q^2 - 1. \end{aligned}$$



The last equality is not important right this moment but will become important in what follows. For now, observe that since  $fe$  kills the highest weight vector  $v$ , and  $hv = nv$ , we have

$$Cv = \frac{1}{2} (n-1)^2 v$$

Thus we conclude

$$\chi_{\mathfrak{p}C\mathfrak{q}} = \frac{1}{2} (n-1)^2 \chi_{\mathfrak{q}}.$$

**Remark 41.8.** In physics one would use the parametrization  $\ell = n/2$  for the representations, and normalize the Casimir to be  $2C$ . In those conventions, the value of the Casimir would be  $\ell(\ell+1)$ .

The example can be generalized. Notice that the procedure of rewriting  $z \in \mathfrak{P} \setminus \mathfrak{Z}(\mathfrak{g})$  in the PBW basis and throwing away terms having any positive root vector on the right, is equivalent to applying the Harish-Chandra homomorphism to  $z$ .

**Proposition 41.9.** *Let  $\mathfrak{g}$  be a semisimple Lie algebra, and  $V$  be a finite-dimensional simple  $U(\mathfrak{g})$ -module. Then the corresponding central character  $\chi$  is given by*

$$\chi(\mathfrak{p}z\mathfrak{q}) = \lambda(\varphi_{\text{HC}}(\mathfrak{p}z\mathfrak{q})) \quad (41.2)$$

where  $\varphi_{\text{HC}}$  denotes the Harish-Chandra homomorphism, and  $\lambda$  has been extended to a character  $U(\mathfrak{h}) \setminus \mathfrak{N}$ .

*Proof.* It once again suffices to check this identity on the highest weight vector  $v$  of  $V$ . By the PBW theorem, we have a direct sum decomposition  $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus \sum_{\alpha \in \mathfrak{p}} U(\mathfrak{g}_{\alpha}) U(\mathfrak{g}_{-\alpha})$ . For any  $u \in U(\mathfrak{g})$ , write  $u = u_0 + u_1$  where  $u_0 \in U(\mathfrak{h})$  and  $u_1 \in \sum_{\alpha \in \mathfrak{p}} U(\mathfrak{g}_{\alpha}) U(\mathfrak{g}_{-\alpha})$ . Clearly  $u \cdot v = u_0 \cdot v$  since  $\mathfrak{n} \cdot v = 0$ . Note also that  $u_0 = \varphi_{\text{HC}}(\mathfrak{p}u\mathfrak{q})$ . Thus we have for any  $z \in \mathfrak{P} \setminus \mathfrak{Z}(\mathfrak{g})$ :

$$z \cdot v = z_0 \cdot v = \varphi_{\text{HC}}(\mathfrak{p}z_0\mathfrak{q}) \cdot v$$

Now, the action of an element  $a$  of  $U(\mathfrak{h})$  on  $v$  is given by  $\lambda(\mathfrak{p}a\mathfrak{q}) \cdot v$ . Thus we obtain (41.2) □

For an  $n$ -dimensional vector space  $V$ , let  $\mathfrak{r}V\mathfrak{s}$  denote the algebra of all functions  $f : V \setminus \mathfrak{N}$  such that when we choose a basis  $\{v_i\}_1^n$  for  $V$ , we have  $f(\mathfrak{p}x_1v_1 + \dots + x_nv_n\mathfrak{q}) = p(x_1, \dots, x_n)$  for some polynomial  $p \in \mathfrak{r}x_1, x_2, \dots, x_n\mathfrak{s}$ . Notice that the property that  $f$  has is independent of the choice of basis, since a linear change of variables maps polynomials to polynomials. Thus, in fact, once we fix a basis we have an isomorphism  $\mathfrak{r}V\mathfrak{s} \cong \mathfrak{r}x_1, x_2, \dots, x_n\mathfrak{s}$ . But  $\mathfrak{r}V\mathfrak{s}$  exists as an object independent of choice of basis. The relation between  $\mathfrak{r}V\mathfrak{s}$  and  $\mathfrak{r}x_1, x_2, \dots, x_n\mathfrak{s}$  is exactly analogous to the relation between  $\text{End}(V)$  and  $M_n(\mathfrak{k})$ .

**Lemma 41.10.** *Let  $V$  be a finite-dimensional vector space. Then there is a canonical isomorphism*

$$\text{Sp}V \cong \mathfrak{r}V\mathfrak{s}.$$

*Proof.* We have an inclusion map  $i : V \setminus \mathfrak{N} \rightarrow \mathfrak{r}V\mathfrak{s}$ . Since  $\mathfrak{r}V\mathfrak{s}$  is commutative, by the universal property of the symmetric algebra there exists a unique algebra homomorphism  $\text{Sp}V \rightarrow \mathfrak{r}V\mathfrak{s}$  whose restriction to  $V$  is  $i$ . Conversely, the inclusion map  $V \rightarrow \mathfrak{r}V\mathfrak{s}$  extends uniquely (using that if  $\{x_1, \dots, x_n\}$  is a basis for  $V$  then  $\mathfrak{r}V\mathfrak{s} \cong \mathfrak{r}x_1, x_2, \dots, x_n\mathfrak{s}$ ) to an algebra map  $\mathfrak{r}V\mathfrak{s} \rightarrow \text{Sp}V$  which is inverse to the previous map. □

Notice that  $U\mathfrak{h} \oplus \mathfrak{S}(\mathfrak{h}) \cong \mathfrak{h} \oplus \mathfrak{S}(\mathfrak{h})$  often one identifies  $U\mathfrak{h}$  with  $\mathfrak{h}$ . With this in mind, the following theorem describes the center of the universal enveloping algebra of any semisimple Lie algebra.

**Theorem 41.11** (Harish-Chandra). *When restricted to the center, the Harish-Chandra homomorphism*

$$\varphi_{\text{HC}} : Z(\mathfrak{g}) \xrightarrow{\sim} \mathfrak{h}^* \quad (41.3)$$

is injective, with image equal to

$$\mathfrak{h}^* \cong \mathfrak{h}^* / W$$

where  $W$  is the Weyl group and  $W \cdot$  refers to the dot action, given on  $\mathfrak{h}^*$  by

$$w \cdot \lambda = w\rho + \lambda - \rho \quad (41.4)$$

where  $\rho$  is the Weyl vector  $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$ , with induced action on  $\mathfrak{h}^*$ :

$$\rho \cdot w^{-1} \lambda = \rho + w\rho - \lambda$$

## 42 Lecture 38: Kac-Moody Algebras

We follow *Introduction to Quantum Groups and Crystal Bases* by Hong and Kang.

Let  $I$  be a finite index set.

**Definition 42.1.** A square matrix  $A = (a_{ij})_{i,j \in I}$  with integer entries is a *generalized Cartan matrix (GCM)* if

1.  $a_{ij} \in \mathbb{Z}$ ,  $a_{ii} = 2$ ,  $i \in I$
2.  $a_{ij} \leq 0$ ,  $i \neq j$
3.  $a_{ij} = 0 \iff a_{ji} = 0$ .

**Definition 42.2.** A GCM  $A$  is called *symmetrizable* if there exists a diagonal matrix  $D = \text{diag}(b_i)_{i \in I}$  with  $b_i \in \mathbb{Z}_{>0}$  such that  $DA$  is symmetric.

**Definition 42.3.** A GCM  $A$  is *indecomposable* if for all partitions  $I = I_1 \sqcup I_2$ ,  $I_j \neq \emptyset$ , there is  $i \in I_1$ ,  $j \in I_2$  such that  $a_{ij} \neq 0$ .

For an  $I \times I$ -matrix, let  $\text{corank } A = |I| - \text{rank } A$ .

**Definition 42.4.** A *Cartan datum* is a quintuple  $(A, \Pi, \Pi^-, P, P^-)$  where

1.  $P^-$  is a free abelian group of rank  $|I| - \text{corank } A$  with  $\mathbb{Z}$ -basis denoted  $\{h_i \mid i \in I\} \cup \{d_s \mid s = 1, 2, \dots, \text{corank } A\}$ ;
2.  $P = \mathbb{Z}\langle \mathfrak{h} \mid \lambda \in P^- \rangle \in \mathbb{Z}\langle \mathfrak{h} \mid \mathfrak{b}_{\mathbb{Z}} P^- \rangle$ ;
3.  $\Pi^- = \{h_i \mid i \in I\}$ ;
4.  $\Pi = \{\alpha_i \mid i \in I\}$  is a linearly independent subset of  $\mathfrak{h}^*$  satisfying

$$\alpha_j \rho_{h_i} = a_{ij} \alpha_j \rho_{d_s} \in \mathbb{Z}, 1 \leq i, j \leq |I|. \quad (42.1)$$

We call

- $\mathfrak{h}$  the *Cartan subalgebra (CSA)*
- $\Pi$  the set of *simple roots*,
- $\Pi^-$  the set of *simple coroots*,
- $P$  the *weight lattice*,
- $P^-$  the *dual weight lattice*

Given a Cartan datum  $\mathfrak{p}A, \Pi, \Pi^-, P, P^-$  we define the *root lattice* to be

$$Q = \sum_{i \in I} \mathbb{Z} \alpha_i \quad (42.2)$$

and the *positive (cone in the) root lattice*

$$Q^+ = \sum_{i \in I} \mathbb{Z}_{\neq 0} \alpha_i. \quad (42.3)$$

and set  $Q = Q^+ \cup Q^-$ . The *fundamental weights*  $\omega_i \in \mathfrak{h}$  for  $i \in I$  are given by

$$\langle \omega_i, \rho_j \rangle = \delta_{ij}, \quad \langle \omega_i, \rho_s \rangle = 0. \quad (42.4)$$

We have a partial order on  $\mathfrak{h}$  given by  $\lambda \preceq \mu \iff \lambda - \mu \in Q^+$ .

The *simple reflections*  $s_i \in \text{GL}(\mathfrak{h})$ ,  $i \in I$ , are given by

$$s_i \rho \lambda = \lambda - \lambda \langle \rho, \alpha_i \rangle, \quad \forall \lambda \in \mathfrak{h}. \quad (42.5)$$

and the *Weyl group*  $W$  is the subgroup of  $\text{GL}(\mathfrak{h})$  generated by the set of simple reflections.

**Definition 42.5.** The Kac-Moody<sup>xi</sup> (KM) algebra  $\mathfrak{g}$  associated with a Cartan datum  $\mathfrak{p}A, \Pi, \Pi^-, P, P^-$  is the Lie algebra with generators

$$\{e_i, f_i\}_{i \in I} \cup P^- \quad (42.6)$$

subject to relations saying that  $P^-$  is a subgroup of  $\mathfrak{g}$  together with

$$\langle \rho h, h^1 \rangle = 0 \quad h, h^1 \in P^-, \quad (42.7)$$

$$\langle \rho e_i, f_j \rangle = \delta_{ij} h_i, \quad (42.8)$$

$$\langle \rho h, e_j \rangle = \alpha_j \langle \rho, h \rangle e_j, \quad (42.9)$$

$$\langle \rho h, f_j \rangle = -\alpha_j \langle \rho, h \rangle f_j, \quad (42.10)$$

$$\langle \rho a_i e_j \rangle^{a_{ij}} \langle \rho e_j \rangle = 0, \quad i \neq j, \quad (42.11)$$

$$\langle \rho a_i f_j \rangle^{a_{ij}} \langle \rho f_j \rangle = 0, \quad i \neq j. \quad (42.12)$$

**Proposition 42.6.**  $\mathfrak{g}$  has a root space decomposition

$$\mathfrak{g} = \sum_{\rho \in Q} \mathfrak{g}_\rho, \quad \mathfrak{g}_\rho = \{x \in \mathfrak{g} \mid \langle \rho, x \rangle = \alpha \langle \rho, h \rangle, \alpha \in \mathbb{C}\} \quad (42.13)$$

and a triangular decomposition

$$\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+, \quad \mathfrak{g}^\pm = \sum_{\rho \in Q^\pm} \mathfrak{g}_\rho. \quad (42.14)$$

---

<sup>xi</sup>Independently discovered in 1967{1968 by three people: V.G. Kac, R. V. Moody and I.L. Kantor

GCMs, hence KM algebras, are classified into three types. For an integer column vector  $u = (u_i)_{i \in I}^T$  we write  $u \geq 0$  (resp.  $u \neq 0$ ) if  $\forall i \in I: u_i \geq 0$  (resp.  $u_i \neq 0$ ).

The following is Theorem 4.3 of Kac's book *Infinite dimensional Lie algebras*

**Theorem 42.7.** *Let  $A$  be an indecomposable GCM. Then exactly one of the following three possibilities hold:*

(Finite Type):  $\text{corank } A = 0$ , and  $\exists u \geq 0: Au = 0$ , and  $\forall v \neq 0 \exists p \geq 0$  or  $v \leq 0$ .

(Affine Type):  $\text{corank } A = 1$ , and  $\exists u \geq 0: Au = 0$ , and  $\forall v \neq 0 \exists \tilde{r} \geq 0$ .

(Indefinite Type):  $\exists u \geq 0: Au = 0$  and  $\exists v \neq 0 \exists q \geq 0$ .

**Example 42.8.** Examples of GCMs of affine type are

$$\begin{array}{ccc} \begin{array}{cc} 2 & 2 \\ 2 & 2 \end{array} & \begin{array}{cc} 2 & 4 \\ 1 & 2 \end{array} & \begin{array}{ccc} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{array} \end{array}$$

An example of a GCM of indefinite type is

$$\begin{array}{cc} 2 & 5 \\ 1 & 2 \end{array}$$

## 42.1 The Dynkin Diagram of a GCM

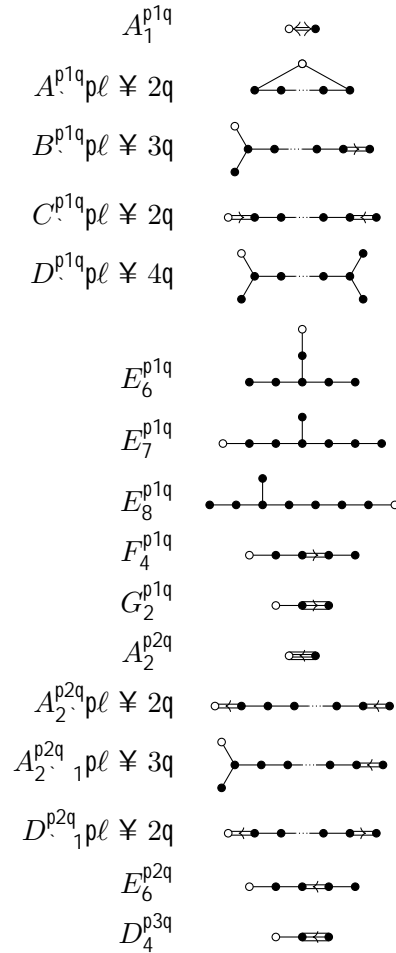
Given a GCM  $A$ , we associate a (generalized) Dynkin diagram  $D(A)$  as follows. The vertex set is  $I$ . For the edges one convention is as follows:

- In the case when  $a_{ij}a_{ji} = 3$  we retain the conventions from finite type (see Definition 31.7).
- When  $a_{ij} = a_{ji} = 2$  we depict the edge as  $\leftrightarrow$ , and when  $\max(a_{ij}, a_{ji}) = p > 4, 1 \leq q$  we draw it as  $\rightleftarrows$ .
- When  $a_{ij}a_{ji} \geq 4$  we draw a bold line between  $i$  and  $j$  and put the label  $\max(a_{ij}, a_{ji})$  above the edge.

The following proposition is not hard to prove, given Kac's theorem above.

**Proposition 42.9.** *Let  $A$  be an indecomposable GCM. Then  $A$  is of affine type if and only if  $\det A = 0$  and all proper subdiagrams of  $D(A)$  are of finite type.*

Using this result one can obtain a complete list of the connected Dynkin diagrams of affine type. The white node corresponds to the unique simple root  $\alpha_0$  with  $\langle \alpha_0, \alpha_0 \rangle = 1$ . The significance of this will be explained in the next theorem.



**Example 42.10.** The three cartan matrices from the previous example correspond to the affine Dynkin diagrams  $A_1^{p1q}$ ,  $A_2^{p2q}$ , and  $A_2^{p2q}$  respectively.

## 42.2 Realization of Affine Kac-Moody Algebras as Extensions of Loop Algebras

A very important fact about the affine KM algebras is their realization as extensions of loop algebras.

**Theorem 42.11.** *The untwisted affine KM algebras are isomorphic to the Lie algebra*

$$\mathfrak{g} \oplus \mathbb{R} \langle t, t^{-1} \rangle \oplus \mathbb{R} \langle c \rangle \oplus \mathbb{R} \langle d \rangle \quad (42.15)$$

where  $\mathfrak{g}$  is the so called underlying finite type Lie algebra obtained by deleting the white node from the Dynkin diagram, and the Lie bracket is given by

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} - m\delta_{m, -n} \rho(x, y) c \quad (42.16)$$

$$[d, x \otimes t^m] = m x \otimes t^m \quad (42.17)$$

$$c \text{ is central} \quad (42.18)$$

The twisted affine KM algebras can be realized as the subalgebra  $\mathfrak{g}$  of all elements fixed by an automorphism  $\theta$  of order 2 or 3, inside an untwisted affine KM algebra  $\mathfrak{g}$ . The automorphism comes from a diagram automorphism of the untwisted affine Dynkin diagram.

## 43 Lecture 39: Hopf Algebras and Quantum Groups

### 43.1 Vector spaces

We make some remarks about the category of vector spaces over  $k$ .

Bilinear maps are not morphisms in the category of vector spaces. This indicates that they are not the correct thing to look at. By the universal property of tensor products, bilinear maps  $U \times V \rightarrow W$  are in bijection with linear maps  $U \otimes V \rightarrow W$ .

Selecting an special element  $v \in V$ , such as an identity element of an algebra, is also not a categorical notion, since we don't have "elements". The solution here is that elements of  $V$  are in bijection with linear maps  $k \rightarrow V$ .

We also recall that  $\otimes$  is a bifunctor, which means that for any two linear maps  $f_1 : V_1 \rightarrow W_1$ ,  $f_2 : V_2 \rightarrow W_2$  we get a linear map  $f_1 \otimes f_2 : V_1 \otimes V_2 \rightarrow W_1 \otimes W_2$ , given by  $(f_1 \otimes f_2)(v_1 \otimes v_2) = f_1(v_1) \otimes f_2(v_2)$  and extended linearly.

The category of vector spaces has a special object,  $k$ , which is the *tensor unit object*. We have natural isomorphisms  $k \otimes V \cong V \cong V \otimes k$  for any vector space  $V$ .

Lastly, the map  $\tau_{V,W} : V \otimes W \rightarrow W \otimes V$ ,  $v \otimes w \mapsto w \otimes v$ , is a natural isomorphism for any vector spaces  $V, W$ .

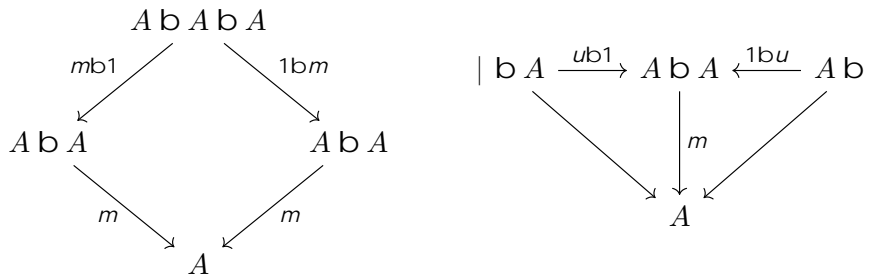
### 43.2 Algebras

An associative algebra is often defined as a vector space together with a bilinear map  $A \otimes A \rightarrow A$ ,  $(a, b) \mapsto ab$  and an element  $1_A \in A$  satisfying some axioms. We reformulate this definition in a way that is purely expressed in terms of objects and morphisms in the category of vector spaces.

**Definition 43.1.** An *algebra* is a triple  $(A, m, u)$  where  $A$  is a vector space with linear maps

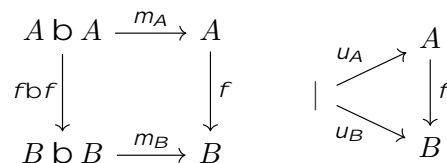
$$m = m_A : A \otimes A \rightarrow A, \quad u = u_A : k \rightarrow A$$

such that these diagrams commute:



The left diagram is *associativity* the right diagram is *unitality*.

An *algebra map*  $f : A \rightarrow B$  is a linear map such that these diagrams commute:



**Example 43.2.**  $k$  is an algebra with  $m_k$  the natural isomorphism  $k \otimes k \cong k$  and  $u_k$  the identity map  $k \rightarrow k$ .

**Example 43.3.** If  $A$  and  $B$  are algebras then the vector space  $A \otimes B$  can be equipped with an algebra structure with  $m_{A \otimes B}$  and  $u_{A \otimes B}$  the unique maps making these diagrams commute:

$$\begin{array}{ccc}
 A \otimes B \otimes A \otimes B & \xrightarrow{m_{A \otimes B}} & A \otimes B \\
 \downarrow 1_A \otimes 1_B & \nearrow m_A \otimes m_B & \\
 A \otimes A \otimes B \otimes B & & 
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \xrightarrow{u_{A \otimes B}} & A \otimes B \\
 \downarrow & \nearrow u_A \otimes u_B & \\
 & & \mathbb{1}
 \end{array}$$

### 43.3 Coalgebras

In category theory any notion has a dual notion. For example the dual notion of products is coproducts. Therefore, now that we have formulated the notion of an algebra in categorical terms, we naturally obtain the definition of a *coalgebra*.

**Definition 43.4.** A *coalgebra* is a triple  $(C, \Delta, \varepsilon)$  where  $C$  is a vector space and

$$\Delta : C \rightarrow C \otimes C, \quad \varepsilon : C \rightarrow \mathbb{1}$$

are linear maps so that these diagrams commute:

$$\begin{array}{ccc}
 & C \otimes C \otimes C & \\
 \nearrow \Delta & & \nwarrow \Delta \\
 C \otimes C & & C \otimes C \\
 \nwarrow \Delta & & \nearrow \Delta \\
 & C & 
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbb{1} \otimes C & \xleftarrow{\varepsilon \otimes 1} & C \otimes C & \xrightarrow{1 \otimes \varepsilon} & C \otimes \mathbb{1} \\
 \nwarrow \Delta & & \uparrow \Delta & & \nearrow \Delta \\
 & & C & & 
 \end{array}$$

A *coalgebra map*  $f : C \rightarrow D$  is a linear map such that these diagrams commute:

$$\begin{array}{ccc}
 C \otimes C & \xleftarrow{\Delta} & C \\
 \downarrow f \otimes f & & \downarrow f \\
 D \otimes D & \xleftarrow{\Delta} & D
 \end{array}
 \qquad
 \begin{array}{ccc}
 & A & \\
 \nwarrow \varepsilon & & \downarrow f \\
 \mathbb{1} & & D
 \end{array}$$

**Example 43.5.** Let  $A$  be a finite-dimensional algebra. Then the dual space  $A^*$  is a coalgebra with  $\Delta = m^*$  and  $\varepsilon = u^*$ . (For this to make sense we must use that the natural injective map  $A \otimes A \rightarrow A \otimes A$  that we always have is surjective in the case that  $A$  is finite-dimensional.)

### 43.4 Bialgebras

**Definition 43.6.** A *bialgebra* is a quintuple  $(B, m, u, \Delta, \varepsilon)$  such that  $(B, m, u)$  is an algebra and  $(B, \Delta, \varepsilon)$  is a coalgebra and one of the following equivalent conditions hold:

- (i)  $\Delta$  and  $\varepsilon$  are algebra maps,
- (ii)  $m$  and  $u$  are coalgebra maps.

A *bialgebra map*  $f : B \rightarrow C$  is a linear map which is both an algebra map and a coalgebra map.

**Example 43.7.** Let  $M$  be a finite monoid, and  $B = \{^M\}$  be the set of all functions  $\xi : M \rightarrow \mathbb{C}$ . This is an algebra with respect to pointwise operations. To define the comultiplication, we note that  $\{^M\} \otimes \{^M\} = \{^M \times M\}$ . We define  $\Delta$  by  $\Delta(\xi)(p, m, n, q) = \xi(pmnq)$ . The counit is given by  $\varepsilon(\xi) = \xi(1_M)$ .

**Example 43.8.** The universal enveloping algebra  $U(\mathfrak{g})$  of any Lie algebra  $\mathfrak{g}$  is not just an algebra but a bialgebra. We show there are algebra maps  $\Delta : U(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$  and  $\varepsilon : U(\mathfrak{g}) \rightarrow \mathbb{C}$  satisfying

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \varepsilon(x) = 0 \quad \forall x \in \mathfrak{g}. \quad (43.1)$$

Define  $\Delta^1 : \mathfrak{g} \rightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$  by  $\Delta^1(x) = x \otimes 1 + 1 \otimes x$  for  $x \in \mathfrak{g}$ . This is a linear map, hence by the universal property of the tensor algebra,  $\Delta^1$  induces an algebra map  $\tilde{\Delta} : T(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$ . One checks that  $\tilde{\Delta}(I) \in I$ , where  $I$  is the ideal defining  $U(\mathfrak{g})$ . Thus  $\tilde{\Delta}$  descends to an algebra map  $\Delta : U(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$ . Similarly one proves the existence of the counit  $\varepsilon : U(\mathfrak{g}) \rightarrow \mathbb{C}$ .

**Warning:** The rule  $\Delta(x) = x \otimes 1 + 1 \otimes x$  only holds for  $x$  in the Lie algebra  $\mathfrak{g}$ , not for all  $x \in U(\mathfrak{g})$ . For example, for  $x, y \in \mathfrak{g}$ ,  $\Delta(xy)$  is defined to be  $\Delta(x)\Delta(y) = (x \otimes 1 + 1 \otimes x)(y \otimes 1 + 1 \otimes y) = xy \otimes 1 + x \otimes y + y \otimes x + 1 \otimes xy$ . A similar remark applies to the counit.

### 43.5 Quantum groups

A quantum group is a certain bialgebra (in fact, Hopf algebra, see next lecture) that we associate to any Cartan datum.

**Definition 43.9.** Let  $F = \mathbb{C}(q)$  be the field of rational functions in an indeterminate  $q$ . Let  $(A, \Pi, \Pi^-, P, P^-)$  be a Cartan datum where  $A$  is a symmetrizable GCM. The associated (*Drinfeld-Jimbo*) quantum group, denoted  $U_q(\mathfrak{g})$  is the  $F$ -algebra with generators  $e_i, f_i$  and  $K_i^{\pm 1}$  satisfying

$$K_i^0 = 1, \quad K_i^h = q^{h_i}, \quad \forall h \in P^-, \quad (43.2)$$

$$q^h e_i q^{-h} = q^{h_i} e_i, \quad \forall h \in P^-, \quad (43.3)$$

$$q^h f_i q^{-h} = q^{-h_i} f_i, \quad \forall h \in P^-, \quad (43.4)$$

$$e_i f_j = f_j e_i + \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad (43.5)$$

$$\sum_{k=0}^{1-a_{ij}} \binom{1-a_{ij}}{k}_q e_i^k e_j e_i^{1-k} = 0, \quad \forall i \neq j, \quad (43.6)$$

$$\sum_{k=0}^{1-a_{ij}} \binom{1-a_{ij}}{k}_q f_i^k f_j f_i^{1-k} = 0, \quad \forall i \neq j. \quad (43.7)$$

where  $K_i = q^{b_i}$  and  $q_i = q^{a_i}$ . The last two relations are the *quantum Serre relations*. The *q-binomial coefficients* are elements of  $F$  defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}, \quad [n]_q! = [n]_q [n-1]_q \cdots [1]_q, \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}. \quad (43.8)$$



The comultiplication and counit on  $U_q(\mathfrak{g})$  are given by

$$\Delta(pq^h) = q^h \otimes q^h, \quad \epsilon(pq^h) = 1, \quad (43.9)$$

$$\Delta(pe_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i, \quad (43.10)$$

$$\Delta(pf_i) = f_i \otimes 1 + K_i \otimes f_i, \quad (43.11)$$

$$\epsilon(pq^h) = 1, \quad \epsilon(pe_i) = 0, \quad (43.12)$$

$$\epsilon(pf_i) = 0, \quad (43.13)$$

$$\epsilon(pf_i) = 0. \quad (43.14)$$

## 44 Lecture 40: Hopf algebras contd.

### 44.1 Convolution Algebras

If  $C$  is a coalgebra and  $A$  is an algebra, then the space  $\text{Hom}(C, A)$  becomes an associative algebra as follows. For  $f, g \in \text{Hom}(C, A)$  we define  $f \circ g$  to be the composition

$$(f \circ g)(c) = \sum (f \otimes g)(\Delta(c)) = \sum f(c_1)g(c_2). \quad (44.1)$$

The identity element of  $\text{Hom}(C, A)$  is the composition of the maps

$$C \xrightarrow{\eta} k \xrightarrow{\epsilon} A.$$

### 44.2 Hopf Algebras

**Definition 44.1.** A Hopf algebra  $H$  is a bialgebra  $(H, m, u, \Delta, \epsilon)$  such that the identity map  $H \xrightarrow{\eta} k \xrightarrow{\epsilon} H$  is invertible in the convolution algebra  $\text{End}(H)$ . That is, if there exists a linear map

$$S : H \rightarrow H \quad (44.2)$$

such that this diagram commutes:

$$\begin{array}{ccccc}
 & & H & & \\
 & \swarrow & \downarrow \eta & \searrow & \\
 H \otimes H & & & & H \otimes H \\
 \downarrow S \otimes 1 & & \downarrow u & & \downarrow 1 \otimes S \\
 H \otimes H & & H & & H \otimes H \\
 \swarrow m & & \downarrow m & & \swarrow m \\
 & & H & & 
 \end{array}$$

(Note that this diagram is self-dual.)  $S$  is called the *antipode*.

Abusing notation one often denotes a Hopf algebra by  $(H, m, u, \Delta, \epsilon, S)$  even though  $S$  is not an additional piece of data, rather it is a property of the data already present.

### 44.3 Examples

- Let  $G$  be a group and let  $k[G]$  be the group algebra defined as the vector space with basis  $G$  and multiplication extended bilinearly from the multiplication in  $G$ . This is a Hopf algebra with

$$\Delta(pg) = g \otimes g, \quad \epsilon(pg) = 1, \quad S(pg) = g^{-1}$$

for all  $g \in G$ . Note that these formulas have to be extended linearly to all of  $k[G]$ . For example  $\Delta(pg) = \sum hq$ ,  $\Delta(pg) = \sum g \otimes h$ ,  $\Delta(pg) = \sum g \otimes h$ ,  $\epsilon(pg) = \sum hg$ ,  $S(pg) = \sum hg$ .

2. Let  $\mathfrak{g}$  be a Lie algebra and  $U\mathfrak{g}$  its universal enveloping algebra. Then  $U\mathfrak{g}$  is a Hopf algebra with

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \varepsilon(x) = 0, \quad S(x) = -x$$

for all  $x \in \mathfrak{g}$ . This time these formulas have to be extended multiplicatively to all of  $U\mathfrak{g}$ . For example  $\Delta(xy) = \Delta(x)\Delta(y) = (x \otimes 1 + 1 \otimes x)(y \otimes 1 + 1 \otimes y) = xy \otimes 1 + x \otimes y + y \otimes x + 1 \otimes xy$ .

3. Let  $G$  be a finite (resp. compact) group, and  $F\mathfrak{g}$  be the algebra of all (resp. continuous) functions from  $G$  to  $\mathbb{C}$  (resp.  $\mathbb{R}$ ). It is known that  $F\mathfrak{g} = G = F\mathfrak{g} \otimes F\mathfrak{g}$ . Thus we may define a comultiplication on  $F\mathfrak{g}$  by

$$\Delta(fg, h) = fgh, \quad @f \in F\mathfrak{g}, @g, h \in G.$$

The counit and antipode are given by

$$\varepsilon(f) = \int f, \quad S(fg) = \int fg^{-1}.$$

#### 44.4 Sweedler Notation

Let  $C$  be a coalgebra. For every  $x \in C$  we have that  $\Delta(x) \in C \otimes C$ . Therefore there are  $N_x \in \mathbb{Z}_{\neq 0}$  and elements  $x_{p1}^i, x_{p2}^i \in C$  such that

$$\Delta(x) = \sum_{k=1}^{N_x} x_{p1}^k \otimes x_{p2}^k. \quad (44.3)$$

However, neither  $N_x$  nor the elements  $x_{pj}^i$  are unique, due to the bilinear nature of  $C \otimes C$ .

**Example 44.2.** Let  $C = \mathbb{C}[G]$  be the group algebra of a group  $G$  and let  $g \in G$ . Then, by definition of the comultiplication,

$$\Delta(g) = g \otimes g$$

With this choice of writing it, we have  $N_g = 1$  and  $g_{p1}^1 = g_{p2}^1 = g$ . However we may also write

$$\Delta(g) = pg^{-1} \otimes g = 1 \otimes g$$

where  $1 \in G$  is the identity element of  $G$ . With this choice,  $N_g = 2$  and  $g_{p1}^1 = g^{-1}, g_{p2}^1 = g, g_{p1}^2 = 1, g_{p2}^2 = g$ .

Due to this non-uniqueness, we have to make sure that any map out of  $C \otimes C$  is actually well-defined.

For example, the map

$$f : C \otimes C, \quad f(x) = \sum_{k=1}^{N_x} x_{p1}^k \otimes x_{p2}^k$$

is not well-defined, because the expression is not bilinear in  $(x_{p1}^k, x_{p2}^k)$ . Concretely, the two different choices of writing  $\Delta(g)$  in the example above would give different results (check!).

Similarly,

$$f : C \otimes C, \quad f(x) = x_{p1}^1 \otimes x_{p2}^1$$

is not well-defined, because it only involves the two factors from the first term in  $\Delta(x)$ , which is not unique.

*Sweedler notation* uses more efficient notation, while at the same time making it harder to write down ill-defined expressions. Instead of (44.3) we write

$$\Delta p x q \quad \overset{\circ}{\underset{p \times q}{x_{p1q} \bowtie x_{p2q}}} \quad (44.4)$$

With the superscripts suppressed, it is impossible to “select” any particular term, and that is a good thing because they are not well-defined anyway.

There is also a “summation-less”<sup>xii</sup> Sweedler notation, wherein we simply write

$$\Delta p x q \quad x_{p1q} \bowtie x_{p2q} \quad (44.5)$$

The summation is implied, because of the subscripts  $p1q$  and  $p2q$  appearing.

Let us practice by writing out the relevant Hopf algebra axioms in summationless Sweedler notation:

Coassociativity reads

$$x_{p1qp1q} \bowtie x_{p1qp2q} \bowtie x_{p2q} \quad x_{p1q} \bowtie x_{p2qp1q} \bowtie x_{p2qp2q} \quad (44.6)$$

while counit axiom is

$$\varepsilon p x_{p1q} q x_{p2q} \quad x \quad x_{p1q} \varepsilon p x_{p2q} q \quad (44.7)$$

and the antipode axiom becomes

$$S p x_{p1q} q x_{p2q} \quad u \varepsilon p x q \quad x_{p1q} S p x_{p2q} q. \quad (44.8)$$

In an algebra known to be associative, we don’t need to use parenthesis when writing an iterated product such as  $xyz$ . The dual version of this is that with a coassociative comultiplication, we write each side of (44.6) simply as

$$x_{p1q} \bowtie x_{p2q} \bowtie x_{p3q}. \quad (44.9)$$

We may iterate this too and write

$$\Delta^n p x q \quad x_{p1q} \bowtie x_{p2q} \bowtie \dots \bowtie x_{p(n-1)q} \quad (44.10)$$

which means we have applied  $\Delta$   $n$  times to various tensor factors; which ones we chose doesn’t matter by coassociativity.

## 44.5 Opposites and Co-opposites; A Property of the Antipode

Recall the flip  $\tau : x \bowtie y \rightsquigarrow y \bowtie x$ .

If  $A$  is an algebra, the *opposite* algebra, denoted  $A^{\text{op}}$  is  $pA, m^{\text{op}}, uq$  where  $m^{\text{op}} = m \circ \tau$ .

Dually, if  $C$  is a coalgebra, the *co-opposite* coalgebra, denoted  $C^{\text{cop}}$  is  $pC, \Delta^{\text{op}}, \varepsilon q$  where  $\Delta^{\text{op}} = \tau \circ \Delta$ .

**Theorem 44.3.** *Let  $H$  be a Hopf algebra. Then  $S$  is a bialgebra map  $H^{\text{op}, \text{cop}} \rightsquigarrow H$ . That is,*

- (i)  $S p x y q = S p y q S p x q$  and  $S \circ u = u$ .
- (ii)  $S p x q_{p1q} S p x q_{p2q} = S p x_{p2q} q S p x_{p1q} q$  and  $\varepsilon \circ S = \varepsilon$ .

<sup>xii</sup>analogous to Einstein’s summation convention in tensor calculus

*Proof.* We only prove  $S\rho xyq = S\rho yqS\rho xq$ , leaving the other statements as an exercise.

Define  $\lambda, \rho \in \text{Hom}(H \otimes H, H)$  in a way that corresponds to the left and right hand sides:

$$\lambda(x \otimes y) = S\rho xyq, \quad \rho(x \otimes y) = S\rho yqS\rho xq.$$

Here we regard  $H \otimes H$  as a coalgebra and  $H$  as an algebra, thus  $\text{Hom}(H \otimes H, H)$  is an associative algebra with respect to the convolution product. To show that  $\lambda = \rho$  it suffices to show that

$$\lambda \circ m = u \circ \epsilon \circ (H \otimes H) \circ m \circ \rho$$

Indeed, then

$$\lambda \circ \lambda = u \circ \epsilon \circ (H \otimes H) \circ \lambda \circ m \circ \rho = u \circ \epsilon \circ (H \otimes H) \circ \rho \circ \rho$$

Using that

$$\Delta_{H \otimes H}(\rho(x \otimes y)) = (\rho_1 \otimes \tau \otimes 1) \Delta(\rho xyq) = \Delta(\rho yq) \otimes (\rho_1 \otimes 1) \Delta(\rho xq)$$

we have by definition of the convolution product

$$\begin{aligned} \rho \lambda \circ m(\rho(x \otimes y)) &= \lambda(\rho_1 \otimes 1) \Delta(\rho xyq) \otimes (\rho_2 \otimes 1) \Delta(\rho xyq) \\ &= \lambda(\rho_1 \otimes 1) \Delta(\rho xyq) \otimes (\rho_2 \otimes 1) \Delta(\rho xyq) \\ &= S\rho_1 \rho_1 q y \rho_1 q \rho_2 q x \rho_2 q y \rho_2 q \\ &= S\rho_1 \rho_1 q y \rho_1 q \rho_2 q x \rho_2 q y \rho_2 q \quad \text{since } \Delta \text{ is an algebra map} \\ &= u \circ \epsilon(\rho xyq) \quad \text{by antipode axiom for } H \\ &= u \circ \epsilon(\rho xq \rho yq) \\ &= u \circ \epsilon \circ (H \otimes H) \circ \rho(x \otimes y). \end{aligned}$$

On the other hand,

$$\begin{aligned} \rho m \circ \rho(x \otimes y) &= m(\rho_1 \otimes 1) \Delta(\rho xyq) \otimes (\rho_2 \otimes 1) \Delta(\rho xyq) \\ &= x \rho_1 q y \rho_1 q S\rho_2 y \rho_2 q S\rho_2 x \rho_2 q \\ &= x \rho_1 q u \circ \epsilon(\rho yq) S\rho_2 x \rho_2 q \quad \text{by antipode axiom} \\ &= u \circ \epsilon(\rho xq u \circ \epsilon(\rho yq)) \quad \text{since } u \circ \epsilon(\rho yq) \in P \text{ and using antipode axiom} \\ &= u \circ \epsilon \circ (H \otimes H) \circ \rho(x \otimes y). \end{aligned}$$

This finishes the proof. □

## 45 Lecture 41: Tensor Products of Modules

The following principle will be key: If  $f : A \rightarrow B$  is an algebra map, and  $V$  is a  $B$ -module, then  $V$  becomes an  $A$ -module by

$$a.v = f(a).v$$

Next we make three observations about left modules over algebras:

- (i) If  $V$  is an  $A$ -module and  $W$  is a  $B$ -module, then  $V \otimes W$  is an  $A \otimes B$ -module via

$$(a \otimes b).(v \otimes w) = (a.v) \otimes (b.w)$$

- (ii)  $V \otimes W$  is obviously a  $(A \otimes B)$ -module.

(iii) If  $V$  is a left  $A$ -module then the dual space  $V^*$  is a left  $A^{\text{op}}$ -module by  $\rho a \cdot \xi q p v q = \xi p a \cdot v q$ .

As a direct corollary we get the following statement about modules over Hopf algebras.

**Lemma 45.1.** *Let  $H$  be a Hopf algebra.*

(i) *If  $V$  and  $W$  are  $H$ -modules (regarding  $H$  as just an algebra), then  $V \otimes W$  is an  $H$ -module with action*

$$h \cdot (v \otimes w) = \rho h_{p1q} \cdot v \otimes \rho h_{p2q} \cdot w$$

(ii) *The vector space  $1 \otimes k$  is an  $H$ -module with action*

$$h \cdot 1 = \varepsilon p h q$$

(iii) *If  $V$  is an  $H$ -module then  $V^*$  is an  $H$ -module with action*

$$\rho h \cdot \xi q p v q = \xi S p h q \cdot v$$

*Proof.* (i) Use that the comultiplication  $\Delta : H \rightarrow H \otimes H$  is an algebra map together with the two observations above.

(ii) Use that the counit  $\varepsilon : H \rightarrow k$  is an algebra map.

(iii) Use that the antipode is an algebra map  $S : H^{\text{op}} \rightarrow H$  □

## 46 Lie Superalgebras

## 47 Reduction Algebras and Extremal Projectors

## 48 Gelfand-Tsetlin Bases for Representations of $\mathfrak{gl}_n$

## 49 Crystal Bases

## 50 Category $\mathcal{O}$

## 51 The Weyl Character Formula

Let  $V = \sum_{\mu \in \mathfrak{h}^*} \rho_{\mu} V_{\mu}$ . Then we define the following,  $\text{ch } V = \sum_{\mu \in \mathfrak{h}^*} \rho_{\mu} \dim V_{\mu} e^{\mu} \in \mathbb{Z}[\mathfrak{h}^*]$  with  $e^{\mu} = e^{\mu_1} \cdots e^{\mu_n}$  and  $e^0 = 1$ .

**Theorem 51.1.**

- 1)  $\text{ch } (V \otimes W) = \text{ch } V \cdot \text{ch } W$
- 2)  $\text{ch } (V \oplus W) = \text{ch } V + \text{ch } W$
- 3) *If  $V, W$  are finite dimensional, then  $\text{ch } (V \oplus W) = \text{ch } V + \text{ch } W$ .*

**Theorem 51.2** (Weyl Character Formula).  $\lambda \in P^+$  then

$$\text{ch } L(\lambda) = \frac{\sum_{w \in W} \epsilon(w) e^{w(\lambda + \rho)}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})}$$

where  $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$

## 52 Appendix: A Brief Introduction to Category Theory

### 52.1 Classes

A *class* is like a set but can be bigger. Every set is a class but not all classes are sets. A class which is not a set is a *proper class*. Just like with sets we can form the cartesian product of classes, consider functions between classes and so on.

The main reason that we need classes is so that we can talk about things like *the class of all sets* because there is no set that contains all sets. (Likewise there is no class that contains all classes, but somehow we don't need to really worry about that!)

### 52.2 Partial binary operations

A *partial binary operation* on a class  $X$  is a function from some subclass of  $X \times X$  to  $X$ . We write  $\cdot : X \times X \dashrightarrow X$  to indicate that the domain of  $\cdot$  may not be all of  $X \times X$ .

### 52.3 Definition of category

**Definition 52.1.** A *category*  $\mathcal{C}$  is a quintuple  $\mathcal{C} = (C_0, C_1, s, t, \cdot)$  where

- $C_0$  is a class whose elements are called the *objects* of  $\mathcal{C}$ ,
- $C_1$  is a class whose elements are called the *morphisms* of  $\mathcal{C}$ ,
- $s : C_1 \rightarrow C_0$  is a map called the *source map*,
- $t : C_1 \rightarrow C_0$  is a map called the *target map*,
- $\cdot : C_1 \times C_1 \dashrightarrow C_1$  is a partial binary operation called *composition (of morphisms)* so that  $\alpha \cdot \beta$  is defined for any morphisms  $\alpha, \beta \in C_1$  with  $t\beta = s\alpha$ ,

subject to the following two axioms:

- (i) (identity) for every object  $x \in C_0$  there exists a morphism  $1_x \in C_1$  with

$$s(1_x) = t(1_x) = x$$

$$\alpha \cdot 1_x = \alpha \quad \text{for all morphisms } \alpha \in C_1 \text{ with } s\alpha = x$$

$$1_x \cdot \beta = \beta \quad \text{for all morphisms } \beta \in C_1 \text{ with } t\beta = x$$

- (ii) (associativity) we have

$$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$$

for any morphisms  $\alpha, \beta, \gamma \in C_1$  with  $t\gamma = s\beta$  and  $t\beta = s\alpha$ .

**Notation 52.2.**  $C_0$  and  $C_1$  are sometimes denoted  $\text{Ob } \mathcal{C}$  and  $\text{Mor } \mathcal{C}$  respectively. You should think of the source and target maps as giving the domain and codomain of a morphism. In this spirit, if  $\alpha \in C_1$  is a morphism with  $s\alpha = x$  and  $t\alpha = y$  we write  $\alpha : x \rightarrow y$ .

## 52.4 Examples

To specify a category we have to say what the objects and morphisms are. The source, target and are almost always the domain, codomain and usual composition.

**Example 52.3.** 1) The category of sets and functions  $\text{Set}$ . This means that by definition  $\text{Set}_0$  is the class of all sets, and  $\text{Set}_1$  is the class of all functions between sets.

2) The category of abelian groups and group homomorphisms  $\text{Ab}$ .

3) For any ring  $R$  the category of left  $R$ -modules and  $R$ -module homomorphisms  $R\text{-Mod}$ .

4) If  $L$  and  $R$  are rings the category of  ${}_{\rho}L, R_{\sigma}$ -bimodules and  ${}_{\rho}L, R_{\sigma}$ -bimodule homomorphisms is denoted by  $L\text{-Mod-}R$ .

5) The category of topological spaces and continuous functions  $\text{Top}$ .

## 52.5 Functors

Just like a group homomorphism is a structure preserving function between groups, a functor is a structure preserving function between categories. Since a category has two underlying classes, a functor needs to be a pair of functions.

**Definition 52.4.** Let  $C$  and  $D$  be categories. A (covariant) functor  $F$  from  $C$  to  $D$  is a pair of maps  $F = \langle F_0, F_1 \rangle$  where  $F_i : C_i \rightarrow D_i$  for  $i = 0, 1$  such that

$$(i) F_1 \langle \rho, 1_x \rangle = 1_{F_0 \langle \rho, x \rangle} \text{ for all } x \in C_0$$

$$(ii) \text{ if } \alpha : x \rightarrow y \text{ then } F_1 \langle \rho, \alpha \rangle : F_0 \langle \rho, x \rangle \rightarrow F_0 \langle \rho, y \rangle$$

$$(iii) F_1 \langle \rho, \alpha \circ \beta \rangle = F_1 \langle \rho, \alpha \rangle \circ F_1 \langle \rho, \beta \rangle \text{ for all morphisms } \alpha, \beta \in C_1 \text{ with } \text{tp}(\beta) = \text{sp}(\alpha).$$

A contravariant functor  $F$  from  $C$  to  $D$  is the same things as a covariant functor except it reverses the direction of morphisms in the sense that (ii) and (iii) are replaced by

$$(ii') \text{ if } \alpha : x \rightarrow y \text{ then } F_1 \langle \rho, \alpha \rangle : F_0 \langle \rho, y \rangle \rightarrow F_0 \langle \rho, x \rangle$$

$$(iii') F_1 \langle \rho, \alpha \circ \beta \rangle = F_1 \langle \rho, \beta \rangle \circ F_1 \langle \rho, \alpha \rangle \text{ for all morphisms } \alpha, \beta \in C_1 \text{ with } \text{tp}(\beta) = \text{sp}(\alpha).$$

**Notation 52.5.** Usually we write  $Fx$  for  $F_0 \langle \rho, x \rangle$  and  $F\alpha$  for  $F_1 \langle \rho, \alpha \rangle$  if no confusion can arise.

## 52.6 Examples

All examples will be a covariant functors. The following examples are related to the universal property of free  $R$ -modules (see next section).

**Example 52.6.** 1) The forgetful functor  $O_R : R\text{-Mod} \rightarrow \text{Set}$  (where  $O$  stands for oblivion) sends any left  $R$ -module  $M$  to the underlying set  $M$ , and any  $R$ -module homomorphism to itself (now regarded as just a function).

2) The free functor  $F_R : \text{Set} \rightarrow R\text{-Mod}$  sends any set  $X$  to the free left  $R$ -module on the set  $X$ , denoted  $F_RX$ . And if  $\alpha : X \rightarrow Y$  then  $F_R\alpha : F_RX \rightarrow F_RY$  is the morphism induced by the composition  $X \rightarrow Y \rightarrow F_RY$ .

The next two examples are important in the context of tensor products (see next section on adjoint functors). Let  $L, S, R$  be rings with 1 and fix an  $\rho S, Rq$ -bimodule  $B$ .

**Example 52.7.** 1)

$$\mathfrak{b}_S B : L\text{-Mod-}S \tilde{\mathfrak{N}} L\text{-Mod-}R$$

is the functor that sends an  $\rho L, Sq$ -bimodule  $A$  to the  $\rho L, Rq$ -bimodule  $A \mathfrak{b}_S B$ , and sends an  $\rho L, Sq$ -bimodule morphism  $\alpha : A \tilde{\mathfrak{N}} A^1$  to the  $\rho L, Rq$ -bimodule morphism  $\alpha \mathfrak{b} 1_B : A \mathfrak{b}_S B \tilde{\mathfrak{N}} A^1 \mathfrak{b}_S B$ .

2) In the opposite direction we have the following functor:

$$\text{Hom}_R \rho B, \mathfrak{q} : L\text{-Mod-}R \tilde{\mathfrak{N}} L\text{-Mod-}S$$

which sends an  $\rho L, Rq$ -bimodule  $A$  to  $\text{Hom}_R \rho B, Aq$ , the set of right  $R$ -module maps  $B \tilde{\mathfrak{N}} A$ .  $\text{Hom}_R \rho B, Aq$  is an  $\rho L, Sq$ -bimodule through

$$\rho \ell \quad \varphi q \rho b q \quad \ell \quad \rho \varphi \rho b q q \quad \ell \quad P \quad L, b \quad P \quad B, \varphi \quad P \quad \text{Hom}_R \rho B, Aq$$

$$\rho \varphi \quad s q \rho b q \quad \varphi \rho s \quad b q \quad @s \quad P \quad S, b \quad P \quad B, \varphi \quad P \quad \text{Hom}_R \rho B, Aq$$

On morphisms the functor  $\text{Hom}_R \rho B, \mathfrak{q}$  takes  $\alpha : A \tilde{\mathfrak{N}} A^1$  to the map  $\alpha : \text{Hom}_R \rho B, Aq \tilde{\mathfrak{N}} \text{Hom}_R \rho B, A^1q$  given by post-composition (push forward):  $\alpha \rho \varphi q \quad \alpha \quad \varphi$ .

## 52.7 Pairs of adjoint functors

### 52.8 Definition

**Definition 52.8.** Given categories  $C$  and  $D$ , and covariant functors  $F : C \tilde{\mathfrak{N}} D$  and  $G : D \tilde{\mathfrak{N}} C$  we say that  $F$  is left adjoint to  $G$  and  $G$  is right adjoint to  $F$  if there is a natural bijection

$$\text{Hom}_C \rho x, Gyq \xrightarrow{x,y} \text{Hom}_D \rho Fx, yq$$

for all  $x \in C_0$  and  $y \in D_0$ . Here  $\text{Hom}_C \rho a, bq$  denotes the class of morphisms in  $C$  from an object  $a$  to an object  $b$ . That the family  $\rho \eta_{x,y} q_{x \in C_0, y \in D_0}$  is “natural” means that whenever  $\alpha : x \tilde{\mathfrak{N}} x^1$  and  $\beta : y \tilde{\mathfrak{N}} y^1$  are morphisms in  $C_1$  and  $D_1$  respectively the following diagram commutes:

$$\begin{array}{ccc} \text{Hom}_C \rho x^1, Gy^1q & \xrightarrow{x^1, y^1} & \text{Hom}_D \rho Fx^1, y^1q \\ \downarrow & & \downarrow F \\ \text{Hom}_C \rho x, Gyq & \xrightarrow{x, y} & \text{Hom}_D \rho Fx, yq \\ \downarrow F \rho \mathfrak{q} & & \downarrow \\ \text{Hom}_C \rho x, Gy^1q & \xrightarrow{x, y^1} & \text{Hom}_D \rho Fx, y^1q \end{array}$$

The commutativity of this diagram makes mathematically precise the vague statement that  $\eta_{x,y}$  should be defined “the same way” regardless of the objects  $x$  and  $y$ . There is an analogous definition for contravariant functors.



## 52.9 Examples

Many universal properties can be expressed in terms of adjoint functors.

**Example 52.9.** 1) In example 52.6, the free functor  $F_R : \text{Set} \xrightarrow{\sim} R\text{-Mod}$  is left adjoint to the forgetful functor  $O_R : R\text{-Mod} \xrightarrow{\sim} \text{Set}$  because

$$\text{Hom}_{\text{Set}}(\rho X, O_R M) \cong \text{Hom}_R(\rho F_R X, M)$$

for any set  $X$  and  $R$ -module  $M$ . In words, any set map  $X \xrightarrow{\sim} M$  extends uniquely to an  $R$ -module morphism  $F_R X \xrightarrow{\sim} M$ . The naturality is a tedious but straightforward exercise.

2) In Example 52.7, the functor  $\otimes_S B$  is left adjoint to  $\text{Hom}_R(\rho B, \_)$ . Let  ${}_L \text{Hom}_R(\rho X, Y)$  denote the set of  $(\rho L, R)$ -bimodule homomorphisms between  $(\rho L, R)$ -bimodules  $X$  and  $Y$ . Then what we are saying is that there is a natural bijection

$${}_L \text{Hom}_S(\rho A, \text{Hom}_R(\rho B, C)) \cong {}_L \text{Hom}_R(\rho A \otimes_S B, C).$$

Taking  $L = R = Z$  the left hand side can be identified with the set of  $S$ -balanced maps  $A \otimes B \xrightarrow{\sim} C$ , so this expresses precisely the universal property of the tensor product.