MATH 365
Exam 2
Spring 2022

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This exam is closed book and closed notes. No electronic devices, including calculators and headphones, are allowed. Drawing pictures to help understand and solve the questions is encouraged. Answer each question completely using exact values. Show your work neatly, including correct notation and showing the steps in your work, as well as writing legibly; **answers without work and/or justifications will not receive credit**. *Circle your final answer for each problem. Each problem is worth* 10 *points. The lowest score will be dropped.*

1	
2	
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DO NOT BEGIN THIS EXAM UNTIL INSTRUCTED TO START

Do not write in these boxes on the exam ⟶

score

1. Let $f = u + iv$ be an analytic function such that $u = x + e^x \cos y$ and $f(0) = 1$. Find v.

Cauchy-Riemann Eqs.

$$\begin{cases} \dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x} = 1 + e^x \cos y \Rightarrow v = y + e^x \sin y + g(x) \\ \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} = -\left(0 + e^x(-\sin y)\right) = e^x \sin y \end{cases}$$

But also

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(y + e^x \sin y + g(x)\right)$$

$$= e^x \sin y + g'(x) \Rightarrow \begin{array}{l} g'(x) = 0 \\ g(x) = C \end{array}$$

$$v = y + e^x \sin y + C$$

$$1 = f(0) = u(0,0) + i v(0,0)$$

$$= e^0 \cos 0 + i\left(0 + e^0 \cdot 0 + C\right)$$

$$= 1 + Ci \Rightarrow C = 0$$

$$\boxed{v = y + e^x \sin y}$$

score

2. Find the power series expansion of $f(z) = e^z(1-z)$ at $z = 1$.

$$f(z) = e \cdot e^{z-1} \cdot (1-z) = (-e)e^{z-1}(z-1) =$$

$$= (-e)\cdot(z-1)\cdot \sum_{n=0}^{\infty} \frac{1}{n!}(z-1)^n =$$

$$= (-e)\sum_{n=0}^{\infty} \frac{1}{n!}(z-1)^{n+1}$$

$$= -e(z-1) - \frac{e}{1!}(z-1)^2 - \frac{e}{2!}(z-1)^3 - \cdots$$

score

3. Compute: $\int_{|z|=1} \frac{(z+i)e^z}{z(z+2i)} dz$

$f(z) = \frac{(z+i)e^z}{z(z+2i)}$ has only one pole inside the unit circle: $z_0 = 0$ of order 1.

By Cauchy's Theorem (or the Residue thm),

$$\int_{|z|=1} \frac{(z+i)e^z}{z(z+2i)} dz = 2\pi i \, Res\left(\frac{(z+i)e^z}{z(z+2i)} ; 0\right) =$$

$$= 2\pi i \frac{(0+i)e^0}{0+2i} = 2\pi i \frac{i}{2i} = \boxed{\pi i}$$

score

4.

a. Find $\operatorname{Res}\left(\frac{\sin z}{z^4}; 0\right)$

$$\frac{\sin z}{z^4} = \frac{1}{z^4}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) =$$

$$= \frac{1}{z^3} - \frac{1}{3!} \cdot \frac{1}{z} + \frac{1}{5!} z - \cdots$$

$$\text{So } \operatorname{Res}\left(\frac{\sin z}{z^4}; 0\right) = -\frac{1}{3!} = \boxed{-\frac{1}{6}}$$

b. Find the Laurent series expansion (at least 3 nonzero terms) of $\frac{1}{z^5 - z^3}$ at $z_0 = 0$.

$$\frac{1}{z^5 - z^3} = \frac{1}{z^3(z^2-1)} = \frac{-1}{z^3} \cdot \frac{1}{1-z^2} =$$

$$= \frac{-1}{z^3} \sum_{n=0}^{\infty} (z^2)^n = \sum_{n=0}^{\infty} -z^{2n-3} =$$

$$= -z^{-3} - z^{-1} - z^1 - z^3 - z^5 - \cdots$$

score

5. Compute: $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+4} dx$

$(z-2i)(z+2i)$
Simple poles

$z = 2i$ is in the upper half-plane

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+4} dx = \operatorname{Re}\left(2\pi i \operatorname{Res}\left(\frac{e^{iz}}{z^2+4}; 2i\right)\right) =$$

$$= \operatorname{Re}\left(2\pi i \frac{e^{i\cdot 2i}}{2i+2i}\right) = \operatorname{Re}\left(\frac{2\pi e^{-2}}{4}\right) = \boxed{\frac{\pi}{2e^2}}$$