MATH 365
Exam 1
Spring 2022

Student name: Key

This exam is closed book and closed notes. No electronic devices, including calculators and headphones, are allowed. Drawing pictures to help understand and solve the questions is encouraged. Answer each question completely using exact values. Show your work neatly, including correct notation and showing the steps in your work, as well as writing legibly; **answers without work and/or justifications will not receive credit**. *Circle your final answer for each problem. Each problem is worth 10 points. The lowest score will be dropped.*

1	
2	
3	
4	
5	

DO NOT BEGIN THIS EXAM UNTIL INSTRUCTED TO START

Do not write in these boxes on the exam →

score

1. Find all complex numbers z such that $z^8 = -2 - 2i$.

(2) $|-2-2i| = \sqrt{4+4} = 2\sqrt{2}$

(2) $\operatorname{Arg}(-2-2i) = -\frac{5}{8}\cdot 2\pi = \frac{5\pi}{4}$

$$z_k = (2\sqrt{2})^{1/8}\left[\cos\left(\frac{5\pi}{4\cdot 8} + \frac{k}{8}\cdot 2\pi\right) + i\sin\left(\frac{5\pi}{4\cdot 8} + \frac{k}{8}\cdot 2\pi\right)\right]$$

(2) (2)

$k = 0, 1, \ldots, 7$

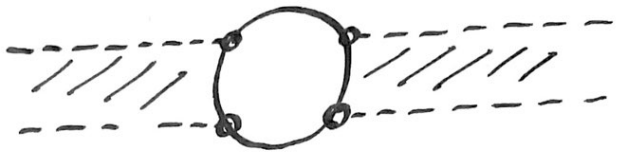
(2)

score

2. Sketch the region D. Determine the boundary of D (be careful with "corner" points). Is D open? closed? connected? convex? a domain? Motivate your answers.

$$D = \{z : -1 < \mathrm{Im}(z) < 1 \text{ and } |z| \geq \sqrt{2}\}$$

D:



3

2

Boundary is:

$\partial D = \{z : -1 \leq \mathrm{Im}(z) \leq 1 \text{ and } |z| = \sqrt{2}\}$

$\cup \{z : |\mathrm{Im}(z)| = 1 \text{ and } |\mathrm{Re}\, z| \geq 1\}$

D open? No, D contains boundary points, $z = \sqrt{2}$. 1

D closed? No, D doesn't contain the four boundary points $\pm 1 \pm i$, $\pm 1 \pm i$ 1

D conn.? no, no path from -2 to 2. 1

D convex? no, not connected. 1

D domain? no, not connected. 1

score

3. Is the series convergent? Why/Why not?

a. $\sum_{n=1}^{\infty} \frac{n}{(1+i)^n}$

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{(1+i)^{n+1}} \cdot \frac{(1+i)^n}{n} = \frac{n+1}{n} \cdot \frac{1}{1+i}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{n} \cdot \frac{1}{|1+i|} = \left(1+\frac{1}{n}\right)\frac{1}{\sqrt{2}} \to \frac{1}{\sqrt{2}} \text{ as } n \to \infty$$

Since $\frac{1}{\sqrt{2}} < 1$, the series converges absolutely by the ratio test.

b. $\sum_{n=1}^{\infty} \left(\frac{7-i}{5+5i}\right)^n$

$$\left|\frac{7-i}{5+5i}\right|^n = \left(\frac{\sqrt{49+1}}{\sqrt{25+25}}\right)^n = 1^n = 1 \not\to 0$$

So the terms do not converge to zero

so the series is not convergent.

score

4. Compute

a. $(2-i\sqrt{12})^{300}$

$|2-i\sqrt{12}| = \sqrt{4+12} = 4$

$2-i\sqrt{12} = 4\left(\frac{1}{2} - \frac{2\sqrt{3}}{4}i\right) = 4\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) =$

$= 4\left(\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right)$

By De Moivre's Formula,

$(2-i\sqrt{12})^{300} = 4^{300}\left(\cos\left(\frac{-\pi}{3}\cdot 300\right) + i\sin\left(\frac{-\pi}{3}\cdot 300\right)\right) =$

$= 4^{300}\left(\cos(-\pi\cdot 100) + i\sin(-\pi\cdot 100)\right)$

$= \boxed{4^{300}}$

b. $\sin(\text{Log}(i))$

$\sin(\text{Log } i) = \sin\left(\ln|i| + i\,\text{Arg}(i)\right)$

$= \sin\left(i\frac{\pi}{2}\right)$

$= \dfrac{e^{i(i\pi/2)} - e^{-i(i\pi/2)}}{2i}$

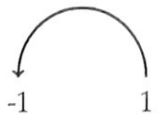
$= \dfrac{e^{-\pi/2} - e^{\pi/2}}{2i} \quad \left(= \frac{i}{2}\left(e^{\pi/2} - e^{-\pi/2}\right)\right)$

score

5. Compute the line integral

$$\int_\gamma \frac{\operatorname{Im} z}{z} dz$$

where γ is the semicircle from 1 to -1 through i:



$$\gamma(t) = e^{it}, \quad 0 \le t \le \pi$$

$$\int_\gamma \frac{\operatorname{Im} z}{z} dz = \int_0^\pi \frac{\sin t}{e^{it}} \, i \cdot e^{it} \, dt =$$

$$= i \int_0^\pi \sin t \, dt = i \left[-\cos t \right]_0^\pi = i(-1-(-1)) = \boxed{2i}$$

score