Math 365 – Study Guide for Final Exam

Sections 1.1, 1.2 Complex Numbers, the Complex Plane; Some Geometry.

- If $z = x + iy$ with x, y real, then $\bar{z} = x - iy$ and $|z| = \sqrt{x^2 + y^2}$, $\operatorname{Re} z = x$, $\operatorname{Im} z = y$
- $z\bar{z} = |z|^2$, $z/w = z\bar{w}/|w|^2$, $|z| - |w| \leq |z + w| \leq |z| + |w|$ (triangle inequality)
- Polar representation: $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$, $r = |z| = \sqrt{x^2 + y^2}$, $\theta = \arg z$ angle from the positive x-axis; $\arg(zw) = \arg(z) + \arg(w)$, $|zw| = |z| \cdot |w|$.
- $\arg(z)$ is multivalued; $\operatorname{Arg} z$ is the value that belongs to $[-\pi, \pi)$.
- The equation $z^n = w = |w|e^{i\psi}$ has n distinct solutions, given by

$$z_k = |w|^{1/n} e^{i\theta_k} = |w|^{1/n}(\cos\theta_k + i\sin\theta_k),\ \theta_k = (\psi + 2\pi k)/n,\ k = 0, 1, 2, \ldots, n-1.$$

Section 1.3 Subsets of the Plane.

- The *open disk of radius R centered at z_0* is given by $\{z : |z - z_0| < R\}$.
- z is a *boundary point* of a set D if every open disk centered at z contains points from D as well as points not in D.

A set D is

- *open* if it contains no boundary points; *closed* if it contains all boundary points;
- *connected* if any two points in D can be joined by a finite number of line segments that lie in D;
- a *domain* if it is open and connected;
- *convex* if the line segment joining any two pairs of points in D is contained in D.

Section 1.4 Functions and Limits.

- $\lim_{n\to\infty} z_n = A$ if whenever $\varepsilon > 0$ there is $N > 0$ so that $|z_n - A| < \varepsilon$ when $n \geq N$.
- $\lim_{z\to z_0} f(z) = A$ if whenever $\varepsilon > 0$ there is $R > 0$ so that $|f(z) - A| < \varepsilon$ when $|z - z_0| < R$.
- f is continuous at z if $f(z)$ is defined and $\lim_{w\to z} f(w) = f(z)$.
- $\sum_{k=0}^{\infty} z_k = \lim_{n\to\infty} \sum_{k=0}^{n} z_k$

Section 1.5 Exp, Log, and Trig Functions.

- $e^{x+iy} = e^x(\cos y + i\sin y)$
- $e^z = 1$ if and only if $z = 2\pi i n$ for some integer n.
- $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$, $\cos(z) = \dfrac{e^{iz} + e^{iz}}{2}$
- $\log z = \ln|z| + i\arg(z)$. A *branch* of $\log z$ means restricting values of $\arg(z)$ to an interval of length 2π. The *principal branch* is $\operatorname{Log} z = \ln|z| + i\operatorname{Arg}(z)$.

Section 1.6 Line Integrals and Green's Theorem.

- A curve γ is *closed* if it's a loop, and *simple* if it doesn't intersect itself.
- The *line integral* of $f(z)$ over γ is $\displaystyle\int_\gamma f(z)dz = \int_a^b f\big(\gamma(z)\big)\gamma'(t)dt$.
- Estimate: $\displaystyle\Big|\int_\gamma f(z)dz\Big| \leq \text{length}(\gamma) \cdot \max_{z\in\gamma} |f(z)|$.
- Green's Theorem: $\displaystyle\int_\gamma f(z)dz = i\iint_\Omega \Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big)dxdy$.

Section 2.1 Analytic and Harmonic Functions; the Cauchy-Riemann Equations. Up to and including Theorem 2. Skip Theorem 3 and Examples 10,11.

- Analytic functions = functions that have derivative in the complex variable sense.
- e^z, $\cos(z)$, $\sin(z)$, (any branch of) $\log(z)$, and rational functions are analytic where defined. The functions $|z|$, $\mathrm{Re}\,(z)$, $\mathrm{Im}\,(z)$ are not analytic.
- Sums, products, quotients of analytic functions are analytic where defined.
- Product rule, quotient rule, chain rule all hold as usual.
- If $f = u+iv$ is analytic then u and v must satisfy the *Cauchy-Riemann Equations*:
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad\qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$
This implies that u is determined by v (and vice versa). See example after Theorem 1.
- Theorem 2: If $f = u + iv$ is analytic and u (or v) or $u^2 + v^2$ is constant then f is constant.

Section 2.2 Power Series. Up to and including Example 9. Skip Example 10.

- *Radius of convergence*. Theorem 2 says
$$R^{-1} = \lim_{n\to\infty} \Big|\frac{a_{n+1}}{a_n}\Big| = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$
Study Examples 1–5.
- Derivative and anti-derivative of power series work as expected, e.g. $\frac{d}{dx}\sum_{n=0}^{\infty} z^n = \sum_{n=1}^{\infty} nz^{n-1}$. In particular, power series are infinitely differentiable, hence define analytic functions in their disk of convergence.
- Multiplying power series: See Theorem 4, Examples 8 and 9.

Section 2.3 Cauchy's Theorem and Cauchy's Formula. Up to and including Example 7. Skip Theorems 2,3 and Examples 8,9,10.

- A domain D is *simply connected* if the inside of any simple closed curve in D is contained in D. (D has "no holes".)
- Roughly, when f is analytic and γ is not encircling any holes:
$$\int_\gamma f(z)dz = 0 \qquad (\textit{Cauchy's Theorem})$$
$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}dz \qquad (\textit{Cauchy's Formula})$$
These are special cases of the Residue Theorem from Section 2.6.
- To solve trig integrals of the type $\int_0^{2\pi} d\theta/(2+\sin\theta)$, substitute $z = e^{i\theta}$ and use Cauchy's Formula (or the Residue Theorem). See Examples 6 and 7.

Section 2.4 Consequences of Cauchy's Theorem. Up to and including The Order of a Zero. Skip the rest of Section 2.4, starting with Morera's Theorem.

- If f is analytic at $z = z_0$ then $f(z)$ has a power series expansion at z_0:
$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \qquad \text{where } a_n = \frac{1}{n!}f^{(n)}(z_0).$$

- If f is analytic in a domain D and all its derivatives vanish at some point z_0 then actually f has to be identically zero everywhere in D.
- If $f(z_0) = 0$, $f'(z_0) = 0$, ..., $f^{(m-1)}(z_0) = 0$ but $f^{(m)}(z_0) \neq 0$ then f *has a zero of order* m *at* z_0. This is the same thing as saying:

$$f(z) = a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \cdots \qquad \text{and } a_m \neq 0.$$

Section 2.5 Isolated Singularities. Up to and including Example 9. Also, skip Examples 4 and 7.

- There are three kinds of singularities: *removable, poles,* and *essential singularities.*
- $f(z)$ has a pole at z_0 of order m precisely when $1/f(z)$ has a zero at z_0 of order m. In this case $f(z)$ has a *Laurent series*:

$$f(z) = a_{-m}(z-z_0)^{-m} + a_{-m+1}(z-z_0)^{-m+1} + \cdots + a_{-1}(z-z_0)^{-1} + a_0 + a_1(z-z_0) + \cdots \qquad \text{and } a_{-m} \neq 0.$$

and the *residue of* f *at* z_0 can be computed in three ways:

$$\operatorname{Res}(f; z_0) = \frac{1}{2\pi i}\int_{|z-z_0|=\varepsilon} f(z)dz = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\big((z - z_0)^m f(z)\big)\Big|_{z=z_0} = a_{-1}$$

In particular, if $m = 1$ we say f has a *simple pole at* z_0 and we have

$$\operatorname{Res}(f; z_0) = \big((z - z_0)f(z)\big)\Big|_{z=z_0}$$

meaning, simplify $(z - z_0)f(z)$ and plug in $z = z_0$.

Section 2.6 The Residue Theorem and Its Applications. Up to and including Example 7. But skip Example 6. Skip everything after Example 7.

- The *Residue Theorem*: If f is analytic in a simply connected domain except for isolated singularities $z_1, z_2, \ldots, z_n$, and γ is a positively oriented closed curve, then

$$\int_\gamma f(z)dz = 2\pi i \sum \operatorname{Res}(f; z_j)$$

where the sum is over all z_j that lie inside γ.

- If $P(x)$ and $Q(x)$ are real polynomials and the $\deg Q(x) \geq 2 + \deg P(x)$ and $Q(x)$ has no real zeroes, then

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}dx = 2\pi i \sum \operatorname{Res}(\frac{P}{Q}; z_j)$$

$$\int_{-\infty}^{\infty} \frac{P(x)\cos x}{Q(x)}dx = \operatorname{Re}\big\{2\pi i \sum \operatorname{Res}\big(\frac{P(z)e^{iz}}{Q(z)}; z_j\big)\big\}$$

$$\int_{-\infty}^{\infty} \frac{P(x)\sin x}{Q(x)}dx = \operatorname{Im}\big\{2\pi i \sum \operatorname{Res}\big(\frac{P(z)e^{iz}}{Q(z)}; z_j\big)\big\}$$

where the sums are over all zeroes z_j of Q that lie in the upper half plane.

Section 3.1 The Zeros of an Analytic Function. Up to and including Theorem 1 on p.173.

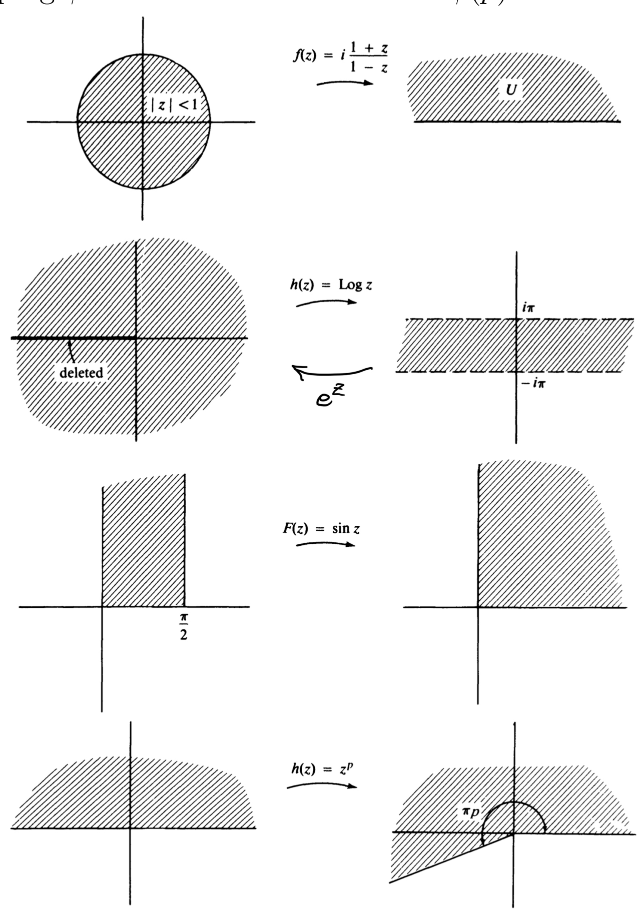
- The zeros of a non-constant analytic function f are *isolated*: If $f(z_0) = 0$ then there is an open disk centered at z_0 which doesn't contain any other zeros of $f(z)$.
- *Counting formula*: $\dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{h'(z)}{h(z)}dz = \begin{pmatrix}\text{number of zeros}\\ \text{of } h \text{ inside } \gamma\end{pmatrix} - \begin{pmatrix}\text{number of poles}\\ \text{of } h \text{ inside } \gamma\end{pmatrix}$

Section 3.3 Linear Fractional Transformations. Up to and including Example 2.

- $T(z) = \dfrac{az+b}{cz+d}$, a, b, c, d complex, $ad - bc \neq 0$. Ex: translation $T(z) = z + a$; dilation $T(z) = rz$ $(r > 0)$; rotation $T(z) = e^{i\theta}z$; inversion $T(z) = 1/z$.
- $T^{-1}(z) = \dfrac{-dz+b}{cw-a}$
- If $T(z) = \left(\dfrac{z-z_1}{z-z_3}\right)\left(\dfrac{z_2-z_3}{z_2-z_1}\right)$ then $T(z_1) = 0,\ T(z_2) = 1,\ T(z_3) = \infty$.

Section 3.5 Riemann Mapping Theorem. Up to and including Example 5.

- Two regions are *conformally equivalent* if there exists a one-to-one analytic mapping from one onto the other.
- Every simply connected domain D, other than $\mathbb{C}$, is conformally equivalent to the open unit disk Δ. Moreover, if p is any point of D, there is a unique one-to-one analytic mapping ϕ from D onto Δ such that $\phi(p) = 0$ and $\phi'(0) > 0$.



Section 4.2. Only the subsection Flow of an Ideal Fluid, pages 261–264 up to and including Example 2.

- If $f = u + iv$ describes the velocity of an ideal fluid then $\bar{f} = u - iv$ is analytic.
- Let $G = \phi + i\psi$ with $G' = \bar{f}$. Then ϕ is the *potential function* and ψ is the *stream function*.
- The *stream lines* are the level curves $\psi = c$, (c a real constant). They describe the path followed by a particle within the flow.
- The *uniform flow* $f(z) = A$, (A a complex constant) has constant velocity and streamlines are straight lines.
- If $f(z)$ is a 1-to-1 analytic mapping from a horizontal strip $a < \operatorname{Im} z < b$ to a domain D, then
$$\Gamma_c = \{f(x + i\tau) : a < \tau < b\}$$
are streamlines in D. Similarly for vertical strips.

Euler's Gamma Function. The posted lecture notes.

- For $\operatorname{Re} z > 0$ the integral $\Gamma(z) = \displaystyle\int_0^\infty e^{-t} t^{z-1} dt$ converges and is analytic in z.
- Recursion relation:
$$\Gamma(z+1) = z\Gamma(z), \qquad \Gamma(1) = 1.$$
- $\Gamma(n+1) = n!$ for all $n = 0, 1, 2, \ldots$
- $\Gamma(z)$ can be extended using $\Gamma(z) = \dfrac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)}$, $\operatorname{Re}(z) > -n$
- $\Gamma(z)$ is analytic everywhere except at $z = 0, -1, -2, \ldots$ where it has simple poles
- $\operatorname{Res}(\Gamma; -n) = (-1)^n/n!$
- Reflection relation: $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$