MATH 365 – STUDY GUIDE FOR EXAM 2

Section 2.1 Analytic and Harmonic Functions; the Cauchy-Riemann Equations. Up to and including Theorem 2. Skip Theorem 3 and Examples 10,11.

- Analytic functions = functions that have derivative in the complex variable sense.
- e^z, $\cos(z)$, $\sin(z)$, (any branch of) $\log(z)$, and rational functions are analytic where defined. The functions $|z|$, $\mathrm{Re}\,(z)$, $\mathrm{Im}\,(z)$ are not analytic.
- Sums, products, quotients of analytic functions are analytic where defined.
- Product rule, quotient rule, chain rule all hold as usual.
- If $f = u+iv$ is analytic then u and v must satisfy the *Cauchy-Riemann Equations*:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad\qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

  This implies that u is determined by v (and vice versa). See example after Theorem 1.
- Theorem 2: If $f = u + iv$ is analytic and u (or v) or $u^2 + v^2$ is constant then f is constant.

Section 2.2 Power Series. Up to and including Example 9. Skip Example 10.

- *Radius of convergence.* Theorem 2 says

$$R^{-1} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

  Study Examples 1–5.
- Derivative and anti-derivative of power series work as expected, e.g. $\frac{d}{dx}\sum_{n=0}^{\infty} z^n = \sum_{n=1}^{\infty} nz^{n-1}$. In particular, power series are infinitely differentiable, hence define analytic functions in their disk of convergence.
- Multiplying power series: See Theorem 4, Examples 8 and 9.

Section 2.3 Cauchy's Theorem and Cauchy's Formula. Up to and including Example 7. Skip Theorems 2,3 and Examples 8,9,10.

- A domain D is *simply connected* if the inside of any simple closed curve in D is contained in D. (D has "no holes".)
- Roughly, when f is analytic and γ is not encircling any holes:

$$\int_\gamma f(z)dz = 0 \qquad (\textit{Cauchy's Theorem})$$

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}dz \qquad (\textit{Cauchy's Formula})$$

  These are special cases of the Residue Theorem from Section 2.6.
- To solve trig integrals of the type $\int_0^{2\pi} d\theta/(2 + \sin\theta)$, substitute $z = e^{i\theta}$ and use Cauchy's Formula (or the Residue Theorem). See Examples 6 and 7.

Section 2.4 Consequences of Cauchy's Theorem. Up to and including The Order of a Zero. Skip the rest of Section 2.4, starting with Morera's Theorem.

- If f is analytic at $z = z_0$ then $f(z)$ has a power series expansion at z_0:
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \text{where } a_n = \frac{1}{n!} f^{(n)}(z_0).$$
- If f is analytic in a domain D and all its derivatives vanish at some point z_0 then actually f has to be identically zero everywhere in D.
- If $f(z_0) = 0$, $f'(z_0) = 0$, ..., $f^{(m-1)}(z_0) = 0$ but $f^{(m)}(z_0) \neq 0$ then f *has a zero of order* m *at* z_0. This is the same thing as saying:
$$f(z) = a_m (z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \cdots \qquad \text{and } a_m \neq 0.$$

Section 2.5 Isolated Singularities. Up to and including Example 9. Also, skip Examples 4 and 7.

- There are three kinds of singularities: *removable, poles,* and *essential singularities.*
- $f(z)$ has a pole at z_0 of order m precisely when $1/f(z)$ has a zero at z_0 of order m. In this case $f(z)$ has a *Laurent series*:
$$f(z) = a_{-m}(z-z_0)^{-m} + a_{-m+1}(z-z_0)^{-m+1} + \cdots + a_{-1}(z-z_0)^{-1} + a_0 + a_1(z-z_0) + \cdots \qquad \text{and } a_{-m} \neq 0.$$
and the *residue of* f *at* z_0 can be computed in three ways:
$$\operatorname{Res}(f; z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=\varepsilon} f(z)dz = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \big((z - z_0)^m f(z)\big)\Big|_{z=z_0} = a_{-1}$$
In particular, if $m = 1$ we say f has a *simple pole at* z_0 and we have
$$\operatorname{Res}(f; z_0) = \big((z - z_0) f(z)\big)\Big|_{z=z_0}$$
meaning, simplify $(z - z_0)f(z)$ and plug in $z = z_0$.

Section 2.6 The Residue Theorem and Its Applications. Up to and including Example 7. But skip Example 6. Skip everything after Example 7.

- The *Residue Theorem*: If f is analytic in a simply connected domain except for isolated singularities $z_1, z_2, \ldots, z_n$, and γ is a positively oriented closed curve, then
$$\int_\gamma f(z)dz = 2\pi i \sum \operatorname{Res}(f; z_j)$$
where the sum is over all z_j that lie inside γ.
- If $P(x)$ and $Q(x)$ are real polynomials and the $\deg Q(x) \geq 2 + \deg P(x)$ and $Q(x)$ has no real zeroes, then
$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \sum \operatorname{Res}(\frac{P}{Q}; z_j)$$
$$\int_{-\infty}^{\infty} \frac{P(x)\cos x}{Q(x)} dx = \operatorname{Re}\left\{2\pi i \sum \operatorname{Res}(\frac{P(z)e^{iz}}{Q(z)}; z_j)\right\}$$
$$\int_{-\infty}^{\infty} \frac{P(x)\sin x}{Q(x)} dx = \operatorname{Im}\left\{2\pi i \sum \operatorname{Res}(\frac{P(z)e^{iz}}{Q(z)}; z_j)\right\}$$
where the sums are over all zeroes z_j of Q that lie in the upper half plane.