Math 365 – Sample Problems for Exam 2

Section 2.1 (Analytic and Harmonic Functions; the Chauchy-Riemann Equations).

1. Let $g(z) = \big(\cos(z^2)\big)^3$. Find $g'(0)$ and $g''(0)$.
2. Let $f = u + iv$ be an analytic function such that $u = x^2 - y^2$ and $f(0) = 0$. Find v.
3. Let $f = u + iv$ be an analytic function such that $u = (e^x + e^{-x})\cos(y)$ and $f(0) = 2$. Find v.

Section 2.2 (Power Series).

4. Find the power series at the origin for $f(z) = e^{-z^2}$.
5. Find the radius of convergence of the power series: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}}(z-1)^n$
6. Find the power series expansion of $f(z) = \frac{1}{z-2}$ at $z_0 = 1$ and determine its radius of convergence.

Section 2.3 (Cauchy's Theorem and Cauchy's Formula).

7. Let γ be the positively oriented square with corners $1+i$, $2+i$, $2+2i$, $1+2i$. Find:
$$\int_\gamma \frac{e^{\cos(z)} + \mathrm{Log}(z)}{z^6}\,dz$$
8. Find:
$$\int_{|z|=4} \frac{e^z + \cos(z)}{z + i\pi}\,dz$$
Would there be any difference if the curve was given by $|z| = 3$ instead?
9. Find: $\displaystyle\int_0^{2\pi} \frac{d\theta}{3 + \sin\theta + \cos\theta}$.

Section 2.4 (Consequences of Cauchy's Formula).

10. Find the order of the zero at $z = 2$ of $f(z) = \frac{(z-2)^4 \mathrm{Log}(z-1)}{\sin(2-z)}$.
11. Suppose that $f(z)$ is an entire function such that all the even order derivatives vanish at the origin, that is: $f^{(2k)}(0) = 0$ for all non-negative integers k. Show that $f(-z) = -f(z)$.

Section 2.5 (Isolated Singularities).

12. Find the order of every pole of the function $f(z) = \dfrac{\cos(z) - 1}{z^2 + z}$. What is $\mathrm{Res}(f; -1)$?
13. Find the residue at $z_0 = -2$ of $h(z) = \dfrac{z+1}{z^2 + 4z + 4}$.
14. Find the Laurent series expansion of $g(z) = \dfrac{\cos(z)}{z^5}$ at $z_0 = 0$. What is $\mathrm{Res}(g; 0)$?
15. Find three terms of the Laurent series expansion of $f(z) = \dfrac{1}{e^z - 1}$ at $z_0 = 0$.

Section 2.6 (Residue Theorem).

16. Compute using residues:
$$\text{(a)} \int_{-\infty}^{\infty} \frac{dx}{x^2 + 4x + 5} \qquad \text{(b)} \int_0^{\infty} \frac{x^2}{x^4 + 2x^2 + 1}\,dx$$
17. Compute using residues:
$$\int_0^{\infty} \frac{x\sin(x)}{(x^2+1)(x^2+3)}\,dx$$