Math 365 – Practice Problems for Final Exam

1. Let $z = ie^{i\pi/5}$ and $w = -2+2i$. Find the polar representation of z/w.
2. Find all complex numbers z such that $z^6 = 2z^3 - 2$.
3. For which values of the real number $r \geq 0$ is the set

$$D_r = \{z : |z| < 2 \text{ and } |z - r| \geq r\}$$

(a) convex? (b) open? (c) simply connected?

4. Consider the power series

$$f(z) = \sum_{n=0}^{\infty} e^{2\pi i n/3} \frac{z^{3n}}{(3n)!}$$

   (a) Find the radius of convergence of $f(z)$.
   (b) Show that $f'''(z) = e^{2\pi i/3} f(z)$.
5. Find the value w of $\log(1+i\sqrt{3})$ satisfying $\pi < \operatorname{Im} w \leq 3\pi$.
6. Show that if ξ is any value of

$$\frac{i}{2}\log\left(\frac{1-iw}{1+iw}\right)$$

   then $\tan\xi = w$. (It might be helpful to put $z = \frac{1-iw}{1+iw}$ for part of this calculation.)
7. Compute the line integral

$$\int_\gamma \left(z + \frac{1}{1+z}\right) dz$$

   where γ is the line segment from 0 to $1+i$.
8. Let $f = u + iv$ be an analytic function with $u = e^x(x\cos y - y\sin y)$, $f(0) = 0$. Find v.
9. Find the power series at the origin for $f'(z)$ if $f(z) = z^2\cos(z)$.
10. Find the power series expansion of $f(z) = \dfrac{1-2iz}{1+2iz}$ at $z = 0$ and determine its radius of convergence.
11. Compute

$$\int_{|z|=2} \frac{z^2}{e^z(z+3i)^2(z+1)} dz$$

12. Suppose that $f(z)$ has an isolated singularity at the origin and satisfies the equation $f(z)f(-z) = 1$ when $z \neq 0$. Show that $f(z)$ cannot have a pole at $z = 0$.
13. Compute $\operatorname{Res}(f;1)$ where $f(z) = \dfrac{\sin(z-1)}{(z-1)^3}$.
14. Find the Laurent series expansion of $g(z) = \dfrac{e^{3z}+e^{-3z}}{z^3}$ at $z = 0$. What is $\operatorname{Res}(g;0)$?
15. Compute:

$$\text{(a)} \int_{-\infty}^{\infty} \frac{dx}{x^2+6x+10} \qquad \text{(b)} \int_{-\infty}^{\infty} \frac{\cos(x)}{(x+1)^2+4} dx \qquad \text{(c)} \frac{1}{2\pi i}\int_{|z|=4} \cot(z) dz.$$

16. Find a fractional linear transformation $f(z) = \dfrac{az+b}{cz+d}$ such that $f(0) = 1$, $f(1) = \infty$ and $f(\infty) = 0$.
17. Let a, b, c, d be real numbers with $ad - bc > 0$. Show that in this case, the fractional linear transformation $f(z) = \dfrac{az+b}{cz+d}$ maps the upper half plane $U = \{z : \operatorname{Im} z > 0\}$ to itself.

18. Let V be the vertical strip
$$V = \{z : |\operatorname{Re} z| < 1\}$$
and let D be the complex plane with the negative imaginary axis deleted:
$$D = \{z : \text{if } \operatorname{Re} z = 0 \text{ then } \operatorname{Im} z > 0\}.$$
    (a) Find a conformal mapping which maps V onto D.
    (b) Use part (a) to write down the equations for the streamlines for the flow of an ideal fluid in the region D.
19. The *Euler-Mascheroni* constant γ can be defined as $\gamma = -\Gamma'(1)$.
    (a) Show that $\Gamma'(2) = 1 - \gamma$ (Hint: differentiate the relation $\Gamma(z+1) = z\Gamma(z)$.)
    (b) Show that $\Gamma'(3) = 3 - 2\gamma$.
    (c) Find a closed formula for $\Gamma'(n)$, in terms of γ, valid for any positive integer n.
20. Show that
$$\lim_{n\to\infty} \int_{|z-1/2|=n} \Gamma(z)dz = 2\pi i e^{-1}$$
Hint: Use that $\operatorname{Res}\big(\Gamma(z); -n\big) = (-1)^n/n!$ for $n \geq 0$.