1. Let $z = ie^{i\pi/5}$ and $w = -2+2i$. Find the polar representation of z/w.

$$-2+2i = 2\sqrt{2}\, e^{i\frac{3}{4}\pi}$$

$$\frac{z}{w} = \frac{ie^{i\frac{\pi}{5}}}{2\sqrt{2}\, e^{i\frac{3}{4}\pi}} = \frac{\sqrt{2}}{4} i\, e^{i\left(\frac{1}{5}-\frac{3}{4}\right)\pi} = \frac{\sqrt{2}}{4} i\, e^{-\frac{11}{20}\pi i}$$

even

odd

$$= \frac{\sqrt{2}}{4} i \left(\cos\left(-\frac{11}{20}\pi\right) + i\sin\left(-\frac{11}{20}\pi\right)\right)$$

$$= \boxed{\frac{\sqrt{2}}{4} i\cos\frac{11}{20}\pi + \frac{\sqrt{2}}{4}\sin\frac{11}{20}\pi}$$

2. Find all complex numbers z such that $z^6 = 2z^3 - 2$.

$$z^6 - 2z^3 + 2 = 0$$

$$z^3 = 1 \pm i = \sqrt{2}\left[\cos\frac{\pi}{4} \pm i\sin\frac{\pi}{4}\right] \qquad \left\{\begin{array}{l} 1+i \Rightarrow \frac{\pi}{4} \\ 1-i \Rightarrow -\frac{\pi}{4} \end{array}\right.$$

$$\| $$

$$|z|^3 \left[\cos 3\theta + i\sin 3\theta\right]$$

$$3\theta = \pm\frac{\pi}{4} + 2k\pi \qquad \therefore \theta = \pm\frac{\pi}{12} + \frac{2}{3}k\pi \quad \& \quad |z| = 2^{1/6}$$

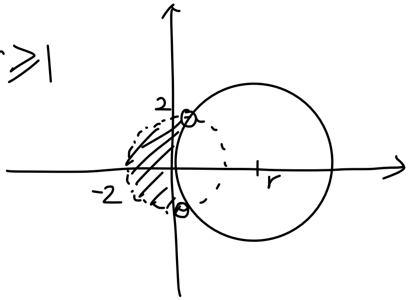
$$\boxed{z = 2^{1/6}(\cos\theta + i\sin\theta) \quad \text{where} \quad \begin{array}{ll} k=0; & \theta = \pm\frac{\pi}{12} \\ k=1; & \theta = \frac{3}{4}\pi, \frac{7}{12}\pi \\ k=2; & \theta = \frac{5}{4}\pi, \frac{17}{12}\pi \end{array}}$$

3. For which values of the real number $r \geq 0$ is the set

$$D_r = \{z : |z| < 2 \text{ and } |z - r| \geq r\}$$

(a) convex? (b) open? (c) simply connected?

$r \geq 1$



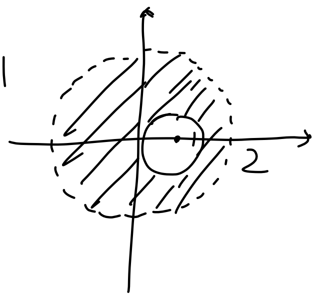
not convex

not open

simply connected

(+2pts)

$0 < r < 1$



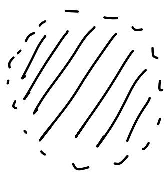
not convex

not open

not simply connected

(+2pts)

$r = 0$;



convex

open

simply connected

(+1 pt)

A problem to think about:

Is $D_\infty = \bigcap_{r=0}^{\infty} D_r$ convex? open? simply connected?

4. Consider the power series

$$f(z) = \sum_{n=0}^{\infty} e^{2\pi i n/3} \frac{z^{3n}}{(3n)!}$$

(a) Find the radius of convergence of $f(z)$. +2 pts
(b) Show that $f'''(z) = e^{2\pi i/3} f(z)$. +3 pts

(a) For any $z \in \mathbb{C}$,

$$\left| \frac{e^{\frac{2}{3}(n+1)\pi i} \frac{z^{3n+3}}{(3n+3)!}}{e^{\frac{2}{3}n\pi i} \frac{z^{3n}}{(3n)!}} \right| = \frac{|z|^3}{(3n+3)(3n+2)(3n+1)} \to 0.$$

$\therefore R = \infty$

(b) $$f'(z) = \sum_{n=1}^{\infty} e^{\frac{2}{3}\pi i n} 3n \frac{z^{3n-1}}{(3n)!} = \sum_{n=1}^{\infty} e^{\frac{2}{3}\pi i n} \frac{z^{3n-1}}{(3n-1)!}$$

$$f''(z) = \sum_{n=1}^{\infty} e^{\frac{2}{3}\pi i n} \frac{z^{3n-2}}{(3n-2)!}$$

$$f'''(z) = \sum_{n=1}^{\infty} e^{\frac{2}{3}\pi i n} \frac{z^{3(n-1)}}{(3(n-1))!} = \sum_{k=0}^{\infty} e^{\frac{2}{3}\pi i (k+1)} \frac{z^{3k}}{(3k)!}$$

$\boxed{n-1=k}$

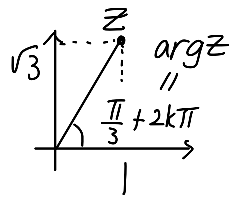
$$= e^{\frac{2}{3}\pi i} \sum_{k=0}^{\infty} e^{\frac{2}{3}\pi i k} \frac{z^{3k}}{(3k)!} = \boxed{e^{\frac{2}{3}\pi i} f(z)}$$

5. Find the value w of $\log(1+i\sqrt{3})$ satisfying $\pi < \operatorname{Im} w \le 3\pi$.

$$w = \log(\overbrace{1+i\sqrt{3}}^{z})$$

$$(*)\ z = 1+i\sqrt{3} = 2e^{\left(\frac{\pi}{3}+2k\pi\right)i}$$

$$= e^{\ln 2 + i\left(\frac{\pi}{3}+2k\pi\right)}$$



$$\Leftrightarrow\ z = 1+i\sqrt{3} = e^{w}$$

$$w = \log z = \ln|z| + i\arg z$$

$$= \ln 2 + i\underbrace{\left(\frac{\pi}{3}+2k\pi\right)}$$

$$\pi < \operatorname{Im} w = \arg z \le 3\pi$$

$$\Rightarrow\ k=1\ \&\ \frac{\pi}{3}+2\pi = \frac{7}{3}\pi$$

$$\therefore\ w = \boxed{\ln 2 + i\,\frac{7}{3}\pi}$$

6. Show that if ξ is any value of

$$\frac{i}{2}\log\left(\frac{1-iw}{1+iw}\right)$$

then $\tan\xi = w$. (It might be helpful to put $z = \frac{1-iw}{1+iw}$ for part of this calculation.)

Set $z = \frac{1-iw}{1+iw}$. Then $z + izw = 1 - iw$

$$1 - z = izw + iw = iw(z+1)$$

$$\therefore\ w = -i\frac{1-z}{1+z} = i\frac{z-1}{z+1} \quad (*)$$

$$\boxed{\tan\xi} = i\,\frac{e^{-i\xi} - e^{i\xi}}{e^{i\xi} + e^{-i\xi}} \quad \text{by definition}$$

$$= i\,\frac{e^{-2i\xi} - 1}{e^{-2i\xi} + 1}$$

$$\xi = \frac{i}{2}\log z$$

$$-2\xi i = \log z$$

$$\Longleftrightarrow\ z = e^{-2\xi i}$$

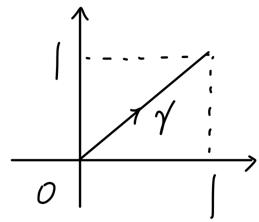
$$= i\,\frac{z-1}{z+1}$$

$$= \boxed{w} \text{ by } (*)$$

7. Compute the line integral

$$\int_\gamma \left(z+\frac{1}{1+z}\right)dz$$

where γ is the line segment from 0 to $1+i$.



$\gamma: z = t + ti, \ 0 \le t \le 1$

$dz = (1+i)\, dt$

$$\int_\gamma z + \frac{1}{1+z}\, dz$$

$$= \int_0^1 \left[t + ti + \frac{1}{1+t+ti} \right] (1+i)\, dt$$

$$= \int_0^1 2it + \underbrace{\frac{1+i}{(1+i)t+1}}_{= \frac{1}{t+\frac{1-i}{2}}} \, dt$$

use $\displaystyle\int \frac{1}{t+\alpha}\, dt = \ln|t+\alpha| + C$

$$= i + \left[\ln \left| t + \frac{1-i}{2} \right| \right]_0^1$$

$$= i + \ln\left|\frac{3-i}{2}\right| - \ln\left|\frac{1-i}{2}\right|$$

$$= i + \ln\left|\frac{3-i}{1-i}\right|$$

$$= i + \ln\sqrt{5} = \boxed{i + \frac{1}{2}\ln 5}$$

8. Let $f = u + iv$ be an analytic function with $u = e^x(x\cos y - y\sin y)$, $f(0) = 0$. Find v.

$u_x = v_y$ & $u_y = -v_x$

$$u_x = e^x(x\cos y - y\sin y) + e^x \cos y = v_y$$
$$u_y = e^x(-x\sin y - \sin y - y\cos y) = -v_x$$

$$\therefore\ v_x = xe^x \sin y + e^x \sin y + ye^x \cos y$$
$$v_y = xe^x \cos y - ye^x \sin y + e^x \cos y$$

By (*),

$$v = \cancel{e^x \sin y} + ye^x \cos y + x\sin y\, e^x - \cancel{e^x \sin y} + \phi(y)$$
$$= xe^x \sin y + \cancel{e^x \sin y} + e^x y\cos y - \cancel{e^x \sin y} + \psi(x)$$
$$= ye^x \cos y + xe^x \sin y + C$$

Since $v(0,0) = C = 0$, $\boxed{\therefore\ v = ye^x \cos y + xe^x \sin y}$.

(*)
$$\int \underbrace{x}_{u}\,\underbrace{e^x\,dx}_{dv} = xe^x - e^x + C$$
$$\int \underbrace{y}_{u}\,\underbrace{\sin y\,dy}_{dv} = -y\cos y + \int \cos y\,dy = -y\cos y + \sin y + C$$

9. Find the power series at the origin for $f'(z)$ if $f(z) = z^2\cos(z)$.

$$f(z) = z^2 \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+2}}{(2n)!}$$

$$= z^2 - \frac{z^4}{2!} + \frac{z^6}{4!} - \cdots$$

$$f'(z) = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n)!} (2n+2)}$$

10. Find the power series expansion of $f(z) = \dfrac{1-2iz}{1+2iz}$ at $z=0$ and determine its radius of convergence.

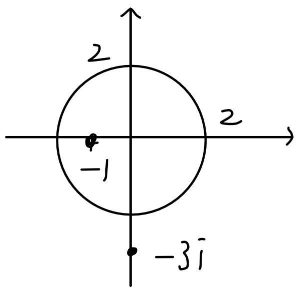
$$\frac{1-2iz}{1+2iz} = \frac{-(2iz+1)+2}{1+2iz} = -1 + 2\,\frac{1}{1+2iz}$$

$$= -1 + 2\,\frac{1}{1-(-2iz)} = \boxed{-1 + 2\sum_{n=0}^{\infty} (-2i)^n z^n}$$

$$\left| \frac{(-2i)^{n+1} z^{n+1}}{(-2i)^n z^n} \right| = |-2i|\,|z| < 1$$

$$\therefore\ |z| < \boxed{\frac{1}{2} = R}$$

11. Compute



$$\int_{|z|=2} \underbrace{\frac{z^2}{e^z(z+3i)^2(z+1)}}_{f(z)} dz$$

$$2\pi i \, \text{Res}(f; -1) = 2\pi i \frac{1}{e^{-1}(-1+3i)^2}$$

$$= 2\pi i \frac{e}{-8-6i}$$

$$= \pi i \frac{e}{25}(-4+3i)$$

$$= \boxed{\frac{e\pi}{25}(-3-4i)}$$

12. Suppose that $f(z)$ has an isolated singularity at the origin and satisfies the equation $f(z)f(-z) = 1$ when $z \neq 0$. Show that $f(z)$ cannot have a pole at $z = 0$.

Suppose that $f(z)$ has a pole at $z=0$.

+1 point

Then $f(z) = \dfrac{g(z)}{z}$ for some g analytic near 0 and $g(0) \neq 0$

Hence $f(-z) = \dfrac{1}{f(z)} = \dfrac{z}{g(z)}$, and by plugging in $-z$,

we have $f(z) = \dfrac{-z}{g(-z)}$.

Since $g(0) \neq 0$,

$f(z) = \dfrac{-z}{g(-z)}$ does not have a pole at $z=0$,

which contradicts the assumption. $\divideontimes$

13. Compute $\text{Res}(f;1)$ where $f(z) = \dfrac{\sin(z-1)}{(z-1)^3}$.

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$f(z) = \frac{1}{(z-1)^3}\left[(z-1) - \frac{(z-1)^3}{3!} + \frac{(z-1)^5}{5!} - \cdots\right]$$

$$= \frac{1}{(z-1)^2} - \frac{1}{3!} + \frac{(z-1)^2}{5!} - \cdots$$

$$\therefore \text{Res}(f;1) = a_{-1} = \boxed{0}$$

14. Find the Laurent series expansion of $g(z) = \dfrac{e^{3z} + e^{-3z}}{z^3}$ at $z = 0$. What is $\mathrm{Res}(g; 0)$?

$$g(z) = \frac{1}{z^3}\left[\sum \frac{3^n}{n!} z^n + \sum \frac{(-3)^n}{n!} z^n\right]$$

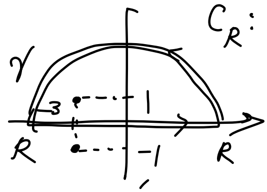
$$= \sum_{n=0}^{\infty} \frac{3^n + (-3)^n}{n!} z^{n-3}$$

$$= \frac{2}{z^3} + \frac{9}{z} + \sum_{n=4}^{\infty} \frac{3^n + (-3)^n}{n!} z^{n-3}$$

$$\mathrm{Res}(g; 0) = \boxed{9}$$

15. Compute:

(a) $\displaystyle\int_{-\infty}^{\infty} \underbrace{\frac{dx}{x^2+6x+10}}_{f(z)}$ (b) $\displaystyle\underbrace{\int_{-\infty}^{\infty} \frac{\cos(x)}{(x+1)^2+4}dx}_{I}$ (c) $\displaystyle\frac{1}{2\pi i}\int_{|z|=4} \cot(z)dz.$



C_R: $|z|=R$, $0 \leq t \leq \pi$

$z = Re^{it}$

$dz = iRe^{it}dt$

(a) $z^2+6z+10=0 \iff z=-3\pm i$

$$\underbrace{\int_{-R}^{R} f(x)\,dx}_{\to \int_{-\infty}^{\infty} f(x)\,dx} + \underbrace{\int_{C_R} f(z)\,dz}_{\to 0\ (*)} = 2\pi i\, \underbrace{\operatorname{Res}(f;-3+i)}_{\lim_{z\to -3+i}\frac{1}{z+3+i} = \frac{1}{2i}}$$

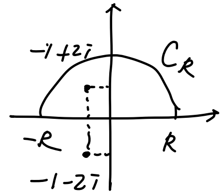
$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \cdot \frac{1}{2i} = \boxed{\pi}$$

(*) $\displaystyle\left|\int_{C_R} f(z)\,dz\right| = \left|\int_0^{\pi} f(Re^{it})\, iRe^{it}\,dt\right|$

$\displaystyle\leq \int_0^{\pi} |f(Re^{it})|\, R\,dt \quad \leftarrow \quad |f(Re^{it})| = \frac{1}{|R^2e^{2it}+6Re^{it}+10|} \leq \frac{2}{R^2}$ for large R

$\displaystyle\leq \pi\frac{2}{R} \longrightarrow 0$ as $R \to \infty$

(b) $f(z) = \dfrac{e^{iz}}{(z+1)^2+4}$ has simple poles at $z = -1 \pm 2i$.



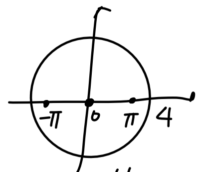
$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \operatorname{Res}(f; -1+2i)$$

$$= 2\pi i \cdot \frac{e^{-2-i}}{4i} = \frac{\pi}{2e^2}(\cos 1 - i \sin 1)$$

$$\underbrace{\int_{-\infty}^{\infty} \frac{\cos x}{(x+1)^2+4}\,dx}_{I} + i\int_{-\infty}^{\infty} \frac{\sin x}{(x+1)^2+4}\,dx$$

$$\therefore\ I = \boxed{\frac{\pi}{2e^2}\cos 1}$$

(c) $\cot z = i\,\dfrac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}} = i\,\dfrac{e^{2iz}+1}{\underbrace{e^{2iz}-1}}$



$e^{2iz} = 1 = e^{2k\pi i}$

$\therefore z = k\pi,\ k \in \mathbb{Z}$

three poles when $k = -1, 0, 1$

$$\frac{1}{2\pi i}\int_{|z|=4} \overbrace{\cot z}^{f(z)}\,dz = \operatorname{Res}(f; -\pi) + \operatorname{Res}(f; 0) + \operatorname{Res}(f; \pi) = \boxed{3}$$

<u>Note</u>: $\dfrac{F}{g'} = i\,\dfrac{e^{2iz}+1}{2ie^{2iz}} = \dfrac{1+e^{2iz}}{2e^{2iz}} = 1$ for $z = -\pi, 0, \pi$

* alternative solution : use the counting formula

Let $h(z) = e^{iz} - e^{-iz}$. Then $\cot z = \dfrac{h'(z)}{h(z)}$.

$$\therefore \frac{1}{2\pi i}\int_{|z|=4} \frac{h'(z)}{h(z)}\,dz = \#\text{zeros} - \#\text{poles} = 3 - 0 = 3$$

16. Find a fractional linear transformation $f(z) = \dfrac{az+b}{cz+d}$ such that $f(0) = 1$, $f(1) = \infty$ and $f(\infty) = 0$.

$$\frac{b}{d} = 1, \qquad \frac{a+b}{c+d} = \infty, \qquad \frac{\infty \cdot a + b}{\infty \cdot c + d} = 0$$

$$b = d \qquad c = -d \qquad a = 0$$

$$f(z) = \frac{d}{-dz+d} = \boxed{\frac{1}{1-z}}$$

17. Let a, b, c, d be real numbers with $ad - bc > 0$. Show that in this case, the fractional linear transformation $f(z) = \dfrac{az+b}{cz+d}$ maps the upper half plane $U = \{z : \operatorname{Im} z > 0\}$ to itself.

For any $z \in U$, $z = x + iy$ for some $x \in \mathbb{R}$ and $y > 0$.

Then $$f(z) = \frac{(ax+b) + iay}{(cx+d) + icy} = \frac{[(ax+b)+iay][(cx+d)-icy]}{(cx+d)^2 + c^2y^2}$$

$$= \frac{(ax+b)(cx+d) + acy^2}{(cx+d)^2 + c^2y^2} + i\,\frac{-cy(ax+b) + ay(cx+d)}{(cx+d)^2 + c^2y^2}$$

Take the imaginary part,

$$\operatorname{Im} f(z) = \frac{\cancel{-acxy} - bcy + \cancel{acxy} + ady}{(cx+d)^2 + c^2y^2} = \frac{\overbrace{(ad-bc)}^{+}\ \overbrace{y}^{+}}{\underbrace{(cx+d)^2 + c^2y^2}_{+}} > 0.$$

18. Let V be the vertical strip

$$V = \{z : |\mathrm{Re}\, z| < 1\}$$

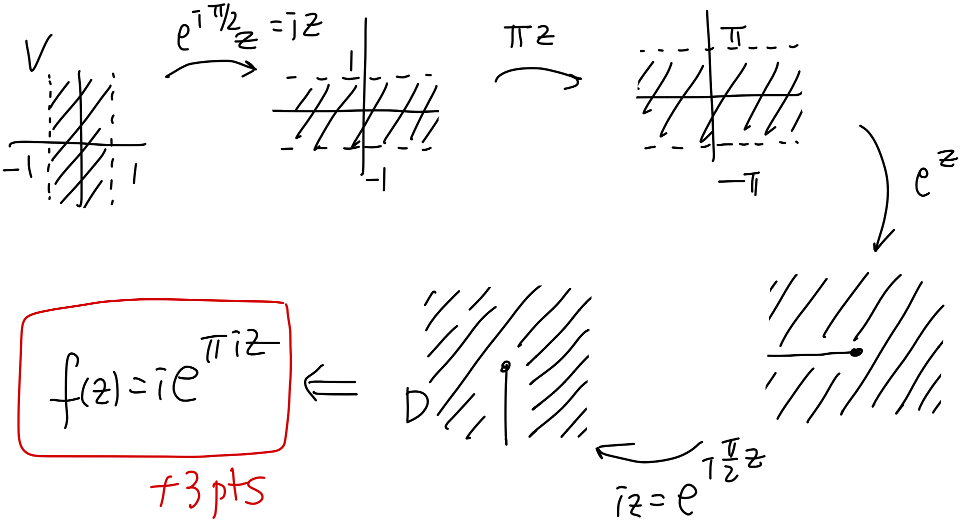
and let D be the complex plane with the negative imaginary axis deleted:

$$D = \{z : \text{if } \mathrm{Re}\, z = 0 \text{ then } \mathrm{Im}\, z > 0\}.$$

(a) Find a conformal mapping which maps V onto D.

(b) Use part (a) to write down the equations for the streamlines for the flow of an ideal fluid in the region D.

(a)



$$\boxed{f(z) = ie^{\pi i z}} \Leftarrow$$

+3pts

(b) The stream line in V is the level curve $x = c$, $-1<c<1$.

In other words, $\Gamma_c = \{c+iy \,;\, y \in \mathbb{R}\}$.

Since $f(c+iy) = i\,e^{\pi i(c+iy)} = ie^{(-y+ic)\pi}$,

the stream line in D is

$$f(\Gamma_c) = \boxed{\{ie^{(-y+ic)\pi} : y \in \mathbb{R}\}}, \quad -1<c<1.$$

+2pts

19. The *Euler-Mascheroni* constant γ can be defined as $\gamma = -\Gamma'(1)$.

(a) Show that $\Gamma'(2) = 1 - \gamma$ (Hint: differentiate the relation $\Gamma(z+1) = z\Gamma(z)$.)

(b) Show that $\Gamma'(3) = 3 - 2\gamma$.

(c) Find a closed formula for $\Gamma'(n)$, in terms of γ, valid for any positive integer n.

(a) Since $\Gamma'(z+1) = \Gamma(z) + z\Gamma'(z)$ and $\Gamma(2) = \Gamma(1) = 1$,

$$\Gamma'(2) = \Gamma(1) + \Gamma'(1) = \boxed{1-\gamma}$$ +1pt

(b) $\Gamma'(3) = \Gamma(2) + 2\Gamma'(2) = 1 + 2(1-\gamma) = \boxed{3-2\gamma}$ +1 pt

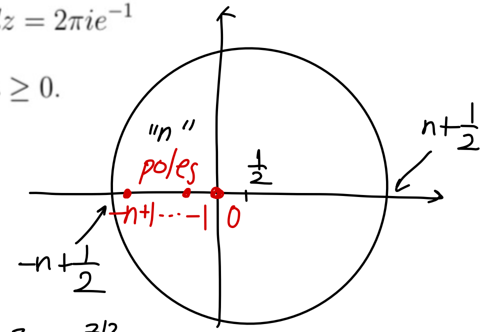
(c)
$$\begin{aligned}
\Gamma'(n+1) &= \Gamma(n) + n\Gamma'(n) \\
&= \Gamma(n) + n\Gamma(n-1) + n(n-1)\Gamma'(n-1) \\
&= \cdots \\
&= \Gamma(n) + n\Gamma(n-1) + n(n-1)\Gamma(n-2) + \cdots \\
&\qquad \cdots + n(n-1)\cdots 2\,\Gamma(1) + n(n-1)\cdots 2\cdot 1\,\Gamma'(1) \\
&= \sum_{k=0}^{n-1} \frac{n!}{(n-k)!}\Gamma(n-k) + n!\,\Gamma'(1) \\
&= \sum_{k=0}^{n-1} \frac{n!}{(n-k)!}(n-k-1)! + n!(-\gamma) \\
&= \boxed{n!\left[\sum_{k=0}^{n-1} \frac{1}{n-k} - \gamma\right]}
\end{aligned}$$

+3pts

20. Show that

$$\lim_{n\to\infty} \int_{|z-1/2|=n} \Gamma(z)dz = 2\pi i e^{-1}$$

Hint: Use that $\mathrm{Res}\big(\Gamma(z); -n\big) = (-1)^n/n!$ for $n \geq 0$.



Since

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^{-1} e^{\frac{z}{n}} = \frac{e^{-\gamma z}}{z} \frac{e^{z}}{1+z} \frac{e^{z/2}}{1+z/2} \cdots,$$

Γ has simple poles at $z = 0, -1, -2, \cdots$

and $z = 0, -1, \cdots, -n+1$ are inside $|z-\frac{1}{2}| = n$.

$$\therefore \int_{|z-\frac{1}{2}|=n} \Gamma(z)\, dz = 2\pi i \sum_{k=0}^{n-1} \mathrm{Res}(\Gamma; -k)$$

$$= 2\pi i \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \longrightarrow \boxed{2\pi i e^{-1}} \text{ as } n \to \infty$$