Exercise 2.5.1

Locate each of the isolated singularities of

$$f(z) = \frac{e^z - 1}{z}$$

and tell whether it is a removable singularity, a pole or an essential singularity. If it is a pole, give the order of the pole.

Theorem 1. *If f has an isolated singularity at z_0, then $z = z_0$ is a removable singularity if and only if*

$$\lim_{z \to z_0} (z - z_0) f(z) = 0.$$

Solution. f is analytic except at $z = 0$, hence it is the only isolated singularity of f. Since

$$\lim_{z \to 0} z f(z) = \lim_{z \to 0} (e^z - 1) = 0,$$

by Theorem 1, $z = 0$ is a removable singularity.

Exercise 2.5.2

Locate each of the isolated singularities of

$$f(z) = \frac{z^2}{\sin z}$$

and tell whether it is a removable singularity, a pole or an essential singularity. If it is a pole, give the order of the pole.

Solution. f has isolated singularities at $z = z_0$ whenever $\sin z = 0$, that is, $z_0 = k\pi$ for any $k \in \mathbb{Z}$. If $z_0 = 0$, then the limit

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{z^2}{\sin z} = \lim_{z \to 0} \frac{z^2}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots} = \lim_{z \to 0} \frac{z}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots} = 0$$

exists, showing that it is a removable singularity. Now suppose that $k \in \mathbb{Z} \setminus \{0\}$. Then the series expansion of $\sin z$ about $z_0 = k\pi$ is

$$\begin{aligned}
\sin z &= (-1)^k (z - k\pi) - \frac{(-1)^k}{3!}(z - k\pi)^3 + \frac{(-1)^k}{5!}(z - k\pi)^5 - \cdots \\
&= (z - k\pi)\left\{(-1)^k - \frac{(-1)^k}{3!}(z - k\pi)^2 + \frac{(-1)^k}{5!}(z - k\pi)^4 - \cdots\right\}
\end{aligned}$$

for any $z \in \mathbb{C}$. Define the function

$$g(z) = (-1)^k - \frac{(-1)^k}{3!}(z - k\pi)^2 + \frac{(-1)^k}{5!}(z - k\pi)^4 - \cdots .$$

Then g is entire with $g(k\pi) = (-1)^k \neq 0$. This leads to

$$f(z) = \frac{z^2}{(z - k\pi) g(z)} = \frac{z^2 / g(z)}{z - k\pi}$$

where $H(z) = z^2/g(z)$ is entire and $H(k\pi) = (-1)^k (k\pi)^2 \neq 0$. Thus, f has a pole of order 1 at $z = k\pi$, $k \neq 0$.

Exercise 2.5.3

Locate each of the isolated singularities of

$$f(z) = \frac{z^4 - 2z^2 + 1}{(z-1)^2}$$

and tell whether it is a removable singularity, a pole or an essential singularity. If it is a pole, give the order of the pole.

Solution. Since

$$f(z) = \frac{z^4 - 2z^2 + 1}{(z-1)^2} = \frac{(z-1)^2(z+1)^2}{(z-1)^2},$$

we see that f has the isolated singularity at $z_0 = 1$. The limit

$$\lim_{z \to 1} f(z) = \lim_{z \to 1} (z+1)^2 = 4$$

exists, hence $z_0 = 1$ is a removable singularity.

Exercise 2.5.5

Locate each of the isolated singularities of

$$f(z) = \frac{2z+1}{z+2}$$

and tell whether it is a removable singularity, a pole or an essential singularity. If it is a pole, give the order of the pole.

Solution. Note that

$$f(z) = 2 - \frac{3}{z+2}.$$

The only isolated singularity occurs at $z_0 = -2$, and is a pole of order 1.

Exercise 2.5.6

Locate each of the isolated singularities of

$$f(z) = \frac{e^z - 1}{e^{2z} - 1}$$

and tell whether it is a removable singularity, a pole or an essential singularity. If it is a pole, give the order of the pole.

Solution. The function

$$f(z) = \frac{e^z - 1}{(e^z - 1)(e^z + 1)}$$

has an isolated singularity at the points where $e^z = \pm 1$, that is, $z_0 = k\pi i$ for $k \in \mathbb{Z}$. If $k = 0$ or k is even, then

$$\lim_{z \to k\pi i} f(z) = \lim_{z \to k\pi i} \frac{1}{e^z + 1} = \frac{1}{2}.$$

Hence $z_0 = k\pi i$ are removable singularities for $k = 0, \pm 2, \pm 4, \cdots$. Now suppose that k is an odd integer. For any $z \in \mathbb{C}$, the series expansion of $e^z + 1$ about $z_0 = k\pi i$ is

$$\begin{aligned} e^z + 1 &= e^{z - k\pi i} e^{k\pi i} + 1 \\ &= -e^{z - k\pi i} + 1 \\ &= -\sum_{n=1}^{\infty} \frac{1}{n!} (z - k\pi i)^n \\ &= (z - k\pi i)\, g(z) \end{aligned}$$

where the function

$$g(z) = -1 - \frac{z - k\pi i}{2!} - \frac{(z - k\pi i)^2}{3!} - \cdots$$

is entire and $g(k\pi i) = -1 \neq 0$. This gives

$$f(z) = \frac{1/g(z)}{z - k\pi i}$$

which shows that $z_0 = k\pi i$ is a pole of order 1 if k is an odd integer.

Exercise 2.5.7

Find the Laurent series for

$$f(z) = \frac{e^z - 1}{z^2}$$

about $z_0 = 0$. Also, give the residue of f at the point.

Solution. By the series expansion of e^z about $z_0 = 0$,

$$\begin{aligned}\frac{e^z - 1}{z^2} &= \frac{1}{z^2}\left(z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\right)\\ &= \frac{1}{z} + \frac{1}{2} + \frac{z}{3!} + \frac{z^2}{4!} + \cdots\end{aligned}$$

for any $z \in \mathbb{C}$. The coefficient of z^{-1} is the residue of f at $z_0 = 0$,

$$\operatorname{Res}(f; 0) = 1.$$

Exercise 2.5.9

Find the Laurent series for

$$f(z) = \frac{\sin z}{(z-\pi)^2}$$

about $z_0 = \pi$. Also, give the residue of f at the point.

Solution. Note that the power series expansion of $\sin z$ about $z_0 = \pi$ is

$$\sin z = -\sin(z-\pi) = -\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(z-\pi)^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!}(z-\pi)^{2k+1}.$$

Using the above expansion, we have

$$\begin{aligned}f(z) &= \frac{\sin z}{(z-\pi)^2}\\ &= \frac{1}{(z-\pi)^2}\left\{-(z-\pi) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!}(z-\pi)^{2k+1}\right\}\\ &= \underbrace{-\frac{1}{z-\pi}}_{\text{principal part}} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!}(z-\pi)^{2k-1}.\end{aligned}$$

The residue of f at $z_0 = \pi$ is

$$\operatorname{Res}(f; \pi) = -1.$$

Exercise 2.5.12

Find the first four terms of the Laurent series for

$$f(z) = \frac{1}{e^z - 1}$$

about $z_0 = 0$. Also, give the residue of f at the point.

Solution. For any $z \in \mathbb{C}$,

$$\begin{aligned} e^z - 1 &= -1 + \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\right) \\ &= z\left(1 + \frac{z}{2} + \frac{z^2}{3!} + \frac{z^3}{4!} + \cdots\right) \\ &= z\, g(z) \end{aligned}$$

where the function g defined by

$$g(z) = 1 + \frac{z}{2} + \frac{z^2}{3!} + \frac{z^3}{4!} + \cdots$$

is analytic and $g(0) \neq 0$. This leads to

$$f(z) = \frac{1}{zg(z)} \tag{1}$$

which shows that f has a pole of order 1 at $z_0 = 0$. Hence the Laurent series for f is in the form

$$f(z) = \frac{c_{-1}}{z} + c_0 + c_1 z + c_2 z^2 + \cdots .$$

We have

$$\begin{aligned} \frac{1}{z} &= f(z)g(z) && \text{by (1)} \\ &= \left(\frac{c_{-1}}{z} + c_0 + c_1 z + c_2 z^2 + \cdots\right)\left(1 + \frac{z}{2} + \frac{z^2}{3!} + \frac{z^3}{4!} + \cdots\right) \\ &= \frac{c_{-1}}{z} + \left(\frac{c_{-1}}{2} + c_0\right) + \left(\frac{c_{-1}}{3!} + \frac{c_0}{2} + c_1\right)z + \left(\frac{c_{-1}}{4!} + \frac{c_0}{3!} + \frac{c_1}{2} + c_2\right)z^2 + \cdots . \end{aligned}$$

Multiplying out the series and equating coefficients of equal powers of z, we obtain

$$c_{-1} = 1, \qquad \frac{c_{-1}}{2} + c_0 = 0, \qquad \frac{c_{-1}}{3!} + \frac{c_0}{2} + c_1 = 0, \qquad \text{and} \qquad \frac{c_{-1}}{4!} + \frac{c_0}{3!} + \frac{c_1}{2} + c_2 = 0.$$

The equations are solved successively for c_{-1}, c_0, c_1 and c_2, yielding

$$c_{-1} = 1, \qquad c_0 = -\frac{1}{2}, \qquad c_1 = \frac{1}{12}, \qquad \text{and} \qquad c_2 = 0.$$

We have the Laurent series

$$f(z) = \frac{1}{z} - \frac{1}{2} + \frac{z}{12} + 0 \cdot z^2 + \cdots ,$$

and the residue of f at $z_0 = 0$

$$\operatorname{Res}(f; 0) = c_{-1} = 1.$$

Exercise 2.5.13

Find the first four terms of the Laurent series for

$$f(z) = \frac{1}{1 - \cos z}$$

about $z_0 = 0$. Also, give the residue of f at the point.

Solution. For any $z \in \mathbb{C}$,

$$\begin{aligned} 1 - \cos z &= 1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) \\ &= z^2\left(\frac{1}{2} - \frac{z^2}{4!} + \frac{z^4}{6!} - \cdots\right) \\ &= z^2 g(z) \end{aligned}$$

where the function g defined by

$$g(z) = \frac{1}{2} - \frac{z^2}{4!} + \frac{z^4}{6!} - \cdots$$

is analytic and $g(0) \neq 0$. This leads to

$$f(z) = \frac{1}{z^2 g(z)} \tag{2}$$

which shows that f has a pole of order 2 at $z_0 = 0$. Hence the Laurent series for f is in the form

$$f(z) = \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + c_0 + c_1 z + \cdots.$$

We have

$$\begin{aligned} \frac{1}{z^2} &= f(z)g(z) && \text{by (2)} \\ &= \left(\frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + c_0 + c_1 z + \cdots\right)\left(\frac{1}{2} - \frac{z^2}{4!} + \frac{z^4}{6!} - \cdots\right) \\ &= \frac{c_{-2}}{2}\frac{1}{z^2} + \frac{c_{-1}}{2}\frac{1}{z} + \left(-\frac{c_{-2}}{4!} + \frac{c_0}{2}\right) + \left(-\frac{c_{-1}}{4!} + \frac{c_1}{2}\right)z + \cdots. \end{aligned}$$

Multiplying out the series and equating coefficients of equal powers of z, we obtain

$$\frac{c_{-2}}{2} = 1, \qquad \frac{c_{-1}}{2} = 0, \qquad -\frac{c_{-2}}{4!} + \frac{c_0}{2} = 0, \qquad \text{and} \qquad -\frac{c_{-1}}{4!} + \frac{c_1}{2} = 0.$$

The equations are solved successively for c_{-2}, c_{-1}, c_0 and c_1, yielding

$$c_{-2} = 2, \qquad c_{-1} = 0, \qquad c_0 = \frac{1}{6}, \qquad \text{and} \qquad c_1 = 0.$$

We have the Laurent series

$$f(z) = \frac{2}{z^2} + 0 \cdot \frac{1}{z} + \frac{1}{6} + 0 \cdot z + \cdots,$$

and the residue

$$\operatorname{Res}(f; 0) = c_{-1} = 0.$$

Exercise 2.5.15

If f is analytic in $0 < |z - z_0| < R$ and has a pole of order m at z_0, show that

$$\operatorname{Res}\left(\frac{f'}{f}; z_0\right) = -\ell.$$

Solution. Since f has a pole of order ℓ at z_0, write

$$f(z) = \frac{g(z)}{(z - z_0)^\ell}$$

where g is analytic in $|z - z_0| < R$ and $g(z_0) \neq 0$. Indeed,

$$\begin{aligned} f'(z) &= \frac{g'(z)(z-z_0)^\ell - \ell g(z)(z-z_0)^{\ell-1}}{(z-z_0)^{2\ell}} \\ \frac{f'(z)}{f(z)} &= \frac{g'(z)}{g(z)} - \frac{\ell}{z - z_0}. \end{aligned}$$

Since $g(z_0) \neq 0$, the function $\dfrac{g'}{g}$ is analytic in $|z - z_0| < R$. Recall that the residue of a function at z_0 is the coefficient of $(z - z_0)^{-1}$ in the power series expansion at z_0. Thus,

$$\operatorname{Res}\left(\frac{f'}{f}; z_0\right) = -\ell.$$

Note : l'Hopital's rule for complex functions

Suppose that the single-variable complex functions $f(z)$ and $g(z)$ are analytic in some neighborhood of $z = z_0$. If $f(z_0) = 0$ and $g(z_0) = 0$, then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \frac{z - z_0}{g(z) - g(z_0)} = \frac{f'(z_0)}{g'(z_0)}$$

since both f' and g' are differentiable at $z = z_0$.

This shows that under some conditions, we may use l'Hopital's rule for complex functions.