Exercise 2.4.1

Give the order of each of the zeros of $f(z) = \dfrac{\sin z}{z}$.

Solution. If $z = 0$, then f is undefined. Suppose that $z \neq 0$. Then $f(z) = 0$ if and only if $\sin z = 0$. By the definition of $\sin z$,

$$\begin{aligned}\frac{e^{iz} - e^{-iz}}{2i} &= 0\\ e^{2iz} &= 1\\ 2iz &= 2k\pi i \quad \text{for nonzero integers } k. \quad \text{(See p.44.)}\end{aligned}$$

Hence the zeroes of f are $z = k\pi$ for all $k \in \mathbb{Z} \setminus \{0\}$. For any nonzero *even* integer k,

$$f'(k\pi) = \frac{z\cos z - \sin z}{z^2} = \frac{\cos k\pi}{k\pi} = \frac{1}{k\pi} \neq 0.$$

Similarly, if k is an odd integer, then

$$f'(k\pi) = -\frac{1}{k\pi} \neq 0.$$

The order of each of the zeros is 1.

Exercise 2.4.2

Give the order of each of the zeros of $f(z) = (e^z - 1)^2$.

Solution. Since

$$(e^z - 1)^2 = 0 \iff e^z = 1,$$

the zeros of f are $z = 2k\pi i$ for any $k \in \mathbb{Z}$. Also, we have

$$\begin{aligned}f'(2k\pi i) &= 2(e^z - 1)e^z\big|_{z=2k\pi i} = 0, \text{ and}\\ f''(2k\pi i) &= 4e^{2z} - 2e^z\big|_{z=2k\pi i} = 2 \neq 0\end{aligned}$$

for any integer k. Therefore each of the zeros has order 2.

Exercise 2.4.3

Give the order of each of the zeros of $f(z) = (z^2 + z - 2)^3$.

Solution. A simple factorization shows that

$$f(z) = (z^2 + z - 2)^3 = (z + 2)^3(z - 1)^3.$$

The zeros are $z = 1$ and $z = -2$. Because the multiple of each root is 3, both zeros have order 3.

Exercise 2.4.6

Give the order of each of the zeros of $f(z) = \mathrm{Log}(1 - z)$ where $|z| < 1$.

Solution. Let $f(z) = 0$. By the definition of Log,

$$1 - z = e^0 = 1$$

Therefore $z = 0$ is the only zero. Note that the function f is analytic in the disc $|z| < 1$ (see Example 10 of Section 2.1), and

$$\frac{d}{dz}\mathrm{Log}(1 - z)\bigg|_{z=0} = -\frac{1}{1 - z}\bigg|_{z=0} = -1 \neq 0.$$

The order of $z = 0$ is 1.

Another solution.
Set $z = x + iy$ where $x, y \in \mathbb{R}$. Then

$$\mathrm{Log}(1 - z) = \ln|1 - z| + i\mathrm{Arg}(1 - z) = 0.$$

In other words, $\mathrm{Arg}(1 - z) = 0$ implies that $y = 0$ and $1 - x > 0$. Plugging in $y = 0$, we get

$$\begin{aligned} \ln|1 - z| = \ln(1 - x) &= 0 \\ x &= 0. \end{aligned}$$

Therefore $z = x + iy = 0$.

Exercise 2.4.9

Find the power series expansion about the origin for $f(z) = z(e^z - 1)$ and the largest disk in which the series is valid.

Solution. Recall that

$$e^z = 1 + \sum_{k=1}^{\infty} \frac{z^k}{k!}.$$

Thus

$$f(z) = z(e^z - 1) = z \sum_{k=1}^{\infty} \frac{z^k}{k!} = \sum_{k=1}^{\infty} \frac{z^{k+1}}{k!}.$$

Because $g(z) = z$ and $h(z) = e^z - 1$ are entire functions, f is entire. The series is valid on the whole complex plane.

Exercise 2.4.10

Find the power series expansion about $z_0 = \pi i$ for $f(z) = e^z$ and the largest disk in which the series is valid.

Solution. Note that

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

Substituting z with $z - \pi i$, we get

$$e^{z-\pi i} = \sum_{k=0}^{\infty} \frac{(z - \pi i)^k}{k!}. \tag{1}$$

Now we obtain the series as follows:

$$\begin{aligned} f(z) = e^z &= e^{z-\pi i} e^{\pi i} \\ &= e^{z-\pi i} \cdot (-1) && \text{since } e^{\pi i} = -1 \\ &= -\sum_{k=0}^{\infty} \frac{(z - \pi i)^k}{k!} && \text{by (1).} \end{aligned}$$

Since f is an entire function, the series is valid everywhere.

Exercise 2.4.11

Find the power series expansion about $z_0 = 1$ for $f(z) = z^3 + 6z^2 - 4z - 3$ and the largest disk in which the series is valid.

Solution. By polynomial long division,

$$\begin{array}{r|rrrr}
1 & 1 & 6 & -4 & -3 \\
 & & 1 & 7 & 3 \\
\hline
1 & 1 & 7 & 3 & 0 \\
 & & 1 & 8 & \\
\hline
1 & 1 & 8 & 11 & \\
 & & 1 & & \\
\hline
 & 1 & 9 & &
\end{array}$$

we have

$$\begin{aligned}
f(z) &= (z-1)(z^2 + 7z + 3) \\
&= (z-1)\big\{(z-1)(z-1+9) + 11\big\} \\
&= (z-1)^3 + 9(z-1)^2 + 11(z-1).
\end{aligned}$$

Since every polynomial is an entire function, the series is valid everywhere.

Exercise 2.4.13

Find the power series expansion about $z_0 = -1$ for $f(z) = \dfrac{z+2}{z+3}$ and the largest disk in which the series is valid.

Solution.

$$\begin{aligned}
\frac{z+2}{z+3} &= 1 - \frac{1}{z+3} \\
&= 1 - \frac{1}{2 + (z+1)} \\
&= 1 - \frac{1}{2} \cdot \frac{1}{1 - \left(\dfrac{z+1}{-2}\right)} \\
&= 1 - \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{z+1}{-2}\right)^k && \text{if } \left|\frac{z+1}{-2}\right| < 1 \\
&= 1 + \sum_{k=0}^{\infty} (-2)^{-k-1} (z+1)^k
\end{aligned}$$

Note that the above series converges only when

$$\left|\frac{z+1}{-2}\right| < 1,$$

that is, $|z+1| < 2$.