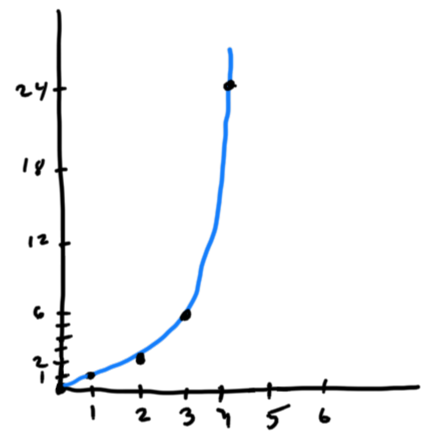
Euler's Gamma Function.

Plotting the factorial function

n	$n!$
1	1
2	2
3	6
4	24
5	120
6	720



Is there some real function $f(x)$ so that $f(n) = n!$ for positive integers n? The answer is yes, and in fact there is a complex function! Euler found the following trick: Consider

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \quad , \ \operatorname{Re} z > 0$$

Then

$$\Gamma(z+1) = \int_0^\infty e^{-t} t^{z}\, dt =$$

$$\overset{\text{IBP}}{=} \underbrace{\Big[-e^{-t} t^{z}\Big]_0^\infty}_{A} - \underbrace{\int_0^\infty -e^{-t} z t^{z-1}\, dt}_{B}$$

$$A = \lim_{t\to\infty}\left(-e^{-t} t^{z}\right) - (-e^{0}\cdot 0) = -\lim_{t\to\infty} \frac{t^{z}}{e^{t}} = 0$$

$$B = z\int_0^\infty e^{-t} t^{z-1}\, dt = z\,\Gamma(z)$$

so $$\boxed{\Gamma(z+1) = z\,\Gamma(z)}$$

What is $\Gamma(1)$?

$$\Gamma(1) = \int_0^\infty e^{-t} t^0 \, dt = \left[-e^{-t}\right]_0^\infty = -\lim_{t\to\infty} e^{-t} + e^0 = 1$$

So $\Gamma(n+1) = n\,\Gamma(n)$ using (*) with $z=n$

$= n(n-1)\Gamma(n-1)$ $\quad z = n-1$

$\vdots$

$= n(n-1)\cdots 1\cdot\Gamma(1)$

$= n!$

So $f(z) = \Gamma(z+1)$ is a complex function

such that $f(n) = n!$

Details. One has to prove that $\Gamma(z)$ actually converges uniformly in z. That is, for every closed & bdd subset U of $\{z : \operatorname{Re} z > 0\}$ and every $\varepsilon > 0$ there is a $N > 0$ so that $\left|\int_N^\infty e^{-t} t^{z-1}\, dt\right| < \varepsilon$ for all z in U.

This implies $\Gamma(z)$ converges and is analytic.

Extending the domain.

$$\Gamma(z+1) = z\Gamma(z) \qquad \operatorname{Re} z > 0$$

$$\Rightarrow \Gamma(z+2) = (z+1)\Gamma(z+1) = (z+1)z\Gamma(z)$$

$$\vdots$$

$$\Gamma(z+N) = (z+N-1)\cdots(z+1)z\Gamma(z).$$

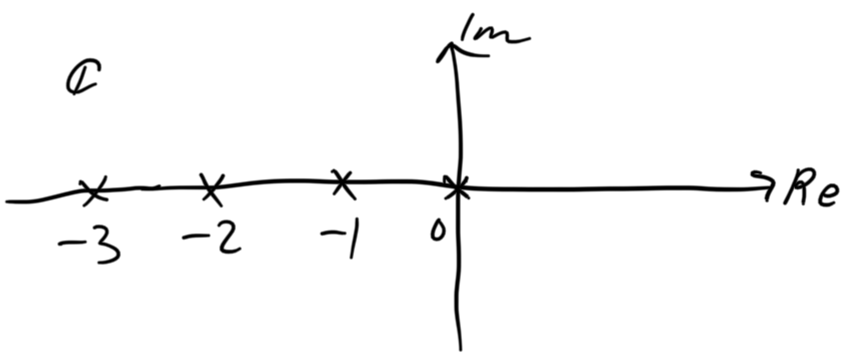
So if we want to define $\Gamma(z)$ for $\operatorname{Re} z > -N$ we put

$$\Gamma(z) = \frac{1}{z(z+1)\cdots(z+N-1)}\Gamma(\underbrace{z+N}_{\operatorname{Re} > 0})$$

$$= \frac{1}{z(z+1)\cdots(z+N-1)}\int_0^\infty e^{-t}\, t^{z+N-1}\, dt$$

We can use this formula to define $\Gamma(z)$ for any $z \neq 0, -1, -2, -3, \ldots$

$\Rightarrow \Gamma(z)$ is analytic at every $z \neq 0, -1, -2 \ldots$ and has simple poles at $0, -1, -2, \ldots$



Let's compute the residue at 0:

$$\mathrm{Res}\left(\Gamma(z);0\right) = z\Gamma(z)\Big|_{z=0} = \Gamma(z+1)\Big|_{z=0} = \Gamma(1) = 1$$

At $z=-n$ $n>0$ we have

$$\mathrm{Res}\left(\Gamma(z);-n\right) \underset{z=w-n}{=} \mathrm{Res}\left(\Gamma(w-n);w=0\right) =$$

$$= \mathrm{Res}\left(\frac{1}{(w-n)(w-n-1)\cdots(w-1)}\Gamma(w);\ w=0\right)$$

$$= \frac{1}{(w-n)\cdots(w-1)}\cdot\underbrace{w\cdot\Gamma(w)}\Big|_{w=0} =$$

$$= \frac{1}{(w-n)\cdots(w-1)}\Gamma(w+1)\Big|_{w=0} = \frac{1}{(-n)(-(n-1))\cdots(-1)}$$

$$= \frac{(-1)^n}{n!} \qquad \boxed{\mathrm{Res}\left(\Gamma(z);-n\right) = \frac{(-1)^n}{n!}} \ \begin{matrix} n>0 \\ \text{integer}\end{matrix}$$

cf. $\Gamma(n+1) = n!$

Q: What is "$(-\frac{1}{2})!$"?

That is, what is $\boxed{\Gamma(\frac{1}{2})?}$

Sol. $\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t}t^{-1/2}\,dt = \left\{\begin{matrix} t=u^2 \\ dt = 2u\,du\end{matrix}\right\}$

$$= \int^\infty e^{-u^2}u^{-1}2u\,du = 2\int^\infty e^{-u^2}du =$$

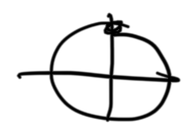
$$= \int_{-\infty}^{\infty} e^{-u^2}\, du \qquad \text{Gaussian integral}$$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy$$

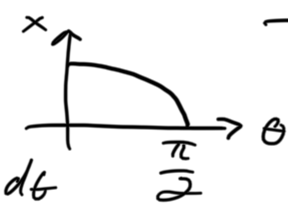
$$= \left\{ \begin{matrix} \text{polar} \\ dA = r\,dr\,d\theta \end{matrix} \right\} \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\, r\, dr\, d\theta = 2\pi \cdot \frac{1}{2}\left[-e^{-r^2}\right]_0^{\infty} = \pi$$

__Application__ $p, q \in \mathbb{R}_{>0}$

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\, dx$$



__sol.__ Set $\begin{cases} x = \cos^2\theta \\ dx = -2\cos\theta\sin\theta\, d\theta \end{cases}$



$$2\int_0^{\pi/2} \cos^{2p-2}\theta \sin^{2q-2}\theta \cos\theta\sin\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \cos^{2p-1}\theta \sin^{2q-1}\theta\, d\theta$$

__thm__ $B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ (Euler)

__Proof__ $\Gamma(p)\Gamma(q) = \int_0^{\infty}\int_0^{\infty} e^{-s}s^{p-1}e^{-t}t^{q-1}\, ds\, dt =$

$$= \int_0^{\infty}\int_0^{\infty} e^{-s-t} s^{p-1} t^{q-1}\, ds\, dt \quad \begin{cases} s = x^2 \\ t = y^2 \end{cases}$$

$$= \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\, x^{2p-2}\, y^{2q-2}\, 2x\, 2y\, dx\, dy$$

$$=4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\, x^{2p-1}\, y^{2q-1}\, dx\, dy$$

Now $x = r\cos\theta$ $\quad y = r\sin\theta$  $0\le\theta\le\frac{\pi}{2}$

$$=4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r^{2p-1}\, r^{2q-1}\cos^{2p-1}\theta\,\sin^{2q-1}\theta\, r\, dr\, d\theta$$

$$=4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r^{2p+2q-1}\cos^{2p-1}\theta\,\sin^{2q-1}\theta\, dr\, d\theta$$

$$=\underbrace{\left(4\int_0^\infty e^{-r^2}\, r^{2p+2q-1}\, dr\right)}\underbrace{\left(\int_0^{\pi/2}\cos^{2p-1}\theta\,\sin^{2q-1}\theta\, d\theta\right)}_{\frac{B(p,q)}{2}}$$

$$\begin{cases} r^2 = t \\ 2r\,dr = dt \end{cases}$$

$$4\int_0^\infty e^{-t}\, t^{p+q}\, \frac{\overbrace{r^{-1}r^{-1}}^{t^{-1}}}{2}\, dt = 4\frac{\Gamma(p+q)}{2}$$

$$\Rightarrow \quad \Gamma(p)\Gamma(q) = \cancel{2}\Gamma(p+q)\,\frac{B(p,q)}{\cancel{2}}$$

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

QED.

Fact: $\Gamma(z)$ has no zeroes. So $\Gamma(z)^{-1}$ entire

Follows from:

$$\boxed{\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \qquad z \in \mathbb{C}}$$

<u>OPTIONAL READING</u>

We provide a proof of this identity.

Take $0 < s < 1$.

$$\Gamma(s)\Gamma(1-s) = \Gamma(s)\int_0^\infty e^{-t}\, t^{1-s-1}\,dt =$$

$$= \int_0^\infty e^{-t}\, t^{-s} \underbrace{\left(\int_0^\infty e^{-u}\, u^{s-1}\,du\right)}_{u = tv} dt$$

$$= \int_0^\infty\int_0^\infty e^{-t}\, t^{-s} \cdot t\; e^{-tv}\, t^{s-1}\, v^{s-1}\,dv\,dt$$

$$= \int_0^\infty\int_0^\infty e^{-t-tv}\, \cancel{t^{-s+1+s-1}}\, v^{s-1}\,dv\,dt$$

$$= \int_0^\infty \left(\int_0^\infty e^{-t(v+1)}\,dt\right) v^{s-1}\,dv$$

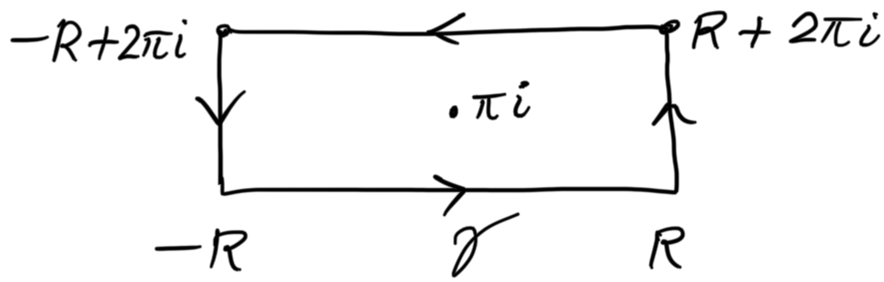
$$= \int_0^\infty \left[\frac{e^{-t(v+1)}}{-(v+1)}\right]_{t=0}^{\infty} v^{s-1}\,dv$$

$$= \int^\infty \quad 1 \quad v^{s-1}\, d \quad = \{v = e^x\}$$

$$= \int_0 \frac{\quad}{v+1} v \quad dv = \{ dv = e^x dx \}$$

$$= \int_{-\infty}^{\infty} \frac{e^{x(s-1)}}{e^x+1} e^x \, dx = \int_{-\infty}^{\infty} \frac{e^{sx}}{e^{x}+1} \, dx$$

Residue calculus using the contour:



$$\int_\gamma f(z)\, dz =$$

$$= \int_{-R}^{R} \frac{e^{sx}}{1+e^x} dx + i \int_0^{2\pi} \frac{e^{s(R+iy)}}{1+e^{R+iy}} dy$$

$$- \int_{-R}^{R} \frac{e^{sx} e^{is2\pi}}{1+e^x} dx - i \int_0^{2\pi} \frac{e^{s(-R+iy)}}{1+e^{-R+iy}} dy =$$

$$= \left(1 - e^{is2\pi}\right) \int_{-R}^{R} \frac{e^{sx}}{1+e^x} dx + \underbrace{i \int_0^{2\pi} \left(\frac{e^{s(R+iy)}}{1+e^{R+iy}} - \frac{e^{s(-R+iy)}}{1+e^{-R+iy}} \right) dx}_{\to 0 \text{ (using estimates)}}$$

On the other hand, by residue thm:

$$\int_\gamma f(z)\, dz = 2\pi i \operatorname{Res}\left(\frac{e^{sz}}{1+e^z} ; \pi i \right)$$

πi — the only pole inside γ

$$= 2\pi i \left. \frac{(z-\pi i) e^{sz}}{1+e^z} \right|_{z=\pi i} =$$

l'Hoptial

$$= 2\pi i \frac{e^{s\pi i}}{e^{\pi i}} = -2\pi i\, e^{s\pi i}$$

both numerator & denom. are analytic

So $$\int_{-\infty}^{\infty} \frac{e^{sx}}{1+e^{x}}\,dx = \frac{-2\pi i\, e^{i\pi s}}{1-e^{i2\pi s}} = \frac{\pi}{\sin(\pi s)}$$

We have shown that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

whenever $z = s$, $0<s<1$. Therefore

$$g(z) = \sin(\pi z)\,\Gamma(z)\,\Gamma(1-z) - \pi$$

is an entire function which vanishes on the real interval $(0,1)$. Since the zeros of a nonzero analytic function are isolated, this implies g is identically zero.

QED