MATH 618 LECTURE 9

READ §3.1.5 and §3.4.1 (cor 3.21).

HW9: Let G be a finite group & $\overline{\mathbb{k}}=\mathbb{k}$, char $\mathbb{k} \nmid |G|$.
Prove that G is abelian iff every irreducible representation is 1-dim'l.

More details on Wedderburn's Thm

$V = V_1 \oplus \cdots \oplus V_n$, $\quad V_\ell \xrightarrow{i_\ell} V \xrightarrow{\pi_k} V_k$

$X \in \mathrm{End}(V)$, $\quad X_{k\ell} := \pi_k \circ V \circ i_\ell$

Then $\quad X_{k\ell} \in \mathrm{Hom}(V_\ell, V_k)$

$$X \longmapsto \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{nn} \end{bmatrix}$$

Matrices like this can be multiplied as usual! We only ever multiply $Y_{ak} \cdot X_{kb}$

$$\mathrm{End}(V) \cong \begin{bmatrix} \mathrm{Hom}(V_1,V_1) & \cdots & \mathrm{Hom}(V_n,V_1) \\ \vdots & \ddots & \vdots \\ \mathrm{Hom}(V_1,V_n) & \cdots & \mathrm{Hom}(V_n,V_n) \end{bmatrix}$$

as algebras

<u>Example.</u> Suppose $\overline{\mathbb{k}} = \mathbb{k}$.

Suppose A is a finite-dim'l alg such that the regular representation is the direct sum of three irreducible subrepresentations:

$$ {}_A A = V_1 \oplus V_2 \oplus V_3 $$

Suppose that $V_1 \cong V_2 \not\cong V_3$.

Then $A \cong \mathrm{End}_A({}_A A)^{op} = \mathrm{End}_A(V_1 \oplus V_2 \oplus V_3)^{op}$

$$ \cong \begin{bmatrix} \mathrm{Hom}_A(V_1, V_1) & \mathrm{Hom}_A(V_2, V_1) & \mathrm{Hom}_A(V_3, V_1) \\ \mathrm{Hom}_A(V_1, V_2) & \mathrm{Hom}_A(V_2, V_2) & \mathrm{Hom}_A(V_3, V_2) \\ \mathrm{Hom}_A(V_1, V_3) & \mathrm{Hom}_A(V_2, V_3) & \mathrm{Hom}_A(V_3, V_3) \end{bmatrix}^{op} $$

$$ \overset{\text{Schur's Lemma}}{\cong} \begin{bmatrix} \mathbb{k} & \mathbb{k} & 0 \\ \mathbb{k} & \mathbb{k} & 0 \\ 0 & 0 & \mathbb{k} \end{bmatrix}^{op} \cong \mathrm{Mat}_2(\mathbb{k}) \times \mathrm{Mat}_1(\mathbb{k}) $$

(Matrix transpose gives an algebra isomorphism $\mathrm{Mat}_n(\mathbb{k})^{op} \to \mathrm{Mat}_n(\mathbb{k})$)

Consequences

Let G be a finite group
$\mathbb{k}$ a field, $\operatorname{char} \mathbb{k} \nmid |G|$, $\overline{\mathbb{k}} = \mathbb{k}$
($\mathbb{k} = \mathbb{C}$ always works)
and $\{V_1, \ldots, V_t\}$ the set of all irreducible representations of G (up to equivalence).

By Maschke's Theorem

$$\mathbb{k}G \cong V_1^{\oplus m_1} \oplus V_2^{\oplus m_2} \oplus \cdots \oplus V_t^{\oplus m_t}$$

where $V^{\oplus m} := V \oplus V \oplus \cdots \oplus V$ (m terms)
and $m_i \in \mathbb{Z}_{\geq 0}$.

m_i is called the multiplicity of V_i (in $\mathbb{k}G$). The m_i are uniquely determined by $\mathbb{k}G$, by Jordan-Hölder's Thm for modules.

Proposition

a) $m_i = \dim V_i$ (!)

b) $|G| = m_1^2 + m_2^2 + \cdots + m_t^2$

Proof a) By Frobenius reciprocity,

$$\begin{array}{c} \mathrm{Hom}_{\mathbb{K}G}(V_i, \mathbb{K}G) \cong \mathrm{Hom}_{\mathbb{K}}(\mathrm{Res}^G_{\{1\}} V_i, \mathbb{K}) \\ \quad\ \ \wr\!\wr \\ \quad\ \ \mathrm{Coind}^G_{\{1\}} \mathbb{K} \end{array}$$

$$\mathrm{RHS} = (V_i)^*$$

$$\mathrm{LHS} \cong \mathrm{Hom}_{\mathbb{K}G}(V_i, V_1^{\oplus m_1} \oplus \cdots \oplus V_t^{m_t})$$

$$\cong \bigoplus_{j=1}^{t} \mathrm{Hom}_{\mathbb{K}G}(V_i, V_j)^{\oplus m_j} \cong \mathbb{K}^{\oplus m_i}$$

By Schur's Lemma.

Taking dimensions on both sides:

$$\dim(\mathbb{K}^{\oplus m_i}) = \dim(V_i^*)$$

$$m_i = \dim V_i$$

b) By Wedderburn's Theorem,

$$\Bbbk G \cong \mathrm{Mat}_{m_1}(\Bbbk) \times \mathrm{Mat}_{m_2}(\Bbbk) \times \cdots \times \mathrm{Mat}_{m_t}(\Bbbk)$$

Taking dimensions on both sides gives the claim. ∎

<u>Example</u>. We know of three irreps of S_3:

$$V_{triv} = \Bbbk 1 \ , \ \sigma . 1 = 1 \ \forall \sigma \in S_3$$

$$V_{sgn} = \Bbbk 1_- \ , \ \sigma . 1_- = \mathrm{sgn}(\sigma) 1_-$$

$$\begin{aligned} V_2 &= \text{standard rep} \\ &= \{(\lambda_1, \lambda_2, \lambda_3) \in \Bbbk^3 \mid \textstyle\sum \lambda_j = 0\} \\ &= \Bbbk(1,-1,0) \oplus \Bbbk(0,1,-1) \end{aligned}$$

Their dimensions are 1, 1, 2.
And $1^2 + 1^2 + 2^2 = 6 = |S_3|$ so these are all the irreps (up to equiv.).

This also implies that

$$\Bbbk S_3 \cong \begin{bmatrix} \Bbbk & & & \\ & \Bbbk & & \\ & & \Bbbk & \Bbbk \\ & & \Bbbk & \Bbbk \end{bmatrix} = \Bbbk \times \Bbbk \times Mat_2(\Bbbk)$$

<u>Example.</u> $|S_4| = 24$

$$24 = \underset{\text{triv}}{1^2} + \underset{\text{sgn}}{1^2} + \underset{\text{stand.}}{3^2} + \ ?$$

$$24 - 11 = 13 = 2^2 + 3^2 \ ?$$

$$\text{OR} \quad = \underbrace{1^2 + 1^2 + \cdots + 1^2}_{13} \ ?$$

Would really help to know the value of $\underline{\underline{t}}$, the <u>number of irreps</u>.

Commutator trick

The (additive) commutator is $[x,y] = xy - yx$.

For subspaces S, T of an algebra A, we put

$$[S,T] = \operatorname{span}\left\{ [x,y] \;\middle|\; \begin{matrix} x \in S \\ y \in T \end{matrix} \right\}$$

Exercise: i) For $A = \mathrm{Mat}_n(\Bbbk)$,

$$[A,A] = \mathfrak{sl}_n(\Bbbk) := \{ a \in A \mid \mathrm{Tr}(a) = 0 \}$$

ii) $[A \times B, A \times B] = [A,A] \times [B,B]$ for any algebras A, B.

($A \times B = A \oplus B$ as vector spaces with componentwise operations.)

Proposition

a) If $A = \mathrm{Mat}_{m_1}(\mathbb{k}) \times \cdots \times \mathrm{Mat}_{m_t}(\mathbb{k})$ then $t = \dim(A/[A,A])$.

b) If G is a finite group and $\overline{\mathbb{k}} = \mathbb{k}$, $\operatorname{char} \mathbb{k} \nmid |G|$, then

$t =$ the number of conjugacy classes in G

$= \dim Z(\mathbb{k}G)$

center of the group algebra of G.

Proof a) By exercise,

$[A,A] = sl_{m_1}(\mathbb{k}) \times \cdots \times sl_{m_t}(\mathbb{k})$

which has dimension

$(m_1^2 - 1) + (m_2^2 - 1) + \cdots + (m_t^2 - 1)$

$= (\dim A) - t$

Now use that $\dim \frac{A}{[A,A]} =$

$= \dim A - \dim [A,A]$.

b) Put $A = \mathbb{k}G$.

$$t = \dim A/[A,A] = \dim \left(A/[A,A]\right)^*$$

$$\left(\frac{A}{[A,A]}\right)^* \cong \{ f \in A^* \mid f([A,A]) = 0 \}$$

$$= \{ f \in A^* \mid f(gh - hg) = 0 \ \forall g, h \in G \}$$

as vector spaces

$$\cong \{ f \in \mathbb{k}^G \mid f(gh) = f(hg) \ \forall g, h \}$$

$$= \{ f \in \mathbb{k}^G \mid f(ghg^{-1}) = f(h) \ \forall g, h \in G \}$$

This is the set of functions $f : G \to \mathbb{k}$ that are constant on conjugacy classes. Each conjugacy class $C \subseteq G$ gives a function $f_C : G \to \mathbb{k}$, $f_C(g) = \begin{cases} 1 & g \in C \\ 0 & g \notin C \end{cases}$.

and $\{f_C\}_{C \in \underbrace{Cl(G)}_{\text{set of conjugacy classes in } G}}$ is a basis for the space of such functions.

As we've mentioned before

$$\left\{ z_C := \sum_{g \in G} g \right\}_{C \in Cl(G)}$$

is a basis for $Z(kG)$. ∎

<u>Ex</u> S_4 again. Partitions of 4 are

(4^1), $(3,1)$, (2^2), $(2,1^2)$, (1^4)

Five in total. Therefore

$$|S_4| = 24 = \underset{\text{triv}}{1^2} + \underset{\text{sgn}}{1^2} + \underset{\text{st}}{3^2} + m^2 + n^2$$

and $m^2 + n^2 = 13$, $m,n > 0 \Rightarrow \begin{matrix} m = 2 \\ n = 3 \end{matrix}$

Thus S_4 has two more ireps! One 2-dim'l and one 3-dim'l not equivalent to the standard one.