Maschke, Schur, Wedderburn

Read:

§1.4.3, §1.4.4 (Wedderburn)

§3.4.1 (Maschke's Thm)

§1.2.5 (Schur's Lemma)

<u>HW8</u>: Let V be a left $\Bbbk G$-module. The set of G-invariants in V is $V^G = \{v \in V \mid g.v = v \ \forall g \in G\}$. Show that V^G is a submodule of V and describe $(\Bbbk G)^G$.

<u>Def</u> A rep (V, ρ) of an alg. A is completely reducible if V is a direct sum of irreducible subreps.

Maschke's Thm: If G is a finite group such that the order, $|G|$, of G is invertible in $\Bbbk$ then every finite-dimensional representation $\rho : G \to \mathrm{Aut}_{\Bbbk}(V)$ is completely reducible.

Proof Let (V, ρ) be a rep of G. If V is irreducible we're done. If not, let $U \subsetneq V$ be a nonzero proper subrep. We claim that $V = U \oplus U'$ for some subrep U' of V.

Let $P: V \to U$ be any linear projection.

Define $\tilde{P} = \frac{1}{|G|} \sum_{g \in G} \rho(g) P \rho(g^{-1})$.

We claim $\tilde{P}$ is a map of representations. $\forall h \in G$:

$$\tilde{P}\rho(h) = \frac{1}{|G|}\sum_{g\in G}\rho(g)P\rho(g^{-1})\rho(h)$$

$$= \frac{1}{|G|}\sum_{g\in G}\rho(g)P\rho(g^{-1}h)$$

$$= \left\{\begin{array}{l}\text{Substitute } g \mapsto hg \\ \text{Then } g^{-1}h \mapsto (hg)^{-1}h = g^{-1}\end{array}\right\}$$

$$= \frac{1}{|G|}\sum_{g\in G}\rho(hg)P\rho(g^{-1})$$

$$= \rho(h)\frac{1}{|G|}\sum_{g}\rho(g)P\rho(g^{-1})$$

$= \rho(h)\tilde{P}$. Also $\tilde{P}(V) = U$. and $(\tilde{P})^2 = \tilde{P}$.

Define $U' = \ker \tilde{P}$.

Then $V = U \oplus U'$

$$v = \tilde{P}(v) + (v - \tilde{P}(v))$$

Repeating this for each of U, U' (& using $\dim V < \infty$) eventually we get

$$V = V_1 \oplus \dots \oplus V_t$$

where V_i are irred. subreps.

$\blacksquare$

Def The regular representation of G is the rep corresponding to the left $\mathbb{k}G$-module $\mathbb{k}G$. Thus

$$\rho_{reg} : G \longrightarrow \mathrm{Aut}(\mathbb{k}G)$$

$$\rho_{reg}(g)\, h = gh \quad \forall g, h \in G.$$

Cor. If $\mathbb{k}^\times \ni |G| < \infty$, then

$$\boxed{\mathbb{k}G = V_1 \oplus \cdots \oplus V_n}$$

for some irred. subreps V_i of $\mathbb{k}G$.

Schur's Lemma.

If $\mathbb{k}$ is algebraically closed and V, W are irreducible A-modules, then

$$\dim_{\mathbb{k}} \operatorname{Hom}_A(V,W) = \begin{cases} 1 & \text{if } V \overset{\text{as } A\text{-modules}}{\cong} W \\ 0 & \text{if } V \not\cong W \end{cases}$$

Pf. Suppose $T: V \to W$ is a nonzero A-module map. We claim $V \cong W$. Since $T \neq 0$, $\ker T$ is a proper A-submodule of V. Since V is irreducible, $\ker T = \{0\}$. Thus T is injective. Similarly, $\operatorname{Im} T$ is a nonzero A-submodule of W, hence

$\operatorname{Im} T = W$ since W is irreducible. So T is bijective, hence an isomorphism of A-modules.

Suppose T_1, T_2 are two isomorphisms $V \to W$. Then $T_1 \circ T_2^{-1} : W \to W$. Since $\overline{\mathbb{k}} = \mathbb{k}$ & $\dim W < \infty$, $T_1 \circ T_2^{-1}$ has an eigenvalue $\lambda \in \mathbb{k}$. Thus

$$T_1 \circ T_2^{-1} - \lambda \, Id_W \in \operatorname{End}_A(W)$$

has nonzero kernel K. Since W is irreducible $K = W$. Thus $T_1 = \lambda T_2$ ∎

Wedderburn's Thm.

Lem $A \cong \mathrm{End}_A({}_AA)^{op}$.

Pf $R: A \to \mathrm{End}_A({}_AA)^{op}$

$$a \longmapsto R_a$$

where $R_a : A \to A$, $R_a(b) = ba$.

- R_a is a left A-module map:
  $R_a(bc) = bca = bR_a(c)$.
- $R_{ab} = R_b \circ R_a$ (Check)
- $R_a(1_A) = 1_A a = a \neq 0$ if $a \neq 0$
  so R is injective.
- Given any $\varphi \in \mathrm{End}_A({}_AA)^{op}$,
  $\varphi(a) = \varphi(a.1_A) = a\varphi(1_A) = R_{\varphi(1_A)}(a)$

so $\varphi = R_{\varphi(1_A)}$ hence R is surjective.

<u>Thm</u> (Wedderburn's Thm for $\mathbb{k}G$)
Let G be a group such that $\operatorname{char} \mathbb{k} \nmid |G| < \infty$.
Then

$$\mathbb{k}G \cong \mathrm{Mat}_{m_1}(\mathbb{k}) \times \cdots \times \mathrm{Mat}_{m_t}(\mathbb{k})$$

for some positive integers t and $m_1, \ldots, m_t$.

<u>Proof</u> Recall that

$$\mathbb{k}G \cong V_1 \oplus \cdots \oplus V_n$$

as left $\mathbb{k}G$-modules, for some irreducible reps V_i.

After arranging the summands,

$$\Bbbk G \cong V_1^{\oplus m_1} \oplus \cdots \oplus V_t^{\oplus m_t}$$

where now $V_i \not\cong V_j$ for $j \neq i$.

By the lemma,

$$(\Bbbk G)^{op} \cong \mathrm{End}_{\Bbbk G}(V_1^{\oplus m_1} \oplus \cdots \oplus V_t^{\oplus m_t})$$

$$\cong \begin{bmatrix} & \vdots & \\ \cdots & \mathrm{Hom}_{\Bbbk G}(V_i^{\oplus m_i}, V_j^{\oplus m_j}) & \cdots \\ & \vdots & \end{bmatrix}$$

By Schur's Lemma:

$$\mathrm{Hom}_{\Bbbk G}(V_i^{\oplus m_i}, V_j^{\oplus m_j}) \cong \begin{cases} 0 & i \neq j \\ \mathrm{Mat}_{m_i}(\Bbbk), & i = j \end{cases}$$

$\blacksquare$