MATH 618 LECTURE 7.

READ: §A.4 (Adjointness)
§3.1.4 (Chainging the group)
§B.2 ($\otimes$-Hom relations)
§1.2.2 Proposition 1.9

HW 7: Let G be a finite group. Show $\mathrm{Ind}_{\{1\}}^G V \cong \mathrm{Coind}_{\{1\}}^G V$ for any vector space V.

Adjoint pairs of functors

Given two functors $L: \mathcal{C} \to \mathcal{D}$, $R: \mathcal{D} \to \mathcal{C}$ between categories $\mathcal{C}, \mathcal{D}$ an adjunction $\alpha: L \dashv R$ is a family of isomorphisms of sets

$$\alpha_{c,d}: \mathrm{Hom}_{\mathcal{D}}(Lc, d) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(c, Rd)$$

natural in $c \in \mathcal{C}$, $d \in \mathcal{D}$.
(i.e. if $c \xrightarrow{f} c'$ in $\mathcal{C}$ & $d \xrightarrow{g} d'$ in $\mathcal{D}$ there is a commutative diagram relating α's). If there exists an adjuction α we say (L, R) is an adjoint pair of functors.

In this case L is called left adjoint to R and R is right adjoint to L.

<u>Ex.</u>

$$\mathrm{Hom}_{\underline{\mathrm{Vect}}_{\mathbb{k}}}(\mathbb{k}X, V) \cong \mathrm{Hom}_{\underline{\mathrm{Set}}}(X, V|_{\underline{\mathrm{Set}}})$$

Thus

$$\begin{aligned} \mathbb{k}\cdot : \underline{\mathrm{Set}} &\longrightarrow \underline{\mathrm{Vect}}_{\mathbb{k}} \\ X &\longmapsto \mathbb{k}X \end{aligned}$$

is left adjoint to the forgetful functor

$$\begin{aligned} \underline{\mathrm{Vect}} &\longrightarrow \underline{\mathrm{Set}}. \\ V &\longmapsto V|_{\underline{\mathrm{set}}} \end{aligned}$$

In diagrams we draw

$$\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D} \quad (\perp)$$

to indicate the existence of an adjuction $\alpha : L \dashv R$.

The $\otimes$-Hom adjunctions.

Left version:

$$\mathrm{Hom}(X\otimes Y, Z) \cong \mathrm{Hom}(Y, \mathrm{Hom}(X,Z))$$

$$(X\otimes -) \dashv \mathrm{Hom}(X,-)$$

More precisely let A, B, C be algebras.

$$X = {}_A X_B \ , \quad Y = {}_B Y_C \ , \quad Z = {}_A Z_C$$

$$\mathrm{Hom}_{A,C}(X \underset{B}{\otimes} Y, Z) \cong \mathrm{Hom}_{B,C}(Y, \underbrace{\mathrm{Hom}_{{}_A\underline{Mod}}(X,Z)})$$

$$\boxed{\begin{array}{l} b\in B,\ c\in C,\ \varphi\in \mathrm{Hom}_{{}_A\underline{Mod}}(X,Z) \\ (b\cdot\varphi\cdot c)(x) = \varphi(x.b).c \end{array}} \longrightarrow \in {}_B\underline{Mod}_C$$

$$(\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \underline{Set})$$

$$\begin{tikzcd} {}_B\underline{Mod}_C \arrow[r, shift left=2, "X \underset{B}{\otimes} -"] \arrow[r, phantom, "\perp"] & {}_A\underline{Mod}_C \arrow[l, shift left=2, "\mathrm{Hom}_{{}_A\underline{Mod}}(X,-)"] \end{tikzcd}$$

Right-handed version

" $\mathrm{Hom}(X \otimes Y, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$ "

Let A, B, C be algebras

Let ${}_A X_B$, ${}_B Y_C$, ${}_A Z_C$ be bimodules.

$$\mathrm{Hom}_{A,C}(X \underset{B}{\otimes} Y, Z) \cong \mathrm{Hom}_{A,B}(X, \underbrace{\mathrm{Hom}_{\underline{Mod}_C}(Y, Z)}_{\in {}_A\underline{Mod}_B}))$$

$$\begin{tikzcd} {}_A\underline{Mod}_B \arrow[r, shift left=2, "- \otimes_B Y"] \arrow[r, phantom, "\perp"] & {}_A\underline{Mod}_C \arrow[l, shift left=2, "\mathrm{Hom}_{\underline{Mod}_C}(Y, -)"] \end{tikzcd}$$

Application (Frobenius Reciprocity)

Restriction functor revisited:

G group, $H \leq G \Rightarrow \Bbbk H \subseteq \Bbbk G$

If V is a representation of G ($\Leftrightarrow$ a $\Bbbk G$-module), then

$$\mathrm{Res}_H^G V = V\downarrow_H = V \text{ as vector space}$$

Note 1: $\mathrm{Res}_H^G V \cong \mathrm{Hom}_{\Bbbk G}(\Bbbk G, V)$

$\psi(1_G) \longleftarrow\!\shortmid\ \psi$

$\Bbbk G$: $(\Bbbk G, \Bbbk H)$-module.

In the left handed adjunction, take $A = \Bbbk G$, $B = \Bbbk H$, $C = \Bbbk$, $X = \Bbbk G$

$$\mathrm{Hom}_{\Bbbk G}(\underbrace{\Bbbk G \otimes_{\Bbbk H} Y}_{\mathrm{Ind}_H^G Y}, Z) \cong \mathrm{Hom}_{\Bbbk H}(Y, \underbrace{\mathrm{Hom}_{\Bbbk G}(\Bbbk G, Z)}_{\cong \mathrm{Res}_H^G Z})$$

$$\boxed{\mathrm{Hom}_G(\mathrm{Ind}_H^G Y, Z) \cong \mathrm{Hom}_H(Y, \mathrm{Res}_H^G Z)}$$

for any $Y \in \mathrm{Rep}\, H$, $Z \in \mathrm{Rep}\, G$.

Note 2: $\quad \mathrm{Res}^G_H V \cong \Bbbk G \otimes_{\Bbbk G} V$

$(\Bbbk H, \Bbbk G)$-bimodule

In left-handed adjunction take

$$A = \Bbbk H, \quad B = \Bbbk G, \quad C = \Bbbk, \quad X = \Bbbk G$$

$$\mathrm{Hom}_{\Bbbk H}(\underbrace{\Bbbk G \otimes_{\Bbbk G} Y}_{\mathrm{Res}^G_H Y}, Z) \cong \mathrm{Hom}_{\Bbbk G}(Y, \underbrace{\mathrm{Hom}_{\Bbbk H}(\Bbbk G, Z)}_{\mathrm{Coind}^G_H Z})$$

$$\Leftrightarrow \quad \mathrm{Hom}_H(\mathrm{Res}^G_H Y, Z) \cong \mathrm{Hom}_G(Y, \mathrm{Coind}^G_H Z)$$

$$\boxed{\mathrm{Ind}^G_H \dashv \mathrm{Res}^G_H \dashv \mathrm{Coind}^G_H}$$

Ex. $\quad H = 1 \leq G, \quad V \in \underline{\mathrm{Rep}}\, H = \underline{\mathrm{Vect}}_{\Bbbk}$

$$\mathrm{Coind}^G_H V = \mathrm{Hom}_{\Bbbk}(\Bbbk G, V)$$
$$= \{F : G \to V\}$$
$$(g \cdot F)(g') = F(g' \cdot g) \quad \forall g' \in G, g \in G.$$