$\{x \in \Bbbk^3 \mid \sum x_i = 0\}$

MATH 618 LECTURE 6

HW6: Find $\rho_W((12))$ where $W = \mathrm{Ind}_{\Bbbk S_3}^{\Bbbk S_4} V_2$

TENSOR PRODUCTS OVER NONCOMMUTATIVE ALGEBRAS.

A – $\Bbbk$-algebra ($\Bbbk$ field or comm. ring)

We wish to generalize the tensor product of vector spaces (i.e. $\Bbbk$-modules) to tensor products of A-modules.

Recall that in $V \otimes W$ we have

$$v \otimes (\lambda w) = (\lambda v) \otimes w \quad \forall \lambda \in \Bbbk, v \in V, w \in W.$$

However, if M and N are left A-modules the requirement

$$m \otimes (a.n) = (a.m) \otimes n \quad \forall a \in A, m \in M, n \in N$$

is "wrong". It would imply

$$\left[\begin{array}{l} m \otimes ((ab).n) = ((ab).m) \otimes n = a.(b.m) \otimes n \\ m \otimes a.(b.n) = (a.m) \otimes (b.n) = b.(a.m) \otimes n \end{array}\right.$$

DEFINITION

Let $X = {}_AX_B$ be an (A,B)-bimodule and $Y = {}_BY_C$ be a (B,C)-bimodule. The tensor product $X \underset{B}{\otimes} Y$ is an (A,C)-bimodule with the following properties:

(1) There exists a map

$$\otimes : X \times Y \longrightarrow X \underset{B}{\otimes} Y$$
$$(x,y) \longmapsto x \otimes y$$

such that

(i) $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$

$x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$

(ii) $(a.x) \otimes y = a.(x \otimes y)$

$x \otimes (y.c) = (x \otimes y).c$

(iii) $x \otimes (b.y) = (x.b) \otimes y$

$\forall x, x_i \in X,\ y, y_i \in Y,\ a \in A,\ b \in B,\ c \in C.$

(2) Whenever ${}_A M_C$ is any (A,C)-module with a map $f : X \times Y \longrightarrow M$ satisfying properties (i)-(iii), there is a unique map of (A,C)-bimodules $\bar{f} : X \underset{B}{\otimes} Y \longrightarrow M$ such that $f(x,y) = \bar{f}(x \otimes y)$

i.e.

$$\begin{tikzcd} X \times Y \arrow[r, "\otimes"] \arrow[dr, "\forall f"'] & X \underset{B}{\otimes} Y \arrow[d, dashed, "\exists ! \bar{f}"] \\ & M \end{tikzcd}$$

Theorem ${}_A X_B \underset{B}{\otimes} {}_B Y_C$ exists is unique up to isomorphism.

Sketch: Existence: Use free construction.

Uniqueness: Abstract nonsense. ∎

Properties.

- $${}_A X_B \underset{B}{\otimes} \left({}_B Y_C \underset{C}{\otimes} {}_C Z_D \right) \cong \left({}_A X_B \underset{B}{\otimes} {}_B Y_C \right) \underset{C}{\otimes} {}_C Z_D$$

- $$\left(\bigoplus_i X_i \right) \underset{B}{\otimes} Y \cong \bigoplus_i \left(X_i \underset{B}{\otimes} Y \right)$$

(Same in right factor)

- $${}_B B_B \underset{B}{\otimes} {}_B Y_C \cong Y_C$$

$$1_B \otimes y \longmapsto y$$

(same in right factor)

Extension of scalars

$\mathbb{k} \subset K$ field extension.

Can view K as a $(K, \mathbb{k})$-bimod.

For any $\mathbb{k}$-vectorspace V (regarded as $(\mathbb{k}, \mathbb{k})$-bimodule) we can consider

$$K \underset{\mathbb{k}}{\otimes} V$$

If $V = A$ an algebra then $K \otimes_{\mathbb{k}} A$ is also an algebra.

Example.

$$K \otimes_{\Bbbk} \Bbbk[x_1, \dots, x_n] \cong K[x_1, \dots, x_n]$$

$$K \otimes_{\Bbbk} Mat_n(\Bbbk) \cong Mat_n(K).$$

$$\mathbb{Q} \otimes_{\mathbb{Z}} A \cong \mathbb{Q}^r$$

A — f.g. abelian group

$$A \cong \mathbb{Z}^r \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/n_k\mathbb{Z}$$

(Since $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$)

INDUCTION.

Take $C = \Bbbk$, and suppose $f : B \to A$ is an algebra map (often: inclusion of a subalg.)

Take ${}_A X_B = A$ with (A, B)-bimodule structure

$$a \cdot x = ax \qquad x \cdot b = x f(b)$$

$\forall a, x \in A, \ b \in B.$

Then for any left B-module ${}_B Y = {}_B Y_{\Bbbk}$ we obtain a left A-module

$${}_A A_B \underset{B}{\otimes} {}_B Y$$

<u>Def</u> $\mathrm{Ind}_B^A \, {}_B Y := {}_A A_B \underset{B}{\otimes} {}_B Y$

or just: $\boxed{\mathrm{Ind}_B^A Y := A \underset{B}{\otimes} Y}$

is the induced module of Y (from B to A)

Lemma If A is free as a right B-module with B-basis $\{a_i\}_{i\in I}$, and $\{y_j\}_{j\in J}$ is a $\Bbbk$-basis for Y then $\{a_i \otimes y_j\}_{(i,j)\in I\times J}$ is a $\Bbbk$-basis for $A \underset{B}{\otimes} Y$.

Pf $A = \bigoplus_{i\in I} a_i B$ $\quad$ $\cong B$ as right B-module

$$\Rightarrow A \underset{B}{\otimes} Y \cong \bigoplus_{i\in I} \left(a_i B \underset{B}{\otimes} Y \right) \overset{\text{v.sp.}}{\cong} \bigoplus_{i \in I} Y$$

Standard module for S_n

S_n acts on $\Bbbk^n$ by permuting coordinates. The standard module for S_n is

$$V_{n-1} = \{x \in \Bbbk^n \mid x_1 + \cdots + x_n = 0\}$$

$$= \ker f, \quad f : \Bbbk^n \longrightarrow \Bbbk$$
$$x \longmapsto \sum x_i$$

$$\mathrm{Ind}_{S_3}^{S_4} V_2 =$$

$$= \Bbbk S_4 \underset{\Bbbk S_3}{\otimes} V_2 \qquad \text{dimension} = [S_4 : S_3] \cdot \dim V_2$$
$$= 4 \cdot 2 = 8$$

A basis for $\Bbbk S_4$ as a right $\Bbbk S_3$-module is $\{(1), (14), (24), (34)\}$

$\Big((i4) S_3 = \{\text{bijections } \{1,2,3\} \to \{1,2,3,4\} \setminus \{i\}\}$
for $i = 1, 2, 3, 4 \Big)$

So $\mathbb{K}S_4 \otimes_{\mathbb{K}S_3} V_2$ has a basis

$$\left\{ (i4) \otimes v_j \;\middle|\; \begin{array}{l} i = 1, 2, 3, 4 \\ j = 1, 2 \end{array} \right\}$$

$v_1 = e_1 - e_2$, $v_2 = e_2 - e_3 \in \mathbb{K}^3$.

<u>HW 6</u> The above construction gives a representation

$\rho : S_4 \longrightarrow GL_8(\mathbb{K})$

<u>Find $\rho((12))$</u>.