MATH 618 LECTURE 5

HW5: 3.1.3(a)

Building representations.

Direct sums, block matrices

Trivial reps;

Sign rep of S_n

2-dim'l rep of dihedral grp

Subreps & irreps

Equivalent reps

Irreps of $S_3 \cong D_3$ dihedral of order 6

Induced reps:

$$\underbrace{S_{\{1,2\}} \times S_{\{3,4\}}}_{\cong \text{Klein Viergroup } V_4} \leq S_4$$

$$\mathrm{Ind}_{V_4}^{S_4} \mathrm{Res}_{V_4}^{S_4} (\mathrm{sgn})$$

Direct sums

If (V, ρ_V) and (W, ρ_W) are reps of a group G, then $(V \oplus W, \rho_{V\oplus W})$ is also a rep, where

$$\rho_{V\oplus W} : G \longrightarrow \mathrm{Aut}(V \oplus W)$$
$$g \longmapsto \big((v,w) \mapsto (\rho_V(g)v, \rho_W(g)w) \big)$$

$\mathbb{K}G$-module notation:

$$a.(v,w) = (a.v, a.w) \quad \forall a \in \mathbb{K}G$$

Choosing bases for V, W we have

$$[\rho_{V\oplus W}(g)] = \left[\begin{array}{c|c} [\rho_V(g)] & 0 \\ \hline 0 & [\rho_W(g)] \end{array}\right]$$

matrix of a linear map

- The trivial rep of a group G is

$$\rho_{triv} : G \longrightarrow \mathrm{Aut}(\mathbb{k}\, v) \cong GL_1(\mathbb{k}) \cong \mathbb{k}^\times$$
$$g \longmapsto 1 \quad \forall g \in G$$

- For S_n we have the sign representation

$$\rho_{sgn} : S_n \longrightarrow \mathbb{k}^\times$$
$$g \longmapsto sgn(g) \quad \forall g \in S_n$$

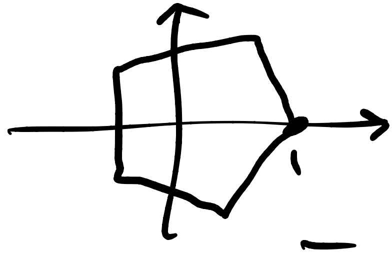
- The dihedral group D_n of order $2n$ has a 2-dim'l rep

$$\rho : \underset{\langle r, s \mid s^2 = r^n = srsr = 1 \rangle}{\overset{\shortparallel}{D_n}} \longrightarrow GL_2(\mathbb{k})$$

given by

$$\rho(r) = \begin{bmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{bmatrix}$$

$$\rho(s) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



$$\left(\cos\frac{2\pi}{n} = \frac{e^{2\pi i/n} + e^{-2\pi i/n}}{2} \in \mathbb{k} \text{ if } \overline{\mathbb{k}} = \mathbb{k} \right.$$

similarly

$$\left. \sin\frac{2\pi}{n} \in \mathbb{k} \quad \text{if } \overline{\mathbb{k}} = \mathbb{k} \right)$$

Def A subrep of a rep (V, ρ) of G is a subspace $U \subset V$ such that

$$\rho(g)U \in U \quad \forall g \in G.$$

Then (U, ρ_U) is a rep of G, where

$$\rho_U : G \to \mathrm{Aut}(U)$$

is defined by

$$\rho_U(g) = \rho(g)\big|_U .$$

Def Two reps are equivalent if there is an invertible map f of reps from one to the other. In matrix form this means

$$[\rho_W(g)] = T[\rho_V(g)]T^{-1}, \text{ where } \underset{f: V \to W}{T = [f]}.$$

<u>Def</u> A rep (V,ρ) of G is irreducible if the only subreps are $\{0\}$, V.

<u>Def</u> A rep. (V,ρ) of G is decomposable if V is the direct sum of two proper subrepresentations.

<u>Def</u> A rep. which isn't decomposable is indecomposable.

<u>Note</u>:

(V,ρ) irreducible $\Rightarrow (V,\rho)$ indecomposable

The other direction holds for finite groups when char $\mathbb{K}=0$ (Maschke's Thm).

Example.

$\rho : D_n \longrightarrow GL_2(\mathbb{k})$ is indecomposable: If

$$\mathbb{k}^2 = \mathbb{k}v_1 \oplus \mathbb{k}v_2$$

where $\mathbb{k}v_i$ are one-dim'l subreps, then

$$\rho(r)v_i = \lambda_i v_i$$
$$\rho(s)v_i = \mu_i v_i$$

So in basis $\{v_1, v_2\}$

$$[\rho(r)] = \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & \lambda_2 \end{array}\right]$$
$$[\rho(s)] = \left[\begin{array}{c|c} \mu_1 & 0 \\ \hline 0 & \mu_2 \end{array}\right]$$

these commute! contradicts that $sr \neq rs$ and ρ is injective.

Induced representations.

Given a representation (V, ρ) of H, where $H \leq G$ we will define a representation $(\mathrm{Ind}_H^G V, \rho_H^G)$ as follows.

① Pick a set of representatives $T = \{t_i\}_{i \in I} \subset G$ for the left cosets of H in G.

This means that

$$G = \bigsqcup_{i \in I} t_i H$$ disjoint union of left cosets

② Define the vector space

$$\mathrm{Ind}_H^G V = \mathbb{K}T \otimes V$$

③ Define the action of G:

$$g.(t_i \otimes v) = t_j \otimes (h.v)$$

where t_j and h are defined uniquely by the condition $g t_i = t_j h$

In representation notation:

$$\rho_H^G : G \longrightarrow \mathrm{Aut}(\mathrm{Ind}_H^G V)$$

is given by

$$\rho_H^G(g)\,(t_i \otimes v) = t_j \otimes \rho(h) v \, .$$

<u>Example.</u>

$$G = S_4$$

$$H = \{(1), (12), (34), (12)(34)\}$$

$$\rho : H \longrightarrow \mathrm{Aut}(\mathbb{k}e_1)$$

$$\sigma \longmapsto (\operatorname{sgn} \sigma)\, \mathrm{Id}_{\mathbb{k}e_1}$$

$$T = \{(1\,2\,3), (1\,3\,2), \ldots\}$$

$$(123)H = \{(123), (13), (1234), (134)\}$$

$$(132)H = \{(132), (23), \ldots\}$$

T has 6 elements since

$$[S_4 : H] = 24/4 = 6$$

$$\mathrm{Ind}_H^G = \left(\Bbbk(123) \oplus \Bbbk(132) \oplus \ldots\right) \otimes \Bbbk e_1$$

For example:

$$(12)\cdot(123)\otimes e_1 = (132)\otimes \overbrace{(12)\cdot e_1}^{=-e_1}$$

$$= -(132)\otimes e_1$$

$$(12)(123) = (23) = (132)(12)$$

$$g \cdot t_i = t_j \cdot h$$