MATH 618 LECTURE 4

HW4 3.1.2(a) $\Bbbk[G\times H]\cong \Bbbk[G]\otimes \Bbbk[H]$

$\otimes$-Hom adjunction.

Any function $f: X\times Y \to Z$ of sets gives for each $x\in X$ a function $f_x : Y \to Z$, $f_x(y) = f(x,y)$ $(*)$

Conversely, given functions $f_x : Y\to Z$ $\forall x$, $(*)$ defines a function $f: X\times Y \to Z$.

$$\mathrm{Fun}(X\times Y, Z) \cong \mathrm{Fun}(X, \mathrm{Fun}(Y,Z)).$$

Linear version of this says:

$$\mathrm{Hom}_{\Bbbk}(U\otimes V, W) \cong \mathrm{Hom}_{\Bbbk}(U, \mathrm{Hom}(V,W)).$$

$$f \longmapsto (u \longmapsto (v \longmapsto f(u,v)))$$

$$(u\otimes v \longmapsto \varphi(u)(v)) \longleftarrow\!\shortmid \varphi$$

Modules vs. Representations.

A algebra.

M A-module with action
$\mu : A \otimes M \longrightarrow M$.

By $\otimes$-Hom adjunction,

$$\mathrm{Hom}_{\mathbb{k}}(A \otimes M, M) \cong \mathrm{Hom}_{\mathbb{k}}(A, \mathrm{End}_{\mathbb{k}}(M))$$

Hence μ corresponds to a linear map

$$\rho = \rho_M : A \longrightarrow \mathrm{End}_{\mathbb{k}}(M)$$

Properties of μ translates to saying ρ is an algebra map.

<u>Def</u> A representation of A is a vector space M together with an algebra map
$\rho = \rho_M : A \longrightarrow \mathrm{End}(M)$.

A **map of representations** $f : M \to N$ is a linear map such that for all $a \in A$:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\rho_M(a)"'] \arrow[dr, phantom, "\circlearrowleft"] & N \arrow[d, "\rho_N(a)"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Module notation:

$$a.m = \mu(a \otimes m)$$

Representation notation:

$$\rho(a)m$$

$$\boxed{a.m = \rho(a)m}$$

$(a \in A, m \in M)$

Example.

$$A = \Bbbk[x]/(x^2-1) = \Bbbk 1_A \oplus \Bbbk \bar{x}$$

Let $V = \Bbbk^n$ and $T : V \to V$ be a linear map such that $T^2 = \mathrm{Id}_V$. Define an algebra map

$$\tilde{\rho} : \Bbbk[x] \longrightarrow \mathrm{End}_{\Bbbk}(V)$$

by requiring that

$$\tilde{\rho}(x) = T.$$

(Here we use that $\Bbbk[x]$ is a free algebra on $\{x\}$)

$$\tilde{\rho}(x^2-1) = \tilde{\rho}(x)^2 - \rho(1) =$$

$$= T^2 - \mathrm{Id}_V = 0.$$

Hence $\tilde{\rho}$ induces an algebra map

$$\rho : A \longrightarrow End(V)$$

Thus (V, ρ) is a rep. of $A = \Bbbk 1_A \oplus \Bbbk \bar{x}$, $(\bar{x})^2 = 1_A$.

Viewing V as an A-module we have

$$a \cdot v = \rho(a) v$$

In particular

$$1_A \cdot v = \rho(1_A) v = Id_V v = v$$

$$\bar{x} \cdot v = \rho(\bar{x}) v = T v \, .$$

Group algebras.

Recall that for each set X we may form a vector space $\mathbb{k}X = \{$ finite formal linear combinations of elements of $X\}$. Moreover

$$\mathrm{Hom}_{\mathbb{k}}(\mathbb{k}X, V) \cong \mathrm{Hom}_{\underline{set}}(X, V)$$

(any set map from X into a vector space V extends uniquely to a linear map $\mathbb{k}X \to V$)

If $X = G$ is a group, we may introduce an algebra structure on $\mathbb{k}G$ as follows.

$$u_{\mathbb{k}G} : \mathbb{k} \longrightarrow \mathbb{k}G$$

$$1_{\mathbb{k}} \longmapsto 1_G$$

$$m_{\mathbb{k}G} : \mathbb{k}G \otimes \mathbb{k}G \longrightarrow \mathbb{k}G$$

$$g \otimes h \longmapsto gh$$

Explicitly

at most finitely many $\neq 0$

$$\left(\sum_{g\in G} \lambda_g g\right)\left(\sum_{h\in G} \mu_h h\right) =$$

$$= \sum_{k\in G} \left(\sum_{\substack{(g,h)\in G\times G \\ gh=k}} \lambda_g \mu_h \right) k$$

<u>Example</u> $C_n = \langle g \rangle$ cyclic group of order n

$$\mathbb{k} C_n = \mathbb{k} g^0 \oplus \mathbb{k} g^1 \oplus \cdots \oplus \mathbb{k} g^{n-1}$$
$$\cong \mathbb{k}[x] / (x^n - 1)$$

<u>Example</u>

$$\mathbb{k} S_3 = \mathbb{k}(1) \oplus \mathbb{k}(12) \oplus \cdots$$

Put $z = (12) + (13) + (23)$

Since $\{(13), (23)\}$ is the conjugacy class of transpositions in S_3 we have $\forall g \in S_3$:

$$g z g^{-1} = g(12)g^{-1} + g(13)g^{-1} + g(23)g^{-1}$$
$$= (12) + (13) + (23) = z$$

This implies that $z \in Z(\mathbb{k} S_3)$.

Universal property of $\Bbbk G$.

$$G \longmapsto \Bbbk G$$

is a functor $\left(\begin{array}{c} G \to H \\ \Rightarrow \Bbbk G \to \Bbbk H \end{array}\right)$

$$\underline{\mathrm{Grp}} \longrightarrow \underline{\mathrm{Alg}}_{\Bbbk}$$

In the other direction we have a functor.

$$(\cdot)^\times : \underline{\mathrm{Alg}}_{\Bbbk} \to \underline{\mathrm{Grp}}$$

$$A \longmapsto A^\times = \{a \in A \mid a \text{ is a } \underset{\substack{\| \\ \text{invertible.}}}{\text{unit}}\}$$

<u>Thm</u> For any group G and algebra A, there is a natural bijection

$$\mathrm{Hom}_{\mathrm{Alg}_{\Bbbk}}(\Bbbk G, A) \cong \mathrm{Hom}_{\mathrm{Grp}}(G, A^\times).$$

Pf $\quad \varphi \longmapsto \varphi|_G$ $\quad$ (note: $\varphi(G) \subset A^\times$ since G is a group)

$\mathbb{k}$-linear extension $\psi_{\mathbb{k}}$ of ψ $\longleftarrow\!\!\shortmid \psi$

Given an algebra map

$\varphi : \mathbb{k}G \longrightarrow A$,

φ is the $\mathbb{k}$-linear extension of $\varphi|_G$. So $\varphi = (\varphi|_G)_{\mathbb{k}}$.

Given a group map

$\psi : G \longrightarrow A^\times$

since $A^\times \subset A$, ψ extends to a linear map

$\psi_{\mathbb{k}} : \mathbb{k}G \longrightarrow A$.

Can check $\psi_{\mathbb{k}}$ is an alg. map. ∎

Application: $A = \mathrm{End}(V)$.

Note $\mathrm{End}(V)^\times = \mathrm{Aut}(V)$.

Thus

$$\mathrm{Hom}_{\mathrm{Alg}_{\Bbbk}}(\Bbbk G, \mathrm{End}(V)) \cong \mathrm{Hom}_{\mathrm{Grp}}(G, \mathrm{Aut}(V))$$

$$\underline{\mathrm{Rep}}\, \Bbbk G \simeq \underline{\mathrm{Rep}}\, G$$

We have seen earlier that

$$_{\Bbbk G}\underline{\mathrm{Mod}} \simeq \underline{\mathrm{Rep}}\, \Bbbk G$$

So $\underline{\mathrm{Rep}}\, G \simeq {}_{\Bbbk G}\underline{\mathrm{Mod}}$.

Example

If G acts on a set X then $\mathbb{k}X$ is a $\mathbb{k}G$-module via bilinear extension of

$$g.x = \text{action of } g \text{ on } x$$

i.e. $\left(\sum_{g \in G} \lambda_g g\right) \cdot \left(\sum_{x \in X} \alpha_x x\right) =$

$$= \sum_{\substack{g \in G \\ x \in X}} \lambda_g \alpha_x \, g.x$$

Example Take $X = G$ with left mult. of G on itself.

$\Rightarrow$ Regular representation.