MATH 618 LECTURE 3.

HW 3: 1.2.1 (a) (Kernels & Res_B^A)

READ: § 1.1.3, § 1.2.2 (restriction)

$$\underbrace{V \times V \times \cdots \times V}_{k} \longrightarrow \Lambda^k V$$

is multilinear and alternating:

- $v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots v_k =$

$= - v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_i \wedge \cdots \wedge v_k$

- $v_1 \wedge \cdots \wedge (\lambda v_i + \mu v_i') \wedge \cdots \wedge v_k =$

$= \lambda\, v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_k +$

$+ \mu\, v_1 \wedge \cdots \wedge v_i' \wedge \cdots \wedge v_k$

$\Rightarrow \Lambda^n(\mathbb{k}^n) = \mathbb{k} \cdot e_1 \wedge \cdots \wedge e_n$

If $f: V \to W$ is linear then $\Lambda(f): \Lambda(V) \to \Lambda(W)$ is given by

$$\Lambda(f)(v_1 \wedge \dots \wedge v_k) = f(v_1) \wedge \dots \wedge f(v_k).$$

$\Lambda(f)$ preserves degree so get $\Lambda^k(f) := \Lambda(f)|_{\Lambda^k(V)} : \Lambda^k(V) \to \Lambda^k(W)$.

In particular, if $A \in Mat_n(\mathbb{k})$ we get a map

$$\Lambda^n(A): \Lambda^n(\mathbb{k}^n) \to \Lambda^n(\mathbb{k}^n)$$

$$e_1 \wedge \dots \wedge e_n \mapsto (Ae_1) \wedge \dots \wedge (Ae_n)$$

Since $\Lambda^n(\mathbb{k}^n)$ is one-dim'l,

$$(Ae_1) \wedge \dots \wedge (Ae_n) = c_A \cdot e_1 \wedge \dots \wedge e_n$$

for some scalar $c_A \in \mathbb{k}$.

<u>Note</u>: The function

$$\begin{aligned} \mathrm{Mat}_n(\mathbb{k}) &\longrightarrow \mathbb{k} \\ A &\longmapsto c_A \end{aligned}$$

is multilinear & alternating in columns & sends $I_n \mapsto 1$

Hence $c_A = \det A$ & we get

$$\Lambda^n(A) = (\det A) \cdot \mathrm{Id}_{\Lambda^n(\mathbb{k}^n)}$$

The top exterior power of a linear endomorphism is multiplication by the determinant.

<u>Ex</u> $n=2$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Lambda^2(A)(e_1 \wedge e_2) = (Ae_1) \wedge (Ae_2)$$

$$= (a e_1 + c e_2) \wedge (b e_1 + d e_2)$$

$$= ab\, \cancel{e_1 \wedge e_1} + cb\, e_2 \wedge e_1$$

$$+ ad\, e_1 \wedge e_2 + cd\, \cancel{e_2 \wedge e_2}$$

$$= (ad - bc)\, e_1 \wedge e_2$$

1.1.3 Modules

A $\mathbb{K}$-algebra.

A (left) A-module is a vector space M with a linear map

$$\mu : A \otimes M \longrightarrow M$$
$$a \otimes m \longmapsto a.m$$

satisfying

$$\begin{tikzcd}
\mathbb{K} \otimes M \arrow[r, "u_A \otimes \mathrm{Id}"] \arrow[dr, "\cong"'] & A \otimes M \arrow[d, "\mu"] \\
& M
\end{tikzcd}$$

$(1_A . m = m \ \forall m \in M)$

and

$$\begin{tikzcd}
A \otimes A \otimes M \arrow[r, "m_A \otimes \mathrm{Id}"] \arrow[d, "\mathrm{Id}_A \otimes \mu"'] & A \otimes M \arrow[d, "\mu"] \\
A \otimes M \arrow[r, "\mu"] & M
\end{tikzcd}$$

$(a.(b.m) = (ab).m)$

An **A-module map** $f: M \to N$ is a linear map such that

$$\begin{array}{ccc} A \otimes M & \xrightarrow{\mu_M} & M \\ {\scriptstyle Id_A \otimes f}\downarrow & & \downarrow{\scriptstyle f} \\ A \otimes N & \xrightarrow{\mu_N} & N \end{array} \qquad \left(a \cdot f(m) = f(a \cdot m) \right)$$

An **A-submodule** N of an A-module M is a subspace such that

$$a \cdot n \in N \quad \forall a \in A \ \forall n \in N$$

The **kernel** (or **annihilator**) of an A-module M is

$$\mathrm{Ker}_A M = \{ a \in A \mid a \cdot m = 0 \ \forall m \in M \}$$

Check: $\mathrm{Ker}_A M$ is a 2-sided ideal of A.

Example. $M = \mathbb{k}^n$, $T \in Mat_n(\mathbb{k})$,
Define a $\mathbb{k}[x]$-module structure on M by $f(x).v = f(T)v$ $\forall p(x) \in \mathbb{k}[x]$, $\forall v \in M$. Then $\operatorname{Ker} M = (m_A(x))$.

min. pol. of A

Right A-modules are defined analogoesly.

If A, B are algebras, an (A,B)-bimodule M is

- a left A-module with ${}'\mu : A \otimes M \to M$
- a right B-module with $\mu' : M \otimes B \to M$

Such that $(a.m).b = a.(m.b)$ i.e.

$$\begin{tikzcd}
A \otimes M \otimes B \arrow[r, "{}'\mu \otimes Id_B"] \arrow[d, "Id_A \otimes \mu'"'] \arrow[dr, phantom, "2"] & M \otimes B \arrow[d, "\mu'"] \\
A \otimes M \arrow[r, "{}'\mu"] & M
\end{tikzcd}$$

Categories

${}_A\mathrm{Mod}$ left A-modules (& A-module maps)

Mod_A right A-modules

${}_A\mathrm{Mod}_B$ (A,B)-modules.

Note ${}_A\mathrm{Mod}_B \cong {}_{A\otimes B^{op}}\mathrm{Mod}$

$(a\otimes b).m := (a.m).b$

Ex. If $W \in {}_A\mathrm{Mod}$, $V \in {}_B\mathrm{Mod}$

Then $\mathrm{Hom}(V,W) \in {}_A\mathrm{Mod}_B$

via $(a.\varphi.b)(v) = a.(\varphi(b.v))$

1.2.2 Changing the algebra

Let $\varphi: A \to B$ be an algebra map, and M be a (left) B-module. Define an A-module structure on M by

$$a \cdot m := \varphi(a).m \quad \forall m \in M \; \forall a \in A.$$

using B-module structure.

This makes M an A-module:

$$1_A \cdot m = \varphi(1_A).m = 1_B.m = m \quad \forall m \in M$$

$$\begin{aligned} a_1 \cdot (a_2 \cdot m) &= \varphi(a_1).(\varphi(a_2).m) \\ &= (\varphi(a_1)\varphi(a_2)).m \\ &= \varphi(a_1 a_2).m = (a_1 a_2).m \end{aligned}$$

We denote this A-module by

$$\mathrm{Res}_A^B M \quad \text{or} \quad \varphi^* M$$

If $f : M \to N$ is a map of B-modules, the same function is a map of A-modules

$$\mathrm{Res}_A^B M \longrightarrow \mathrm{Res}_A^B N$$

Indeed

$$f(a.m) = f(\varphi(a).m)$$

$$= \varphi(a).f(m)$$

since f is a B-module map

$$= a.f(m)$$

by def of A-mod structure on $\mathrm{Res}_A^B N$.

Thus we have a functor

$$\varphi^* = \mathrm{Res}_A^B : {}_B\mathrm{Mod} \longrightarrow {}_A\mathrm{Mod}.$$

called the restriction functor along φ.

Remark If $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ are algebra maps then

$$(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$$

or $\mathrm{Res}_A^C = \mathrm{Res}_A^B \circ \mathrm{Res}_B^C$

Hides dependence of algebra maps.

explains reason for notation φ^*.