MATH 618 LECTURE 28

<u>HW 28</u> Find the dimension of the irrep

$$V(\underbrace{1,1,\ldots,1}_{k},\underbrace{0,0,\ldots,0}_{n-k})$$

of $\mathfrak{gl}_n$. $(0 \le k \le n)$

Gelfand-Tsetlin Theory for gl_n

① The center of $U(gl_n)$

Harish-Chandra map:

$$\varphi : U(gl_n) \longrightarrow U(\mathfrak{h})$$

projection along PBW basis

$$gl_n = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

Ex $n=3$

$$\varphi\left(E_{31}^{a_{31}} E_{32}^{a_{32}} E_{21}^{a_{21}} E_{11}^{a_{11}} E_{22}^{a_{22}} E_{33}^{a_{33}} E_{12}^{a_{12}} E_{23}^{a_{23}} E_{13}^{a_{13}}\right)$$

$$= \begin{cases} E_{11}^{a_{11}} E_{22}^{a_{22}} E_{33}^{a_{33}} & \text{if } a_{ij}=0 \ \forall i\neq j \\ 0 & \text{otherwise} \end{cases}$$

($\mathfrak{h}$ = all $n\times n$ diagonal matrices)
CSA of gl_n

<u>Notation</u>: $Z(\mathfrak{gl}_n) := Z(U(\mathfrak{gl}_n))$.

Harish-Chandra proved:

<u>Thm</u> $\varphi|_{Z(\mathfrak{gl}_n)}$ is an injective alg map with image

$U(\mathfrak{h})^{S_n}$ $\quad x_i := E_{ii} - i + 1$, $i = 1, \dots, n$

$\sigma(x_i) = x_{\sigma(i)}$ $\quad \forall \sigma \in S_n$, $i = 1, \dots, n$.

<u>Ex</u> $n = 2$

$$U(\mathfrak{h})^{S_2} = \mathbb{C}[x_1, x_2]^{S_2} =$$

$$= \mathbb{C}[x_1 + x_2, x_1 x_2]$$

$$x_1 + x_2 = E_{11} + E_{22} - 1$$

$$x_1 x_2 = E_{11}(E_{22} - 1)$$

Ex The Casimir element of $U(\mathfrak{gl}_n)$ is

$$C = \sum_{1 \le i,j \le n} E_{ij} E_{ji}$$

One can show $CE_{k\ell} = E_{k\ell} C \ \forall k,\ell$ in $U(\mathfrak{gl}_n)$.
hence C belongs to the center of $U(\mathfrak{gl}_n)$

For $n=2$ we have

$$C = E_{11}^2 + E_{12}E_{21} + E_{21}E_{12} + E_{22}^2$$

$\varphi(C) = ?$ We write C in the PBW basis wrt ordered basis $(E_{21}, E_{11}, E_{22}, E_{12})$

$$C = E_{11}^2 + E_{21}E_{12} + [E_{12}, E_{21}] + E_{21}E_{12} + E_{22}^2 =$$

$$= E_{11}^2 + 2E_{21}E_{12} + E_{11} - E_{22} + E_{22}^2$$

So $\varphi(C) = E_{11}^2 + E_{11} - E_{22} + E_{22}^2$

Expressing this in

$x_1 = E_{11}$, $x_2 = E_{22} - 1$

we have

$$\varphi(C) = x_1^2 + x_1 - (x_2+1) + (x_2+1)^2 =$$

$$= x_1^2 + x_1 - x_2 - 1 + x_2^2 + 2x_2 + 1$$

$$= x_1^2 + x_1 + x_2 + x_2^2$$ symmetric

$$\in \mathbb{C}[x_1, x_2]^{S_2}$$

Coumn determinant of a matrix $A = (a_{ij})$ with noncommuting entries is

$$\operatorname{cdet} A := \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma)\, a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n}$$

(expansion along leftmost column)

Capelli Determinant $\mathcal{C}(u) :=$

$$\operatorname{cdet}\begin{bmatrix} u+E_{11} & E_{12} & \cdots & E_{1n} \\ E_{21} & u+E_{22}-1 & & E_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ E_{n1} & E_{n2} & \cdots & u+E_{nn}-(n-1) \end{bmatrix}$$

Entries of matrix are from the noncomm. algebra $U(\mathfrak{gl}_n)[u]$.

Thus $\mathcal{C}(u) \in U(\mathfrak{gl}_n)[u]$.

<u>Thm</u> $\mathcal{C}(u) = u^n + z_1 u^{n-1} + \cdots + z_n$ where $z_k \in Z(\mathfrak{gl}_n)$ and

$$\varphi(z_k) = \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}$$

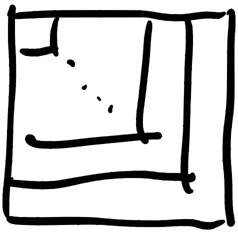
degree k elementary symmetric polynomial

Corollary

$Z(\mathfrak{gl}_n)$ is a polynomial algebra in $z_1, z_2, \ldots, z_n$.

Consider

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \cdots \subset \mathfrak{gl}_n$$



This gives a seq of subalgebras

$$U(\mathfrak{gl}_1) \subset U(\mathfrak{gl}_2) \subset \cdots \subset U(\mathfrak{gl}_n)$$

<u>Def</u> The Gelfand-Tsetlin (GZ) subalgebra of $U(\mathfrak{gl}_n)$ is

$$\Gamma = \mathbb{C}[Z_1, \ldots, Z_n] \text{ where}$$

$$Z_i = Z(\mathfrak{gl}_i) := Z(U(\mathfrak{gl}_i)).$$

Lemma

$$Z_1 \otimes Z_2 \otimes \cdots \otimes Z_n \longrightarrow \Gamma$$

is an isomorphism.

Cor. $\Gamma \cong \mathbb{C}[x_{ki} \mid 1 \leq i \leq k \leq n]$

polynomial algebra in $\frac{n(n+1)}{2}$ variables.

② Irreps of $\mathfrak{gl}_n$ are in 1-1 correspondence with

$$P_+ := \{ \lambda \in \mathbb{C}^n \mid \lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0} \ \forall i = 1, 2, .., n-1 \}$$

$$P_+ \cong \mathbb{C} \times (\mathbb{Z}_{\geq 0})^{n-1}$$

Ex $\lambda = (3, 2, 1) \rightsquigarrow V_\lambda$

$\lambda' = (3+\sqrt{2}, 2+\sqrt{2}, 1+\sqrt{2}) \rightsquigarrow V_{\lambda'}$

By Weyl's Thm, any fd rep of $\mathfrak{gl}_n$ is completely reducible.

Elements of P_+ are called dominant integral weights for $\mathfrak{gl}_n$.

The branching rule $\mathfrak{gl}_n \downarrow \mathfrak{gl}_{n-1}$.

For any fd irrep V_μ of $\mathfrak{gl}_{n-1}$ and fd irrep V_λ of $\mathfrak{gl}_n$

$$\dim \operatorname{Hom}_{\mathfrak{gl}_{n-1}}(V_\mu, V_\lambda\downarrow_{\mathfrak{gl}_2}) \le 1$$

with equality iff the following interleaving condition holds:

$$\mu \to \lambda : \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \cdots \geq \mu_{n-1} \geq \lambda_n$$

where $\underset{\alpha,\beta\in\mathbb{C}}{\alpha \geq \beta}$ means $\alpha-\beta \in \mathbb{Z}_{\geq 0}$

Ex $V(3,2,1)\downarrow_{\mathfrak{gl}_2} \cong V(3,2) \oplus V(3,1) \oplus$
$\oplus V(2,2) \oplus V(2,1)$

As with S_n we have a branching graph for $\{\mathfrak{gl}_n\}_{n=1}^{\infty}$. Part of this graph looks as follows:

$$\begin{tikzcd}
\vdots & & & & \\
\mathfrak{gl}_3 & V(0,0,0) & V(1,0,0) & V(1,1,0) & V(2,0,0) \\
\mathfrak{gl}_2 & V(0,0) \arrow[u] \arrow[ur] & V(1,0) \arrow[u] \arrow[ur] \arrow[urr] & V(2,0) \arrow[ur] & \cdots \\
\mathfrak{gl}_1 & V(0) \arrow[u] \arrow[ur] \arrow[urr] & V(1) \arrow[u] \arrow[ur] & V(2) \arrow[u] & \cdots
\end{tikzcd}$$

This can answer questions about bases for all fd irreps of $\mathfrak{gl}_n$: We have

$$V(\lambda_1, \lambda_2, \ldots, \lambda_n) \otimes V(c, c, \ldots, c)$$

$$\cong V(\lambda_1 + c, \lambda_2 + c, \ldots, \lambda_n + c) \quad \forall c \in \mathbb{C}$$
$$\forall \lambda \in P_+$$

so WLOG $\lambda_n = 0$

Ex A path in this branching graph can be encoded in a Gelfand-Tsetlin pattern: For ex

$$V(2,1,0) \leftarrow V(1,1) \leftarrow V(1) \quad \longleftrightarrow \quad \begin{array}{|c|c|c|}\hline 2 & 1 & 0 \\ \hline 1 & 1 & \\ \hline 1 & & \\ \hline \end{array}$$

We have

$$V(2,1,0) = \mathbb{C}\begin{smallmatrix}2&&1&&0\\&2&&1&\\&&2&&\end{smallmatrix} \oplus \mathbb{C}\begin{smallmatrix}2&&1&&0\\&2&&1&\\&&1&&\end{smallmatrix} \oplus$$

$$\oplus\, \mathbb{C}\begin{smallmatrix}2&&1&&0\\&2&&0&\\&&2&&\end{smallmatrix} \oplus \mathbb{C}\begin{smallmatrix}2&&1&&0\\&2&&0&\\&&1&&\end{smallmatrix} \oplus \mathbb{C}\begin{smallmatrix}2&&1&&0\\&2&&0&\\&&0&&\end{smallmatrix}$$

$$\oplus\, \mathbb{C}\begin{smallmatrix}2&&1&&0\\&1&&1&\\&&1&&\end{smallmatrix} \oplus \mathbb{C}\begin{smallmatrix}2&&1&&0\\&1&&0&\\&&1&&\end{smallmatrix} \oplus \mathbb{C}\begin{smallmatrix}2&&1&&0\\&1&&0&\\&&0&&\end{smallmatrix}$$

(In fact, $V(2,1,0) \cong \mathfrak{sl}_3$ with adjoint action of $\mathfrak{gl}_3$.)

<u>Note</u> As with S_n, the Gelfand-Tsetlin subalgebra $\Gamma \subset U(\mathfrak{gl}_n)$ acts diagonally in the GZ basis of any $V(\lambda)$.