MATH 618 LECTURE 27

READ §6.3 (CSAs, Root space decomp.)

§6.4.2, §6.4.3 (Classical Lie algs)

Thm 7.13

Prop 8.2(a)

HW 27 Let V be any representation of $\mathfrak{sl}_2$ and let

$$V^+ = \{ v \in V \mid \rho(e)v = 0 \ \& \ \rho(h)v \in \mathbb{k}v \}.$$

Show that $\forall v \in V^+$, $\sum_{k=0}^{\infty} \mathbb{k}\rho(f)^k v$ is a subrepresentation of V.

<u>Note</u> V^+ is called the subspace of primitive vectors of V.

Simple Lie algs.

V vector space with bilinear form $B : V \times V \to \mathbb{k}$.

$$\mathfrak{g}(V,B) = \{ x \in \mathfrak{gl}(V) \mid B(x.v,w) + B(v,x.w) = 0 \ \forall v,w \in V \}$$

$\bar{\mathbb{k}} = \mathbb{k}$, char $\mathbb{k} = 0$:

For B symmetric & nondegenerate

$so(V) := \mathfrak{g}(V,B)$ orthogonal Lie alg

Fixing an orthonormal basis for V,

$so(V) \cong so_n := \{ x \in \mathfrak{gl}_n \mid x^T = -x \}$

For B skew-symmetric & non-degenerate, ($\Rightarrow \dim V$ is even)

$sp(V) := \mathfrak{g}(V,B)$ symplectic Lie alg

Choosing appropriate basis for V,

$$sp(V) \cong sp_{2n} := \left\{ x = \left(\begin{array}{c|c} a & b = b^T \\ \hline c = c^T & -a^T \end{array} \right) \right\}$$

From now on: $(\overline{\mathbb{k}}=\mathbb{k},\ \mathrm{char}\,\mathbb{k}=0)$

Classification Thm

Every fin. dim simple Lie alg is isomorphic to exactly one of the following:

	Type:
• $\mathfrak{sl}_{n+1}$, $n \geq 1$	A_n
• $\mathfrak{so}_{2n+1}$, $n \geq 2$	B_n
• $\mathfrak{sp}_{2n}$, $n \geq 3$	C_n
• $\mathfrak{so}_{2n}$, $n \geq 4$	D_n
• five exceptional Lie algebras	E_6 E_7 E_8 F_4, G_2

Ex $G_2 = \mathrm{Der}\,\mathbb{O}$ Lie alg of derivations of the octonions.

Representation theory.

Def i) An abelian Lie subalg $\mathfrak{h} \subset \mathfrak{g}$ is toral (=torus-like) if $\operatorname{ad} \mathfrak{h} := \{\operatorname{ad} h \mid h \in \mathfrak{h}\}$ is simultaneously diagonalizable.

ii) A Cartan subalgebra (CSA) $\mathfrak{h} \subset \mathfrak{g}$ is maximal element (wrt inclusion) of the family of toral subalgebras.

Ex. Unitary group

$$U(n) = \{x \in GL_n(\mathbb{C}) \mid x^* x = 1\}$$

$$\cup$$

$$T(n) = \left\{ \begin{pmatrix} e^{it_1} & & \\ & \ddots & \\ & & e^{it_n} \end{pmatrix} \;\middle|\; t_i \in \mathbb{R} \right\}$$

$T(2)$ is homeomorphic to the two-torus $S^1 \times S^1$

$$\underbrace{\operatorname{Lie} T(n)}_{\text{a CSA of } U(n)} \cong \mathbb{R}^n \hookrightarrow \operatorname{Lie} U(n) = \underbrace{\{x \mid x^* + x = 0\}}_{\text{Skew-hermitian matrices}}$$

Ex $\mathfrak{g} = \mathfrak{sl}_n$

$$\mathfrak{h} = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \middle/ \sum a_i = 0 \right\} \subset \mathfrak{g}$$

$\mathfrak{h}$ is a CSA of $\mathfrak{sl}_n$.

Let $h_i = E_{ii} - E_{i+1,i+1}$, $i = 1, 2, \ldots, n-1$.
$\{h_i\}_{i=1}^{n-1}$ is a basis for $\mathfrak{h}$.

$$(\mathrm{ad}\, h_i)(E_{k\ell}) = [E_{ii} - E_{i+1,i+1}, E_{k\ell}]$$
$$= \underbrace{(\delta_{ik} - \delta_{\ell i} - \delta_{i+1,k} + \delta_{\ell,i+1})}_{=:\, \alpha_{k\ell}(h_i)} \cdot E_{k\ell}$$

Extending $\alpha_{k\ell}$ linearly to $\mathfrak{h}$ we have

$$(\mathrm{ad}\, h)(E_{k\ell}) = \alpha_{k\ell}(h) E_{k\ell} \quad \forall h \in \mathfrak{h}.$$

This shows that $\mathfrak{h}$ is a toral subalgebra of $\mathfrak{sl}_n$. One can show it is a CSA.

(cont.) $\longrightarrow$

Ex on $\mathfrak{sl}_n$ (contd.) We have

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid (\mathrm{ad}\, h)(x) = \alpha(h)x \ \forall h \in \mathfrak{h}\}$

and

$$\Phi = \{\alpha_{k\ell} \mid 1 \le k, \ell \le n,\ k \ne \ell\}$$

because

$$\mathfrak{g}_\alpha = \begin{cases} \mathbb{k} E_{k\ell}, & \alpha = \alpha_{k\ell} \in \Phi \\ 0, & \alpha \notin \Phi \end{cases}$$

Prop

Any simple f.d. Lie alg $\mathfrak{g}$ has a CSA $\mathfrak{h}$, unique up to an automorphism of $\mathfrak{g}$.

Root space decomposition

$\mathfrak{g}$ simple fd Lie alg $/ \mathbb{k}$

$\mathfrak{h} \subset \mathfrak{g}$ CSA

For $\alpha \in \mathfrak{h}^*$ put

$$\mathfrak{g}_\alpha = \{ x \in \mathfrak{g} \mid [h, x] = \alpha(h) x \ \forall h \in \mathfrak{h} \}$$

Then $\mathfrak{h} = \mathfrak{g}_0$ (Lemma) and hence

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha$$

$\alpha \neq 0$ is a root if $\mathfrak{g}_\alpha \neq 0$

$\Phi = \{\text{roots}\} \subset \mathfrak{h}^* \setminus \{0\}.$

Weight representations.

$\mathfrak{g}$ simple Lie alg

$\mathfrak{h}$ fixed CSA of $\mathfrak{g}$.

V a representation of $\mathfrak{g}$.

Def $\lambda \in \mathfrak{h}^*$ is a weight of V if $\exists v \in V \setminus \{0\}$ s.t.

$$h.v = \lambda(h)v \quad \forall h \in \mathfrak{h}.$$

Ex. The roots of $\mathfrak{g}$ are exactly the nonzero weights of the adjoint representation.

Put $V_\lambda = \{v \in V \mid h.v = \lambda(h)v \ \forall h \in \mathfrak{h}\}$

Def V is a weight representation if $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$.

V_λ are called weight spaces

Elements of V_λ are weight vectors (of weight λ).

<u>Prop</u> Any fin.-dim'l irrep V of $\mathfrak{g}$ is a weight representation.

<u>Proof</u> $\{\rho(h) \mid h \in \mathfrak{h}\}$ is a family of commuting linear operators on V hence has a common eigenvector $v \neq 0$. So

$\rho(h)v = \lambda(h)v$ for some function $\lambda : \mathfrak{h} \to \mathbb{k}$. Now ρ is linear $\Rightarrow \lambda$ is linear i.e. $\lambda \in \mathfrak{h}^*$.

Put $$V' = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu .$$

We just saw $V' \neq 0$. We claim V' is a subrep of V.

By root space decomposition it suffices to show

$$\rho(\mathfrak{g}_\alpha) V_\mu \subset V'$$

$\forall \alpha, \mu \in \mathfrak{h}^*$. We have

$$\forall h \in \mathfrak{h}, \forall x \in \mathfrak{g}_\alpha, \forall v \in V_\mu :$$

$$\rho(h)\rho(x)v = \rho(x)\rho(h)v + \rho([h,x])v =$$

$$= \mu(h)\rho(x)v + \alpha(h)\rho(x)v$$

$$= (\mu+\alpha)(h) \cdot \rho(x)v.$$

Thus $\rho(\mathfrak{g}_\alpha) V_\mu \subseteq V_{\mu+\alpha}$.

So V' is a nonzero subrep.

By irreducibility $V' = V$.

QED.

Triangular decompositions:

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

as vector spaces, where

$\mathfrak{h}$ is a CSA

$\mathfrak{n}_\pm$ are Lie subalgebras

such that

$$[\mathfrak{h}, \mathfrak{n}_\pm] \subseteq \mathfrak{n}_\pm$$

$$\mathrm{ad}\, x\big|_{\mathfrak{g}} \text{ are nilpotent } \forall x \in \mathfrak{n}_\pm$$

Thm Every simple f.d. Lie alg $\mathfrak{g}$ has a triangular decomp. Moreover $\exists$ decomp $\Phi = \Phi_+ \sqcup \Phi_-$ of the roots such that

$$\mathfrak{n}_\pm = \bigoplus_{\alpha \in \Phi_\pm} \mathfrak{g}_\alpha$$

Ex.

$$\mathfrak{sl}_n = \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ * & & 0 \end{pmatrix} \oplus \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \oplus \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$$

$$\mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

$$\mathfrak{n}_+ = \bigoplus_{i<j} \mathbb{k} E_{ij}$$

$$\mathfrak{n}_- = \bigoplus_{i>j} \mathbb{k} E_{ij}$$

Def A highest weight representation is a weight rep V generated by a vector v satisfying

$$\rho(\mathfrak{n}_+)\, v = 0$$

The weight of v is called the heighest weight of V.

thm Any f.d. irrep of $\mathfrak{g}$ is a highest weight rep and is characterized up to equivalence by its highest weight.

Ex. Possible highest weights for fd irreps of $\mathfrak{sl}_n$ are

$$\{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0} \ \forall i = 1, \dots, n-1\}$$

where $h_i = E_{ii} - E_{i+1,i+1}$.

Thus

$$\operatorname{Irr} \mathfrak{sl}_n \longleftrightarrow (\mathbb{Z}_{\geq 0})^{n-1}$$

In general

$$\operatorname{Irr} \mathfrak{g} \longleftrightarrow (\mathbb{Z}_{\geq 0})^{\operatorname{rk} \mathfrak{g}}$$

where $\operatorname{rk} \mathfrak{g}$ = rank of $\mathfrak{g}$

:= dim of a CSA.