MATH 618 LECTURE 26

READ: §5.4

HW26: Exercise 5.4.4(c).

Universal enveloping algebra

Thm-Def

The functor $\underline{Alg} \longrightarrow \underline{Lie}$ sending an associative algebra A to the Lie algebra A_{Lie} ($=A$ with bracket $[a,b]=ab-ba$) has a left adjoint, denoted

$$U: \underline{Lie} \longrightarrow \underline{Alg}$$
$$\mathfrak{g} \longmapsto U(\mathfrak{g}).$$

The associative algebra $U(\mathfrak{g})$ is called the universal enveloping algebra of $\mathfrak{g}$.

Thus for every Lie alg $\mathfrak{g}$ and assoc. alg A there is a natural isomorphism

$$\mathrm{Hom}_{\underline{Alg}}(U(\mathfrak{g}), A) \cong \mathrm{Hom}_{Lie}(\mathfrak{g}, A_{Lie}).$$

Proof Define

$$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/I$$ where

I is the ideal generated by

$$\{x \otimes y - y \otimes x - [x,y] \mid x, y \in \mathfrak{g}\}.$$

Consider a Lie algebra map

$$f : \mathfrak{g} \longrightarrow A_{Lie}$$

Since f is a linear map to (the underlying vector space of) an associative algebra, f induces an algebra map

$$\tilde{f} : T(\mathfrak{g}) \to A \quad \text{s.t. } \tilde{f}|_{\mathfrak{g}} = f.$$

We have

$$\tilde{f}(x \otimes y - y \otimes x - [x,y]) =$$

$$\tilde{f}(x)\tilde{f}(y) - \tilde{f}(y)\tilde{f}(x) - \tilde{f}([x,y]) =$$

$= f(x)f(y) - f(y)f(x) - f([x,y])$

$= [f(x), f(y)]_{A_{Lie}} - f([x,y]_{g}) = 0$

since f is a Lie alg map.

Thus $\tilde{f}$ induces an alg map

$$F : U(g) \longrightarrow A$$

satisfying

$$F(x+I) = f(x) \quad \forall x \in g.$$

Conversely, given an alg map

$$F : U(g) \to A$$

define $f : g \to A_{Lie}$ by

$$f(x) = F(x+I) \quad \forall x \in g$$

Then f is linear and

$f([x,y]) = F([x,y]+I) = \underbrace{F(x \otimes y - y \otimes x + I)}_{\text{by def of } I}$

$= F(x+I)F(y+I) - F(y+I)F(x+I) =$

$= [f(x), f(y)]_{A_{Lie}} \quad \forall x, y \in \mathfrak{g}.$

Hence f is a Lie alg map.
Clearly $f \mapsto F$, $F \mapsto f$ are inverses of each other.
Naturality is left as an exercise.

QED.

Note If $\mathcal{B}$ is a basis for $\mathfrak{g}$, then $T(\mathfrak{g}) \cong \underbrace{\Bbbk\langle \mathcal{B} \rangle}_{\text{free alg.}}$

and $I = \langle x \otimes y - y \otimes x - [x,y] \mid x, y \in \mathcal{B} \rangle$

Ex. $U(sl_2) = \dfrac{\Bbbk\langle e, f, h \rangle}{I}$

where

$I = \langle ef - fe - h,\ he - eh - 2e,\ hf - fh + 2f \rangle$

Note Since $\mathfrak{gl}(V) = \mathrm{End}(V)_{Lie}$ by definition, we have

$$\mathrm{Hom}_{Lie}(\mathfrak{g}, \mathfrak{gl}(V)) \cong \mathrm{Hom}_{Alg}(U(\mathfrak{g}), \mathrm{End}(V))$$

$$\Rightarrow \quad \mathrm{Rep}\,\mathfrak{g} \cong \mathrm{Rep}\,U(\mathfrak{g})$$

Lie algebra representations of $\mathfrak{g}$ are equivalent to associative algebra representations of $U(\mathfrak{g})$.

Note Picking a total order on a basis $\mathcal{B}$ for $\mathfrak{g}$, it's easy to see that $U(\mathfrak{g})$ is spanned by ordered monomials in $\mathcal{B}$:

$$U(\mathfrak{g}) = \mathrm{Span}_{\Bbbk}\left\{ \bar{x}_1 \cdots \bar{x}_n \;\middle|\; \begin{matrix} x_i \in \mathcal{B} \\ x_1 \leq \cdots \leq x_n \end{matrix} \right\}$$

where $\bar{x}_i := x_i + I$

$$\left(\begin{array}{l}\text{If } \mathcal{B} = \{x_1, \dots, x_n\} \text{ finite} \\ U(\mathfrak{g}) = \mathrm{Span}_k\{\bar{x}_1^{k_1} \cdots \bar{x}_n^{k_n} \mid k_i \in \mathbb{Z}_{\geq 0}\}\end{array}\right)$$

Indeed $\bar{x}\bar{y} = \bar{y}\bar{x} + \overline{[x,y]}$

implies that

$$\bar{x}_1 \cdots \bar{x}_i \bar{x}_{i+1} \cdots \bar{x}_n =$$
$$= \bar{x}_1 \cdots \bar{x}_{i+1} \bar{x}_i \cdots \bar{x}_n +$$
$$+ \bar{x}_1 \cdots \underbrace{\overline{[x_i, x_{i+1}]}}_{\text{expand in } \mathcal{B}} \cdots \bar{x}_n$$

Formally, put

$$U(\mathfrak{g})_{(n)} = \mathrm{Span}\left\{\bar{x}_1 \cdots \bar{x}_k \;\middle|\; \begin{array}{c} 0 \leq k \leq n \\ x_i \in \mathfrak{g} \end{array}\right\}$$

Then $U(\mathfrak{g})_{(0)} \subset U(\mathfrak{g})_{(1)} \subset \cdots$

$$U(\mathfrak{g}) = \bigcup_{n \geq 0} U(\mathfrak{g})_{(n)} \text{ and}$$

$$U(\mathfrak{g})_{(m)} U(\mathfrak{g})_{(n)} \subset U(\mathfrak{g})_{(m+n)}$$

i.e. $\{U(\mathfrak{g})_{(n)}\}_{n=0}^{\infty}$ is a filtration of $U(\mathfrak{g})$.

Previous calculation implies

$$\bar{x}_1 \cdots \bar{x}_n + U(\mathfrak{g})_{(n-1)} = \bar{x}_{\sigma(1)} \cdots \bar{x}_{\sigma(n)} + U(\mathfrak{g})_{(n-1)}$$

$\forall \sigma \in S_n \quad \forall x_i \in \mathfrak{g}$.

This means that we have an algebra epimorphism

$$\varphi : S(\mathfrak{g}) \longrightarrow \underbrace{\mathrm{gr}\, U(\mathfrak{g}) := \bigoplus_{n=0}^{\infty} \frac{U(\mathfrak{g})_{(n)}}{U(\mathfrak{g})_{(n-1)}}}_{\text{associated graded algebra}}$$

where $U(\mathfrak{g})_{(-1)} := \{0\}$,

given by

$$\varphi : x_1 \cdots x_n \longmapsto \bar{x}_1 \cdots \bar{x}_n + U(\mathfrak{g})_{(n-1)}$$

$\forall x_i \in \mathfrak{g}, \forall n \geq 0$

PBW Thm

ψ is an algebra isomorphism.

Proof Uses Diamond Lemma, see book for details. The crucial step is that ambiguities are resolvable.

$$\begin{tikzcd}
 & zyx \arrow[dl] \arrow[dr] & \\
yzx+[z,y]x \arrow[d] & & zxy+z[y,x] \arrow[d] \\
yxz+y[z,x]+[z,y]x \arrow[d] & & xzy+[x,z]y+z[y,x] \arrow[d] \\
xyz+[y,x]z+y[z,x]+[z,y]x \arrow[rr, phantom, "="] & & xyz+x[z,y]+[x,z]y+z[y,x]
\end{tikzcd}$$

By Jacobi Id.

∎

$\underline{\text{Ex}}$ $U(\mathfrak{sl}_2) = \bigoplus_{k,l,m \geq 0} \mathbb{C} f^k h^l e^m$

$$ef = fe + h$$

$$\begin{aligned} h^2 f &= h(fh + [hf]) = \underline{hf}h - 2hf \\ &= (fh + [hf])h - 2hf \\ &= fh^2 - 2\underline{fh} - 2hf \\ &= fh^2 - 2(hf - 2f) - 2hf \\ &= fh^2 - 4hf + 4f \end{aligned}$$

<u>Warning</u> Product in $U(\mathfrak{g})$ is <u>not</u> matrix multiplication but juxtaposition of <u>words</u> (subject to relations).

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 = 0$ in $M_2(\mathbb{k})$ but

$e^2 = ee$ is a nonzero word in $U(\mathfrak{sl}_2)$.