MATH 618 LECTURE 25

READ: 5.7.1 – 5.7.3

HW 25: Exercise 5.2.6.

<u>Def</u> A Lie alg $\mathfrak{g}$ is simple if 1) $\mathfrak{g}$ is non-abelian and 2) $\mathfrak{g}$ has no proper non-zero ideals.

<u>Note</u> 1) can be replaced by

1′) $\dim \mathfrak{g} > 1$

Indeed if 1′) and 2) hold but $\mathfrak{g}$ was abelian, then any nonzero $x \in \mathfrak{g}$ would span an ideal, hence $\mathfrak{g} = \Bbbk x$ by 2), contradicting 1′).
Conversely, 1) ⟹ 1′) because any one-dim'l Lie alg is abelian.

Def A $\mathfrak{g}$-module is a vector space V with a bilinear map
$$\mathfrak{g} \times V \to V, \quad (g,v) \mapsto g.v$$
satisfying
$$x.(y.v) - y.(x.v) = [x,y].v$$
$\forall x,y \in \mathfrak{g}\ \forall v \in V$.

(*) $\boxed{\rho(x)v = x.v}$ $\quad \forall x \in \mathfrak{g}, v \in V$

Given a rep $(V, \rho : \mathfrak{g} \to \mathfrak{gl}(V))$
(*) defines a $\mathfrak{g}$-module structure on V. (RHS := LHS)

Conversely, given a $\mathfrak{g}$-module $(V, \cdot : \mathfrak{g} \times V \to V)$, (*) defines a representation ρ (LHS := RHS)

$\boxed{\mathrm{Rep}\,\mathfrak{g} \cong \mathfrak{g}\text{-}\underline{\mathrm{Mod}}}$

<u>Ex</u> $\mathfrak{g}$ is a $\mathfrak{g}$-module via

$$x.y := [x,y] \quad \forall x,y \in \mathfrak{g}.$$

Let us check this: $\forall x,y,z \in \mathfrak{g}$:

$$\begin{aligned} x.(y.z) - y.(x.z) &= [x,[y,z]] - [y,[x,z]] = \\ &= [x,[y,z]] + [y,[z,x]] \quad \text{(anticommutativity)} \\ &= -[z,[x,y]] \quad \text{(Jacobi Identity)} \\ &= [[x,y],z] \quad \text{(anticomm.)} \\ &= [x,y].z \quad \text{(def. of action)} \end{aligned}$$

This $\mathfrak{g}$-module is called the adjoint $\mathfrak{g}$-module.

The corresponding rep is

$$\rho : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

$$\rho(x)y = [x,y] \quad \forall x,y \in \mathfrak{g}.$$

and is called the adjoint representation of $\mathfrak{g}$.

Note The kernel of the adjoint rep is $\{x \in \mathfrak{g} \mid [x,y]=0 \ \forall y \in \mathfrak{g}\} = \mathfrak{z}(\mathfrak{g})$ the center of $\mathfrak{g}$.

Special linear Lie alg.

The trace may be regarded as a Lie alg map

$$tr: \mathfrak{gl}_n \to \mathbb{k}_{Lie}$$

where $\mathfrak{gl}_n = M_n(\mathbb{k})_{Lie}$.

$$tr([x,y]) \underset{\substack{\uparrow \\ \text{def of bracket} \\ \text{in } M_n(\mathbb{k})_{Lie}}}{=} tr(xy-yx) = \underbrace{tr(xy)-tr(yx)}_{=0}$$

while $[tr\, x, tr\, y]=0$ since $\mathbb{k}_{Lie}$ is abelian.

Def The special linear Lie alg is $\mathfrak{sl}_n = \ker(tr) = \{x \in \mathfrak{gl}_n \mid tr(x)=0\}$

Remark One can show that $\mathfrak{sl}_n = \mathrm{Lie}\, SL_n$. This is related to the matrix identity

$$\exp(\mathrm{tr}\, A) = \det(\exp(A))$$

where $\exp(A) = \sum_{n \geq 0} \frac{1}{n!} A^n$

(Writing $A = D + N$, D = diagonalizable, N = nilpotent, $DN = ND$, we reduce to the cases $A = D$, $A = N$)

$$\mathfrak{sl}_2 = \Bbbk e \oplus \Bbbk h \oplus \Bbbk f$$

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$[h, e] = 2e, \; [h f] = -2f, \; [e, f] = h$$

Proposition: $\mathfrak{sl}_n$ is a simple Lie algebra. $\boxed{\text{char}\,\Bbbk = 0}$

Proof when $n=2$:

$[e,f] = h \neq 0$ so $\mathfrak{sl}_2$ is non-abelian. Let $\mathfrak{a} \subseteq \mathfrak{sl}_2$ be a nonzero ideal. We show that $\mathfrak{a} = \mathfrak{sl}_2$. We consider 3 cases

① $e \in \mathfrak{a}$

Then $\underbrace{[f,e]}_{= -[e,f] = -h} \in [\mathfrak{sl}_2, \mathfrak{a}] \subseteq \mathfrak{a} \Rightarrow h \in \mathfrak{a}$

and $\underbrace{[f,h]}_{= -[h,f] = 2f} \in [\mathfrak{sl}_2, \mathfrak{a}] \subseteq \mathfrak{a} \Rightarrow f \in \mathfrak{a}$

$\Rightarrow \mathfrak{a} = \mathfrak{sl}_2$

② $\lambda h + \mu e \in \mathfrak{a}$, some $\lambda, \mu \in \Bbbk$, $\lambda \neq 0$.

Then $\underbrace{[e, \lambda h + \mu e]}_{= -2\lambda e} \in \mathfrak{a} \Rightarrow e \in \mathfrak{a}$, done by ①.

③ $\lambda f + \mu h + \nu e \in \mathfrak{a}$, for some $\lambda, \mu, \nu \in \mathbb{k}$, $\lambda \neq 0$. Then

$$[e, \lambda f + \mu h + \nu e] \in \mathfrak{a}$$

$$\parallel$$

$$\lambda h - 2\mu e \Rightarrow \text{Done by ②.}$$

QED

Some sl_2-modules.

For $n \in \mathbb{Z}_{\geq 0}$ let

$$V(n) = \bigoplus_{k=0}^{n} \Bbbk x^k y^{n-k} = \begin{array}{l}\text{space of}\\ \text{homogeneous}\\ \text{polynomials}\\ \text{in } x, y \text{ of}\\ \text{degree } n.\end{array}$$

$\dim V(n) = n+1.$

Define a linear map

$\rho : sl_2 \to gl(V(n))$ by

$$e \mapsto x\frac{\partial}{\partial y}$$

$$f \mapsto y\frac{\partial}{\partial x}$$

$$h \mapsto x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

<u>Proposition</u> The map ρ is a Lie alg map $sl_2 \to gl(V(n))$ i.e. a rep of sl_2. Thus $V(n)$ is an sl_2-module.

Theorem

1) $V(n)$ is a simple sl_2-module for every $n \in \mathbb{Z}_{\geq 0}$.
2) Every finite-dimensional simple sl_2-module is isomorphic to $V(n)$ for some $n \in \mathbb{Z}_{\geq 0}$.
3) Every finite-dimensional sl_2-module is a direct sum of simple sl_2-modules.