MATH 618 LECTURE 24

READ §5.1

HW24: Exercise 5.1.6 (Witt algebra)

Def A **Lie algebra** (over a field $\mathbb{k}$) is a vector space $\mathfrak{g}$ with a bilinear operation $[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ called the **bracket** on $\mathfrak{g}$, satisfying

$$[x,x]=0 \quad \forall x\in\mathfrak{g} \quad \text{(Alternating Law)}$$

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0 \quad \text{(Jacobi Identity)}$$

$\forall x,y,z\in\mathfrak{g}$.

Def A **Lie algebra map** $\varphi:\mathfrak{g}\to\mathfrak{h}$ is a linar map s.t. $\varphi([x,y]_{\mathfrak{g}})=[\varphi(x),\varphi(y)]_{\mathfrak{h}}$ $\forall x,y\in\mathfrak{g}$.

$\underline{\mathrm{Lie}}_{\mathbb{k}}$ category of Lie algebras and Lie alg maps.

Note $0=[x+y,x+y]=[x,y]+[y,x]$
$\Rightarrow [x,y]=-[y,x] \quad \forall x,y\in\mathfrak{g} \quad (*)$

If $\operatorname{char}\mathbb{k}\neq 2$, $(*)$ implies Alternating Law.

Ex $\mathbb{R}^3$ with cross product is a Lie algebra.

Ex Any $\mathbb{K}$-vector space V becomes a Lie alg by defining
$[u,v] := 0 \quad \forall u,v \in V$.

Def A Lie alg $\mathfrak{g}$ is **abelian** if $[x,y]=0 \quad \forall x,y \in \mathfrak{g}$.

Notation For any subsets $X,Y \subseteq \mathfrak{g}$ we put $[X,Y] = \mathrm{Span}_{\mathbb{K}}\left\{[x,y] \,\middle|\, \begin{smallmatrix} x\in X \\ y \in Y \end{smallmatrix}\right\}$

Def A subspace $\mathfrak{a} \subseteq \mathfrak{g}$ is a

- (Lie) **subalgebra** if $[\mathfrak{a},\mathfrak{a}] \subseteq \mathfrak{a}$
- (Lie) **ideal** if $[\mathfrak{g},\mathfrak{a}] \subseteq \mathfrak{a}$.

Note $[X,Y]=[Y,X]$ $\forall$ subsets $X,Y \subseteq \mathfrak{g}$.
Subalgebras are Lie algebras wrt. restriction of bracket. Ideals are subalgs.

Def If $\mathfrak{a} \subseteq \mathfrak{g}$ is an ideal, the quotient space $\mathfrak{g}/\mathfrak{a}$ is naturally a Lie alg wrt.

$$[x+\mathfrak{a}, y+\mathfrak{a}] = [x,y] + \mathfrak{a}$$

$\forall x,y \in \mathfrak{g}$.

Isomorphism Thm If $\varphi: \mathfrak{g} \to \mathfrak{h}$ is a Lie alg map then $\varphi(\mathfrak{g})$ is a subalg of $\mathfrak{h}$, $\ker\varphi$ is an ideal of $\mathfrak{g}$, and $\mathfrak{g}/\ker\varphi \cong \varphi(\mathfrak{g})$ via

$$x + \ker\varphi \mapsto \varphi(x) \quad \forall x \in \mathfrak{g}.$$

Def The center of a Lie alg $\mathfrak{g}$ is $\mathfrak{z}(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x,y] = 0 \ \forall y \in \mathfrak{g}\}$

(lower case script z)

Note: $[\mathfrak{g}, \mathfrak{z}(\mathfrak{g})] = 0 \subseteq \mathfrak{z}(\mathfrak{g}) \Rightarrow \mathfrak{z}(\mathfrak{g})$ is an ideal of $\mathfrak{g}$.

Def The derived subalg $\mathfrak{g}'$ of a Lie alg is $\mathfrak{g}' := [\mathfrak{g},\mathfrak{g}]$

Note: $[\mathfrak{g},\mathfrak{g}'] \subseteq [\mathfrak{g},\mathfrak{g}] = \mathfrak{g}'$

since $\mathfrak{g}' \subseteq \mathfrak{g}$

so $\mathfrak{g}'$ is also an ideal of $\mathfrak{g}$.

If $\mathfrak{a}$ is any ideal of $\mathfrak{g}$, then

$\mathfrak{g}/\mathfrak{a}$ is abelian

$\iff [x,y] \in \mathfrak{a}$ for all $x,y \in \mathfrak{g}$

$\iff \mathfrak{g}' \subseteq \mathfrak{a}$.

The functor $\underline{Alg}_{\mathbb{k}} \to \underline{Lie}_{\mathbb{k}}$

Let A be an associative alg. Define a bracket on the underlying vector space of A by

$$[a,b] = ab - ba \quad \forall a,b \in A.$$

Clearly $[a,a] = 0 \ \forall a \in A$.
We further have $\forall a,b,c \in A$:

$$\begin{aligned}[a,bc] &= abc - bca \\ &= abc - bac + bac - bca \\ &= [a,b]c + b[a,c]\end{aligned}$$

hence

$$\begin{aligned}[a,[b,c]] &= [a,b]c + b[a,c] \\ &\quad - [a,c]b - c[a,b] \\ &= [[a,b],c] + [b,[a,c]] = \\ &= -[c,[a,b]] - [b,[c,a]]\end{aligned}$$

$\Rightarrow$ Jacobi identity holds.

Thus $(A, [\cdot,\cdot])$ is a Lie alg. We denote it by A_{Lie}.

<u>Note</u> $\mathfrak{z}(A_{Lie}) = Z(A)$ (as v. spaces) hence A_{Lie} is abelian iff A is commutative.

<u>Ex</u> $A = \mathrm{End}_{\mathbb{k}}(V)$, V vector sp.

The Lie alg A_{Lie} is denoted $\mathfrak{gl}(V)$ and is called the general linear Lie algebra.

<u>Def</u> A representation of a Lie alg $\mathfrak{g}$ is a Lie alg map

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V).$$