MATH 618 LECTURE 23

HW23: Prove that the gradient of the determinant function $\det : M_n(\mathbb{R}) = \mathbb{R}^{n^2} \longrightarrow \mathbb{R}$ is nonzero on every matrix of determinant 1. Conclude that

$$SL_n(\mathbb{R}) = \{ A \in M_n(\mathbb{R}) \mid \det A = 1 \}$$

is a manifold.

Lie groups.

<u>Def</u> A (real) Lie group G is a group which is also a manifold such that

$$G \times G \to G \quad (g,h) \mapsto gh$$
$$G \to G \quad g \mapsto g^{-1}$$

are smooth. (Throughout, smooth mean real analytic)

<u>Facts from diff geom/Calculus</u>:

1) Any open subset of $\mathbb{R}^n$ is a manifold

2) If $f : \mathbb{R}^n \to \mathbb{R}^m$ is smooth and $p \in f(\mathbb{R}^n)$ is a regular value i.e. the Jacobian $\left(\frac{\partial f_i}{\partial x_j}\right)_{ij}$ has full rank ($= \min\{m,n\}$) at every point in $f^{-1}(\{p\})$, then $M := f^{-1}(\{p\})$ is a manifold.

Ex. $GL_n(\mathbb{R})$ general linear group
$= \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\}$

$SL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det A = 1\}$
Special linear group

Consider

$$\det : M_n(\mathbb{R}) = \mathbb{R}^{n^2} \longrightarrow \mathbb{R}$$
$$A \longmapsto \det(A)$$

which is smooth (being a pol. fcn)

Then

$$GL_n(\mathbb{R}) = \det^{-1}(\underbrace{\mathbb{R} \setminus \{0\}}_{\text{open}})$$

$\Rightarrow GL_n(\mathbb{R})$ open subset of $\mathbb{R}^{n^2}$
$\Rightarrow GL_n(\mathbb{R})$ is a manifold.

$SL_n(\mathbb{R}) = \det^{-1}(\{1\})$ can check 1 is a regular value
$\Longrightarrow SL_n(\mathbb{R})$ is a manifold.

Moreover multiplication & inverse matrix are given by polynomials (resp. rational fcns) which are smooth.

$\Rightarrow GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$ are Lie groups.

One-parameter subgroups are smooth group homs

$$\gamma : \mathbb{R} \to G$$

Then
$$\begin{cases} \gamma(0) = 1_G \\ \gamma(s+t) = \gamma(s)\gamma(t) \\ \gamma(-s) = \gamma(s)^{-1} \end{cases} \quad (*)$$

$\forall s, t \in \mathbb{R}$

Ex. $s \mapsto \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$, $s \mapsto \begin{bmatrix} e^s & 0 \\ 0 & e^{-s} \end{bmatrix}$

are one-parameter subgroups of $SL_2(\mathbb{R})$ $\quad (e = 2.718\ldots\ldots)$

One-parameter subgroups of matrix Lie groups can be expanded as

$$\gamma(s) = X_0 + X_1 s + X_2 s^2 + \dots$$

By properties ⊛ one checks that $X_0 = I_n$, $X_2 = \frac{1}{2} X_1^2$, ...

i.e. $\gamma(s) = \exp(Xs)$ for some matrix $X \in M_n(\mathbb{R})$.

In fact, this illustrates:

If $\gamma : \mathbb{R} \to G$ is any smooth function then there is a unique one-parameter subgroup $\tilde{\gamma} : \mathbb{R} \to G$ such that $\frac{d\tilde{\gamma}}{dt} = \frac{d\gamma}{dt}$.

Explicitly, $\tilde{\gamma}(t) = \exp\left(t \cdot \frac{d\gamma}{dt}\right)$.

Def The Lie algebra of a matrix group $G \subset M_n(\mathbb{R})$ is

$$\mathfrak{g} = \mathrm{Lie}\, G = \left\{ \frac{d\gamma}{dt} \,\middle/\, \gamma : \mathbb{R} \to G \text{ smooth fcn.} \right\}$$

$$= \left\{ X \in M_n(\mathbb{R}) \,\middle|\, \exp(tX) \in G \ \forall t \in \mathbb{R} \right\}$$

Note 1) $\mathfrak{g}$ is closed under

$$X \mapsto \lambda X, \quad \forall \lambda \in \mathbb{R} :$$

Indeed, if $X \in \mathfrak{g}$ then $\exists \gamma : \mathbb{R} \to G$:

$$\gamma(t) = X_0 + Xt + \cdots$$

Define $\gamma_\lambda(t) = \gamma(\lambda t)$. Then γ_λ is smooth & $\frac{d\gamma_\lambda}{dt} = \lambda X$

2) $\mathfrak{g}$ is closed under addition:

Let $X, Y \in \mathfrak{g}$. Let

$$\gamma(t) = 1 + Xt + \cdots$$

$$\delta(t) = 1 + Yt + \cdots$$

Then $t \mapsto \gamma(t)\delta(t)$ is smooth and $\gamma(t)\delta(t) = 1 + (X+Y)t + \cdots \Rightarrow X+Y \in \mathfrak{g}$.

3) $\mathfrak{g}$ is closed under the bracket $[x,y] = xy - yx$:

Let $x, y \in \mathfrak{g}$. Let

$$f(s,t) = e^{tx} e^{sy} e^{-tx} e^{-sy}$$

Then γ is a smooth function $\mathbb{R} \times \mathbb{R} \to G$.

Direct computation shows

$$f(s,t) = 1 + (xy - yx)st + \dots$$

$$= \exp(st[x,y])$$

Put $\gamma(s) = f(1,t)$. Then γ is a smooth fcn $\mathbb{R} \to G$ with $\frac{d\gamma}{dt} = [x,y]$.

Hence $[x,y] \in \mathfrak{g}$.