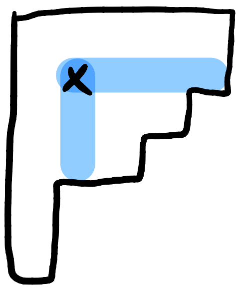
MATH 618 LECTURE 22

READ §4.3.5, §4.3.6

HW22: Exercise 4.6.1(a)

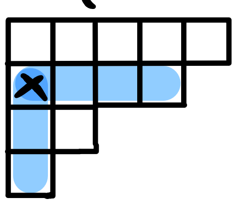
Hook-length formula

The **hook** at location x in a Young diagram λ is the set of boxes to the right of x, directly below x, and x itself



The **hook length** at x is

$$h(x) = h_\lambda(x) = \#\{\text{boxes in the hook at } x\}$$

Ex. If $\lambda =$ 

$h_\lambda(x) = 7$

Thm For any $\lambda \vdash n$,

$$\dim V^\lambda \left(= \#\{\text{SYTs of shape } \lambda\}\right) = \frac{n!}{\prod_x h_\lambda(x)}$$

where x runs over all boxes in the Young diagram of λ.

Ex The standard rep V_{n-1} of S^n is equivalent to V^λ, $\lambda = (n-1,1)$ whose Young diagram is

$$\dim V^\lambda = \frac{n!}{n\cdot(n-2)!\cdot 1} = n-1$$ as expected.

Murnaghan-Nakayama Rule

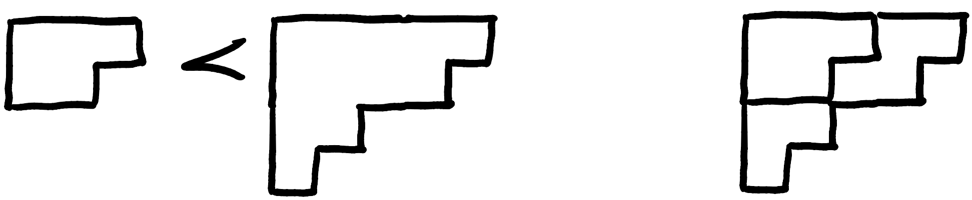
Gives the character table for the symmetric group S_n.

We need some notation:

1) If $\lambda, \mu \in \mathcal{P}$ we write

$$\mu < \lambda$$

if the Young diagram of μ is contained in that of λ:



Formally, $<$ defines a partial order on $\mathcal{P}$ and

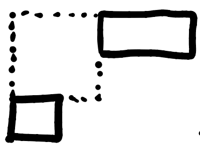
$$\mu < \lambda \iff \mu_i \leq \lambda_i \ \forall i \geq 1$$

where $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$
$\mu = (\mu_1 \geq \mu_2 \geq \ldots)$

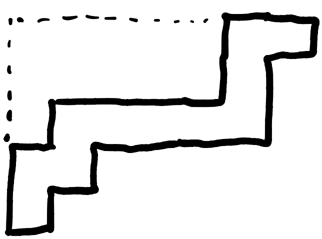
2) If $\mu < \lambda$ the **skew shape** λ/μ is defined as the "complement of μ in λ":

The **height** of a skew shape:

$$h(\lambda/\mu) = \#\{\text{rows occupied by } \lambda/\mu\} - 1$$

<u>Ex.</u> $h\Big($  $\Big) = 1$

3) A **skew hook** is a skew shape λ/μ of "thickness" 1



Murnaghan-Nakayama Rule:

$\lambda, \alpha \vdash n, \quad \alpha = (\alpha_1 \geq \cdots \geq \alpha_\ell > 0)$

Let $\chi^\lambda = \chi_{V^\lambda}$ and χ^λ_α be the value of χ^λ at an element $\sigma \in S_n$ of cycle type α.

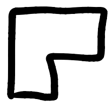
Then:

$$\chi^\lambda_\alpha = \sum_{\Lambda} (-1)^{h(\Lambda)}$$

where Λ runs over all sequences $\Lambda = (\emptyset = \lambda^0 \prec \lambda^1 \cdots \prec \lambda^\ell = \lambda)$ of partitions λ^i such that λ^i/λ^{i-1} is a skew hook of height α_i for $1 \leq i \leq \ell$, and $h(\Lambda) = \sum_{i=1}^{\ell} h(\lambda^i/\lambda^{i-1})$.

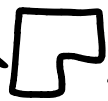
Ex. S_3

	(1)	(12)	(123)
$\mathbb{1}$	1	1	1
sgn	1	-1	1
V_2	2	0	-1

Take $\lambda = (2,1) =$  and $\alpha = (3)$

$\Lambda = (\varnothing = \lambda^0 < \lambda^1 = \lambda)$

is the only sequence possible.

$h(\lambda^1/\lambda^0) = h($  $) = 2 - 1 = 1$

$\Rightarrow \chi^\lambda_\alpha = (-1)^1 = -1$ ok

See Examples 4.23, 4.24 in the book for more examples.

Beyond finite groups

Infinite groups too hard to study.

Topology is a way to deal with infinity.

Def A topological group is a group G which is also a topological space, such that

mult $G \times G \to G$, $(g,h) \mapsto gh$

& inverse $G \to G$, $g \mapsto g^{-1}$

are continuous.

$\underline{\text{Ex}}$ p-adic integers $\hat{\mathbb{Z}}_{(p)}$

consists of all sequences

$$a = (a_1, a_2, \dots) \in \mathbb{Z}/_{(p)} \times \mathbb{Z}/_{(p)^2} \times \dots$$

such that $a_{n+1} \subset a_n \quad \forall n \geq 1.$

$$a_n = \alpha_n + (p)^n \qquad \alpha_n \in \mathbb{Z}$$

$$\alpha_{n+1} + (p)^{n+1} \subset \alpha_n + (p)^n$$

$$\Leftrightarrow \qquad \alpha_{n+1} \equiv \alpha_n \mod p^n$$

$$\pi_n : \hat{\mathbb{Z}}_{(p)} \longrightarrow \mathbb{Z}/_{(p)^n}, \quad a \longmapsto a_n$$

Requiring there are continuous

wrt discrete top on $\mathbb{Z}/_{(p)^n}$

gives $\pi_n^{-1}(\{0\})$ open

In general a group hom

$$\varphi : \hat{\mathbb{Z}}_{(p)} \longrightarrow H$$

(H discrete top) is continuous iff φ factors through some

$$\mathbb{Z}/_{(p)^n}$$

Thus "$\mathrm{Rep}\, \hat{\mathbb{Z}}_{(p)} = \bigcup_{n=1}^{\infty} \mathrm{Rep}\, \mathbb{Z}/_{(p)^n}$"

$$\text{Irreps of } \hat{\mathbb{Z}}_{(p)} \longleftrightarrow \left\{ \xi \in \mathbb{C} \;\middle|\; \begin{array}{l} \xi^{p^n} = 1 \\ \text{for some} \\ n \geq 1 \end{array} \right\}$$

$\hat{\mathbb{Z}}_{(p)}$ are special cases of profinite groups, defined as projective limits of finite groups.