MATH 618 LECTURE 21

On the graph isomorphism $\mathbb{B} \cong \mathbb{Y}$.

READ: §4.2.5

§§4.3.1 – 4.3.4

§4.4 (Skip the proofs).

HW21: Exercise 4.3.2.

Fix a positive integer n.

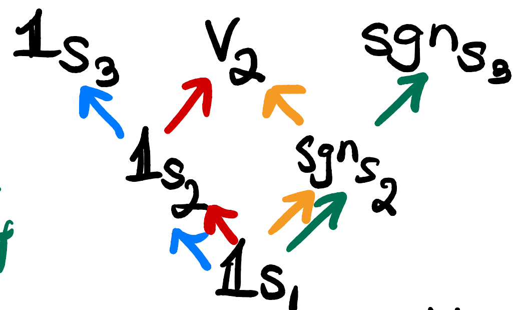
Algebraic Picture.

The set of paths T in the branching graph $\mathbb{B}$ from $\mathbb{1}_{S_1}$ to any vertex in $\operatorname{Irr} S_n$ parametrizes the GZ basis $\{v_T\}_T$ for $\displaystyle\bigoplus_{V \in \operatorname{Irr} S_n} V$.

Ex $n=3$

Four paths:

T_1 T_2 T_3 T_4



Correspondingly there are 4 GZ-basis vectors for $\mathbb{1}_{S_3} \oplus V_2 \oplus sgn_{S_3}$:

$$\{\underbrace{v_{T_1}}_{\mathbb{1}_{S_3}}, \underbrace{v_{T_2}, v_{T_3}}_{V_2}, \underbrace{v_{T_4}}_{sgn_{S_3}}\}$$

Since the GZ-basis $\{v_T\}_T$ diagonalizes the GZ-subalgebra $\mathcal{GZ}_n$, we may consider the joint spectrum of the JM elements on $\bigoplus_{V \in \mathrm{Irr}\, S_n} V$:

$$\mathrm{Spec}(n) := \left\{ \left(\varphi_T(X_1), \varphi_T(X_2), \dots, \varphi_T(X_n) \right) \right\}_T \subseteq \mathbb{k}^n$$

where $\varphi_T(X_i) \in \mathbb{k}$ is the eigenvalue of X_i acting on v_T:

$$X_i \cdot v_T = \varphi_T(X_i)\, v_T$$

Define an equivalence relation $\approx$ on $\mathrm{Spec}(n)$ by

$$\left(\varphi_T(X_1), \dots, \varphi_T(X_n)\right) \approx \left(\varphi_{T'}(X_1), \dots, \varphi_{T'}(X_n)\right)$$

$\overset{\text{def}}{\iff}$ The paths T and T' end at the same vertex.

Ex $n=3$. JM elements $\begin{cases} X_1 = 0 \\ X_2 = (12) \\ X_3 = (13)+(23) \end{cases}$

Since v_{T_1} spans the trivial rep $\mathbb{1}_{S_3}$ we have $\sigma.v_{T_1} = v_{T_1}$ $\forall \sigma \in S_3$. Thus:

$X_1.v_{T_1} = 0.v_{T_1}$, $X_2 v_{T_1} = 1 v_{T_1}$, $X_3 v_{T_1} = 2 v_{T_1}$

$\Rightarrow (0,1,2) \in \mathrm{Spec}(3)$.

Similarly, v_{T_4} spans the sign rep of S_3, hence we get

$(0,-1,-2) \in \mathrm{Spec}(3)$

Recall that for the standard rep V_2 we have $v_{T_2} = (1,1,-2)$, $v_{T_3} = (1,-1,0)$ and S_3 acts on $V_2 = \mathbb{k} v_{T_2} \oplus \mathbb{k} v_{T_3}$ by permuting coordinates.

$X_1 v_{T_2} = 0 v_{T_2}$, $X_2 v_{T_2} = 1 \cdot v_{T_2}$, $X_3 v_{T_2} = (-1) \cdot v_{T_2}$

$X_1 v_{T_3} = 0 \cdot v_{T_3}$, $X_2 v_{T_3} = (-1) v_{T_3}$, $X_3 v_{T_3} = 1 \cdot v_{T_3}$

This shows that

$$\mathrm{Spec}(3) = \{(0,1,2), (0,1,-1), (0,-1,1), (0,-1,-2)\}$$

with three equivalence classes:

$$\mathrm{Spec}(3)/\!\approx\; = \{\{(0,1,2)\}, \{(0,1,-1), (0,-1,1)\}, \{(0,-1,-2)\}\}$$

corresponding to the three irreps of S_3.

Thus we have the diagram:

$$\begin{tikzcd}
T \arrow[rr, mapsto] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & & (\varphi_T(X_1), \ldots, \varphi_T(X_n)) \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
\left\{ \begin{array}{c} \text{Paths in } \mathbb{B} \\ T : \mathbb{1}_{S_1} = W_1 \to \cdots \to W_n \end{array} \right\} \arrow[rr, "\cong"] \arrow[d, two heads, "\text{Take endpoint}"'] & & \mathrm{Spec}(n) \arrow[d, two heads] \\
\mathrm{Irr}\, S_n \arrow[rr, "\cong"] \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & & \mathrm{Spec}(n)/\!\approx \\
W_n & &
\end{tikzcd}$$

Additional feature of this diagram is that

$$
\begin{tikzcd}
\{T : \mathbb{1}_{S_1} \to \cdots \to W_{n-1} \to W_n\} \arrow[r, "\cong"] \arrow[d, "\substack{\text{Delete last} \\ \text{vertex } W_n \\ \text{from path}}"] & \mathrm{Spec}(n) \arrow[d, "\substack{\text{proj on} \\ \text{first } n-1 \\ \text{components}}"] \\
\{T' : \mathbb{1}_{S_1} \to \cdots \to W_{n-1}\} \arrow[r, "\cong"] & \mathrm{Spec}(n-1)
\end{tikzcd}
$$

commutes. This is because for $1 \leq i \leq n-1$:

$$
\begin{aligned}
\varphi_{T'}(X_i)v_{T'} &= X_i \cdot v_{T'} \\
&= X_i \cdot v_T \\
&= \varphi_T(X_i)v_T \\
&= \varphi_T(X_i)v_{T'}
\end{aligned}
$$

GZ-basis $\{v_{T'}\}_{T'}$ is a subset of $\{v_T\}_T$.

for any path $T : \mathbb{1}_{S_1} \to \cdots \to W_n$.

Combinatorial Picture

A partition is a sequence

$$\lambda = (\lambda_1, \lambda_2, \ldots)$$

of weakly decreasing non-negative integers with $|\lambda| = \sum_{i=1}^{\infty} \lambda_i < \infty$

(i.e. $\lambda_N = 0$, $N \gg 0$)

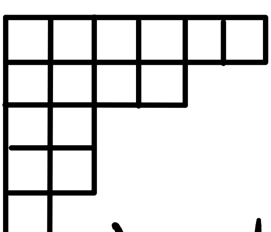
$$\mathcal{P} = \{\text{all partitions}\}$$

$$\mathcal{P}_n = \{\lambda \in \mathcal{P} \mid |\lambda| = n\}$$

If $\lambda \in \mathcal{P}_n$ we say λ is a partion of n, written $\lambda \vdash n$.

The λ_i are called the parts of λ.

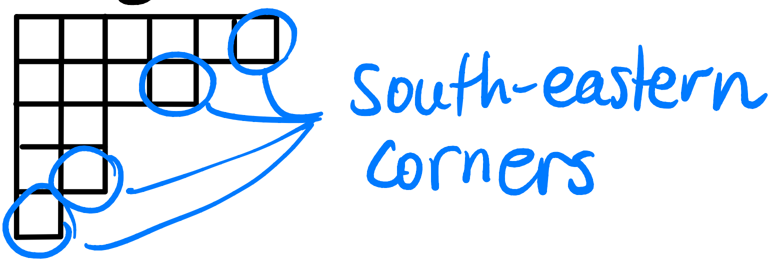
A partition can be drawn as a Young diagram



$\longleftrightarrow \lambda = (6, 4, 2, 2, 1)$

λ_i left-adjusted boxes in row i.

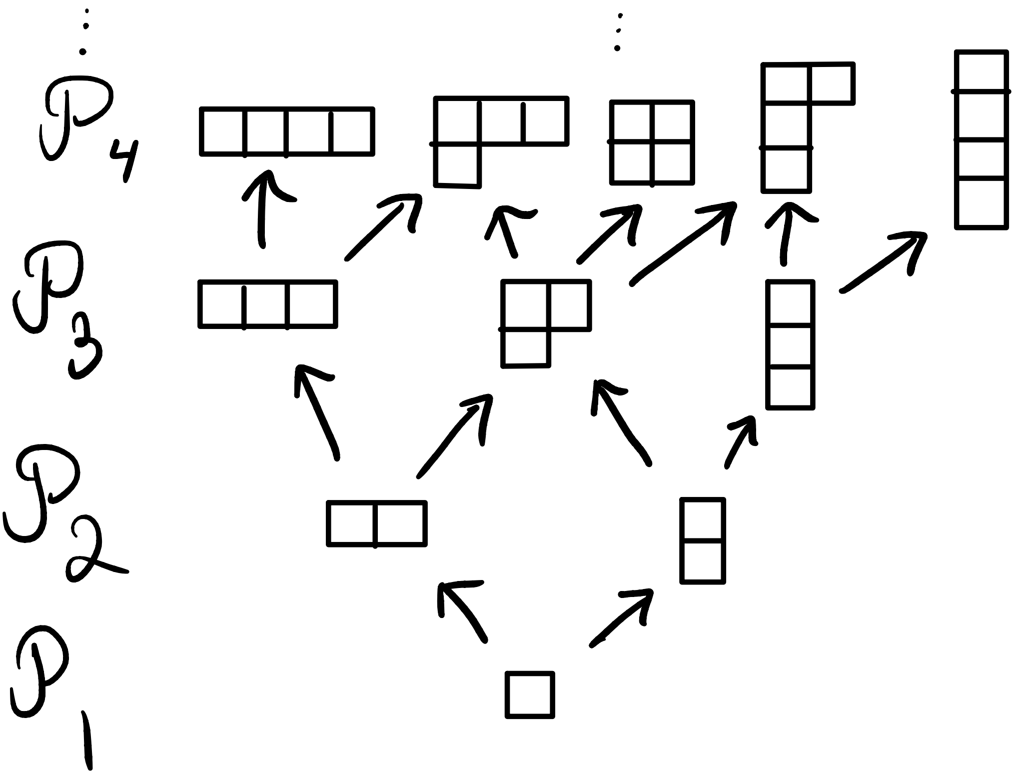
A south-eastern corner of a Young diagram is a box which can be deleted to form a Young diagram with 1 fewer box:



<u>Def</u> The Young Graph $\mathbb{Y}$ is the directed graph with vertex set $\mathcal{P}$ and an arrow

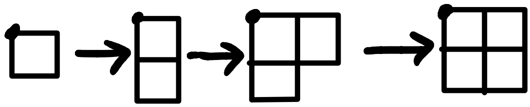
$$\mu \longrightarrow \lambda$$

iff μ can be obtained from λ by deleting a (south-eastern) box.



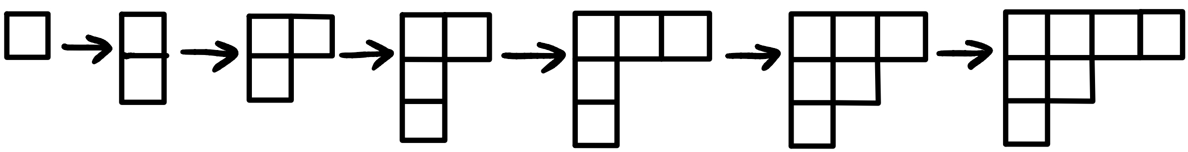
<u>Paths</u> in $\mathbb{Y}$ from the bottom vertex $\square$ to any Young diagram λ can be recorded in a Standard Young Tableau (SYT) obtained from λ by inserting $\{1, 2, \ldots, |\lambda|\}$ into the boxes so that each row and each column is strictly increasing.

__Ex__ The path in $\mathbb{Y}$:



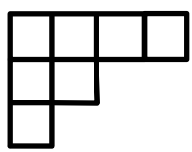
corresponds to the SYT $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$

__Ex__ The SYT $\begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$ corresponds to the path in $\mathbb{Y}$:



The shape of a standard Young tableau T is the underlying Young diagram:

$$\text{Shape}\left(\begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}\right) =$$

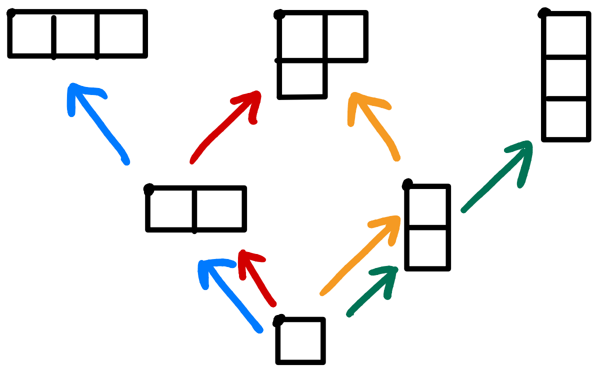


Note that this corresponds to taking the endpoint of T, when viewed as a path in $\mathbb{Y}$.

Let $SYT(n)$ be the set of all standard Young tableaux having n boxes.

<u>Ex.</u> $SYT(3) = \left\{ \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\,,\ \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} \right\}$

These SYTs represent the paths:



There is a surjection

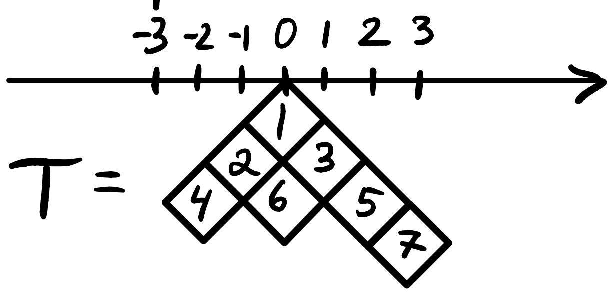
$$\begin{array}{ccc} SYT(n) & \ni & T \\ \big\downarrow & & \big\downarrow \\ \mathcal{P}_n & \ni & \mathrm{Shape}(T). \end{array}$$

given by deleting the numbers.

To complete the picture we need an analog of $\mathrm{Spec}(n)$.

<u>Def</u> The contents of $T \in SYT(n)$ is a vector
$$c(T) = (c_1, c_2, \dots, c_n) \in \mathbb{Z}^n$$
defined as follows.

① Tilt T by $45°$ clockwise and attach top corner to a coordinate axis:



② c_i = coordinate of the box containing i:

$$c(T) = (0, -1, 1, -2, 2, 0, 3)$$

We define $\mathrm{Cont}(n) = \{ c(T) \mid T \in SYT(n) \}$.

Ex

$$\mathrm{Cont}(3) = \left\{ c\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}\right), c\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\right), c\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right), c\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}\right) \right\}$$

$$= \{(0,1,2), (0,1,-1), (0,-1,1), (0,-1,-2)\}.$$

(compare with $\mathrm{Spec}(3)$!)

Define an equivalence relation $\sim$ on $\mathrm{Cont}(n)$ by

$$c(T) \sim c(T')$$

iff $\sigma . c(T) = c(T')$ for some $\sigma \in S_n$

Note that $c(T) \sim c(T')$ iff $\mathrm{shape}(T) = \mathrm{shape}(T')$.

We have the following commutative diagram

$$\begin{tikzcd}
\mathrm{SYT}(n-1) \arrow[r, "\cong"] & \mathrm{Cont}(n-1) \\
\mathrm{SYT}(n) \arrow[u, "\text{Delete } n\text{:th box}"] \arrow[r, "\cong"] \arrow[d, "\mathrm{Shape}(\cdot)"] & \mathrm{Cont}(n) \arrow[u, "\text{Project onto first } n-1 \text{ components}"'] \arrow[d] \\
\mathcal{P}_n \arrow[r, "\cong"] & \mathrm{Cont}(n)/\sim
\end{tikzcd}$$

The key result is now:

<u>Thm</u> $\mathrm{Spec}(n) = \mathrm{Cont}(n)$, and $\approx \; = \; \sim$.

The proof is based on explicitly describing these sets. See book for details.

Corollary We have a commutative diagram

$$\begin{tikzcd}
\left\{\begin{matrix}\text{Paths in } \mathbb{B} \\ \mathbb{1}_{S_1} \to \cdots \to w_{n-1}\end{matrix}\right\} \arrow[r, "\cong"] & \mathrm{Spec}(n-1) = \mathrm{Cont}(n-1) & \mathrm{SYT}(n-1) \arrow[l, "\cong"'] \\
\left\{\begin{matrix}\text{Paths in } \mathbb{B} \\ \mathbb{1}_{S_1} \to \cdots \to w_n\end{matrix}\right\} \arrow[r, "\cong"] \arrow[u, two heads] \arrow[d, two heads] & \mathrm{Spec}(n) = \mathrm{Cont}(n) & \mathrm{SYT}(n) \arrow[l, "\cong"'] \arrow[u, two heads] \arrow[d, two heads, "\text{Shape}"] \\
\mathrm{Irr}\, S_n \arrow[r, "\cong"] & \mathrm{Spec}(n)/\approx \; = \; \mathrm{Cont}(n)/\sim & \mathcal{P}_n \arrow[l, "\cong"']
\end{tikzcd}$$

Corollary (Graph Isomophism Thm) $\mathbb{B} \cong \mathbb{Y}$.

Proof The bottom row gives a bijection $\mathrm{vert}\, \mathbb{Y} \cong \mathrm{vert}\, \mathbb{B}$, $\lambda \mapsto V^\lambda$. There is an arrow $\mu \to \lambda$ in $\mathbb{Y}$ iff $\exists$ path $\square \to \cdots \to \mu \to \lambda$ in $\mathbb{Y}$. By diagram this holds iff $\exists$ path $\mathbb{1}_{S_1} \to \cdots \to V^\mu \to V^\lambda \iff \exists\, V^\mu \to V^\lambda$ in $\mathbb{B}$.

QED.