MATH 618 LECTURE 20

READ:
- EXAMPLE 4.3 IN §4.2.3
- §4.2.4 (GZ_n properties).
- §1.4.4 (Idempotents)

<u>HW 20</u>: Exercise 4.2.4 (orthogonality of GZ-bases).

GZ-basis for V_{n-1}

$\mathrm{Irr}\, S_n \ni V_{n-1} = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{k}^n \mid \sum_{i=1}^{n} \lambda_i = 0\}$

Note that $V_{n-1} \downarrow_{S_{n-1}}$ contains two subreps:

$V'_{n-2} := \{(\lambda_1, \ldots, \lambda_{n-1}, 0) \in \mathbb{k}^n \mid \sum_{i=1}^{n-1} \lambda_i = 0\}$

and

$\mathbb{k} v_{n-1}$, $v_{n-1} := (\underbrace{1, 1, \ldots, 1}_{n-1}, -(n-1))$

Note that $V'_{n-2} \cong V_{n-2}$ and $\mathbb{k} v_{n-1} \cong \mathbb{1}_{S_{n-1}}$.

$\dim V'_{n-2} = n-2$ so

$$V_{n-1} \downarrow_{S_{n-1}} = V'_{n-2} \oplus \mathbb{k} v_{n-1}$$

is the decomposition into irreps of S_{n-1}.

Restricting further to S_{n-2}:

$$V_{n-1}\downarrow_{S_{n-2}} = V'_{n-2}\downarrow_{S_{n-2}} \oplus (\mathbb{k}v_{n-1})\downarrow_{S_{n-2}}$$

As with $V_{n-1}\downarrow_{S_{n-1}}$ we may decompose

$$V'_{n-2}\downarrow_{S_{n-2}} = V'_{n-3} \oplus \mathbb{k}v_{n-2}$$

where $V'_{n-3} = \{(\lambda_1, \ldots, \lambda_{n-2}, 0, 0) \mid \sum_{i=1}^{n-2} \lambda_i = 0\}$

and $v_{n-2} = (\underbrace{1, 1, \ldots, 1}_{n-2}, -(n-2), 0)$

Continuing like this we get

$$V_{n-1}\downarrow_{S_1} = \mathbb{k}v_1 \oplus \mathbb{k}v_2 \oplus \cdots \oplus \mathbb{k}v_{n-1}$$

where $v_k = (\underbrace{1, 1, \ldots, 1}_{k}, -k, 0, \ldots, 0)$

Thus $\{v_1, \ldots, v_{n-1}\}$ is the GZ-basis for V_{n-1}.

Example $n=5$. Consider the following subgraph of $\mathbb{B}$:

$$\begin{tikzcd}
\operatorname{Irr} S_5 & & V_4 & & & \\
\operatorname{Irr} S_4 & \mathbb{1}_{S_4} \arrow[ur] & & V_3 \arrow[ul] & & \\
\operatorname{Irr} S_3 & & \mathbb{1}_{S_3} \arrow[ul] \arrow[ur] & & V_2 \arrow[ul] & \\
\operatorname{Irr} S_2 & & & \mathbb{1}_{S_2} \arrow[ul] \arrow[ur] & & V_1 \cong \operatorname{sgn}_{S_2} \arrow[ul] \\
\operatorname{Irr} S_1 & & & & \mathbb{1}_{S_1} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Each of the 4 GZ-basis vectors of V_4 corresponds to a path in $\mathbb{B}$ from $\mathbb{1}_{S_1}$ to V_4:

v_4	$\mathbb{1}_{S_1} \to \mathbb{1}_{S_2} \to \mathbb{1}_{S_3} \to \mathbb{1}_{S_4} \to V_4$
v_3	$\mathbb{1}_{S_1} \to \mathbb{1}_{S_2} \to \mathbb{1}_{S_3} \to V_3 \to V_4$
v_2	$\mathbb{1}_{S_1} \to \mathbb{1}_{S_2} \to V_2 \to V_3 \to V_4$
v_1	$\mathbb{1}_{S_1} \to V_1 \to V_2 \to V_3 \to V_4$

The following theorem connects the GZ-basis with the GZ-subalgebra.

<u>Thm</u> a) $\mathcal{GZ}_n$ consists of all $a \in \mathbb{k}S_n$ such that for every irrep V of S_n, $\rho_V(a)$ is diagonal in the GZ-basis.

b) $\mathcal{GZ}_n$ is maximal commutative in $\mathbb{k}S_n$ (i.e. maximal among all commutative subalgebras, w.r.t. inclusion).

c) $\mathcal{GZ}_n$ is semisimple:

$$\mathcal{GZ}_n \cong \underbrace{\mathbb{k} \times \mathbb{k} \times \cdots \times \mathbb{k}}_{d_n}$$

where $d_n = \sum\limits_{V \in \mathrm{Irr}\, S_n} \dim V$

<u>Proof</u> Let

$$\mathcal{D} = \{ a \in \Bbbk S_n \mid \forall V \in \operatorname{Irr} S_n : \rho_V(a) \text{ is diagonal in the GZ-basis} \}$$

① $\mathcal{D}$ is a maximal commutative subalgebra of $\Bbbk S_n$:

By Wedderburn's Thm

$$\Bbbk S_n \cong \prod_{V \in \operatorname{Irr} S_n} \operatorname{End}_{\Bbbk}(V)$$

In each V, choose the GZ-basis.
Then we get an algebra isomorphism

$$\varphi : \Bbbk S_n \xrightarrow{\cong} \prod_{V \in \operatorname{Irr} S_n} \operatorname{Mat}_{\dim V}(\Bbbk)$$

<u>Ex</u> $n=3$

$$\varphi : \Bbbk S_3 \xrightarrow{\cong} \begin{bmatrix} \Bbbk & & & \\ & \Bbbk & & \\ & & \Bbbk & \Bbbk \\ & & \Bbbk & \Bbbk \end{bmatrix}$$

Then $\mathcal{D} = \varphi^{-1}\left(\begin{Bmatrix}\text{all diagonal}\\ \text{matrices}\end{Bmatrix}\right)$

Any matrix that commutes with all diagonal matrices is itself a diagonal matrix.

So $\varphi(\mathcal{D})$ is maximal commutative in $\prod\limits_{V\in \mathrm{Irr}\, S_n} \mathrm{Mat}_{\dim V}(\mathbb{k})$.

Since φ is an algebra isomorphism $\mathcal{D}$ is maximal commutative in $\mathbb{k}S_n$.

② $\mathcal{GZ}_n \subseteq \mathcal{D}$.

Since $\mathcal{GZ}_n$ is generated by the centers $Z_k = Z(\mathbb{k}S_k)$ for $k=1,2,\dots,n$, it suffices to show that $Z_k \subseteq \mathcal{D}$ for each $k=1,\dots,n$.

Let $z\in Z_k$ and let $V\in \mathrm{Irr}\, S_n$.

Let $T: \mathbb{1}_{S_1} = W_1 \to W_2 \to \cdots \to W_n = V$

be a path in $\mathbb{B}$, and v_T the corresponding GZ-basis vector. Then v_T spans W_1 hence $v_T \in W_i$ for $i=1,\dots,n$. In particular $v_T \in W_k$.
By Schur's Lemma,

$$\rho_V(z)v_T = \rho_{W_k}(z)v_T = \xi v_T$$

for some $\xi \in \mathbb{k}$. Since T was arbitrary this proves that $\rho_V(z)$ is diagonal in the GZ-basis. Thus $z \in \mathcal{D}$. Since z was arbitrary, $Z_k \subseteq \mathcal{D}$.

③ $\mathcal{D} \subseteq GZ_n$:

For each $V \in \operatorname{Irr} S_n$, let

$$e(V) = \varphi^{-1}\left(\begin{bmatrix} 0 & & \\ & I_V & \\ & & 0 \end{bmatrix}\right)$$

I_V = Identity matrix in slot V

Ex $\quad n=3$

$$e(\mathbb{1}_{S_3}) = \varphi^{-1}\left(\begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & 0 \\ & & 0 & 0 \end{bmatrix}\right)$$

$$e(\mathrm{sgn}_{S_3}) = \varphi^{-1}\left(\begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 0 & 0 \\ & & 0 & 0 \end{bmatrix}\right)$$

$$e(V_2) = \varphi^{-1}\left(\begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 1 & 0 \\ & & 0 & 1 \end{bmatrix}\right)$$

Properties: for all $V \in \mathrm{Irr}\, S_n$:

i) $e(V)^2 = e(V)$ idempotent

ii) $e(V) \in Z_n$ central

iii) $e(V)e(V') = \begin{cases} e(V) & V \cong V' \\ 0 & V \not\cong V' \end{cases}$ mutually orthogonal

iv) $\displaystyle\sum_{V \in \mathrm{Irr}\, S_n} e(V) = 1_{\Bbbk S_n}$ complete set

primitive

v) If $e(V) = e' + e''$ for some central mutually orthogonal idempotents e', e'' then $e'=0$ OR $e''=0$

These properties hold because the $\varphi(e(V))$ have those properties in $\prod_{V \in \operatorname{Irr} S_n} \mathrm{Mat}_{\dim V}(\mathbb{K})$.

Now we exhibit a basis for $\mathcal{D}$ and show it is contained in $\mathcal{GZ}_n$.
Pick a path in $\mathbb{B}$:
$T: \mathbb{1}_{S_n} = W_1 \to W_2 \to \cdots \to W_n = V$.
and consider the following element:

$$e_T := \underbrace{e(W_1)}_{\in Z_1} \underbrace{e(W_2)}_{\in Z_2} \cdots \underbrace{e(W_n)}_{\in Z_n} \in \mathcal{GZ}_n$$

Then $\varphi(e_T)\, v_{T'} = \begin{cases} v_T & T' = T \\ 0 & T' \neq T \end{cases}$

for every path T' in $\mathbb{B}$ from $\mathbb{1}_{S_1}$ to $V' \in \operatorname{Irr} S_n$.

Ex. $n=3$

$$\begin{tikzcd}
\operatorname{Irr} S_3 & \mathbb{1}_{S_3} & & V_2 & & \operatorname{sgn}_{S_3} \\
\operatorname{Irr} S_2 & & \mathbb{1}_{S_2} \arrow[ul] \arrow[ur] & & \operatorname{sgn}_{S_2} \arrow[ul] \arrow[ur] & \\
\operatorname{Irr} S_1 & & & \mathbb{1}_{S_1} \arrow[ul, shift left] \arrow[ul, shift right] \arrow[ur, shift left] \arrow[ur, shift right] & &
\end{tikzcd}$$

$T_1: \mathbb{1}_{S_1} \to \mathbb{1}_{S_2} \to \mathbb{1}_{S_3}$

$T_2: \mathbb{1}_{S_1} \to \operatorname{sgn}_2 \to \operatorname{sgn}_3$

$T_3: \mathbb{1}_{S_1} \to \mathbb{1}_{S_2} \to V_2$

$T_4: \mathbb{1}_{S_1} \to \operatorname{sgn}_{S_2} \to V_2$

} All paths from $\mathbb{1}_{S_1}$ to 3rd level of $\mathbb{B}$

$$e_{T_1} = e(\mathbb{1}_{S_1})e(\mathbb{1}_{S_2})e(\mathbb{1}_{S_3}) \overset{\varphi}{\longmapsto} \begin{array}{c} \begin{matrix} T_1 & T_2 & T_3 & T_4 \end{matrix} \\ \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & 0 \\ & & 0 & 0 \end{bmatrix} \end{array}$$

$$e_{T_2} = e(\mathbb{1}_{S_1})e(\operatorname{sgn}_2)e(\operatorname{sgn}_3) \overset{\varphi}{\longmapsto} \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 0 & 0 \\ & & 0 & 0 \end{bmatrix}$$

$$e_{T_3} = e(\mathbb{1}_{S_1})e(\mathbb{1}_{S_2})e(V_2) \overset{\varphi}{\longmapsto} \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 1 & 0 \\ & & 0 & 0 \end{bmatrix}$$

$$e_{T_4} = e(\mathbb{1}_{S_1})e(\operatorname{sgn}_{S_2})e(V_2) \overset{\varphi}{\longmapsto} \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & 0 \\ & & 0 & 1 \end{bmatrix}$$

Thus $\{e_T\}$ is a basis for $\mathcal{D}$, where T ranges over all paths from $\mathbb{1}_{S_1}$ to n:th level.

$\Rightarrow \mathcal{D} \subseteq GZ_n$ (since all $e_T \in GZ_n$)

(4) We have shown $\mathcal{D} = GZ_n$ and that $\mathcal{D} \cong \underbrace{\mathbb{k} \times \cdots \times \mathbb{k}}_{d_n}$ and that $\mathcal{D}$ is maximal commutative in $\mathbb{k}S_n$.

Therefore a), b), c) from the theorem hold.

QED.