MATH 618 LECTURE 2

$\boxed{\text{HW2: Exercise 1.1.7}}$

Tensor algebras

vector space $V \rightsquigarrow T(V)$ algebra

Put $T^0(V) = \mathbb{k}$

$$T^n(V) = \underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ factors}}$$

and $T(V) = \bigoplus_{n=0}^{\infty} T^n(V)$

We equip this vector space with multiplication and unit maps.

$$u : \mathbb{k} \xrightarrow{\mathrm{Id}} T^0(V) \hookrightarrow T(V)$$

$$m_{ij} : T^i(V) \otimes T^j(V) \xrightarrow{\mathrm{Id}} T^{i+j}(V)$$

$$m = \bigoplus_{i,j=0}^{\infty} m_{ij}$$

Here we use $T(V) \otimes T(V) \cong \bigoplus_{i,j=0}^{\infty} T^i(V) \otimes T^j(V)$

Note: We have a linear map $i : V \to T(V)$ given by
$$V \xrightarrow{\mathrm{Id}} T^1(V) \hookrightarrow T(V).$$

<u>Universal property of $T(V)$</u>:

Any linear map $j : V \to A$, where A is an algebra, extends uniquely to a homomorphism $f : T(V) \to A$.

("extends" means that $f|_V = j$

i.e.
$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "j"'] & T(V) \arrow[d, dashed, "\exists! f"] \arrow[dl, phantom, "\circlearrowleft"] \\ & A \end{tikzcd}$$
)

$$\mathrm{Hom}_{\mathrm{Alg}_{\mathbb{k}}}(T(V), A) \cong \mathrm{Hom}_{\mathrm{Vect}_{\mathbb{k}}}(V, \underbrace{A|_{\mathrm{Vect}_{\mathbb{k}}}}_{\text{forget alg structure}})$$

<u>Remark</u> If $\{x_1, \dots, x_n\}$ basis for V then $\mathbb{k}\langle x_1, \dots, x_n \rangle \cong T(V)$ as algebras.

Symmetric algebras.

We had
$$T : \mathrm{Vect}_{\Bbbk} \longrightarrow \mathrm{Alg}_{\Bbbk}$$
$$V \longmapsto T(V)$$

Similarly there is a functor

$$S : \mathrm{Vect}_{\Bbbk} \longrightarrow \mathrm{CommAlg}_{\Bbbk}$$
$$V \longmapsto S(V)$$

with universal properties.

$$S(V) \stackrel{\text{def}}{=} T(V)/\underline{I}$$

where I is the 2-sided ideal generated by $\{v \otimes v' - v' \otimes v \mid v, v \in V\}$.

$$\mathrm{Hom}_{\mathrm{CommAlg}_{\Bbbk}}(S(V), A) \cong \mathrm{Hom}_{\mathrm{Vect}_{\Bbbk}}(V, A|_{\mathrm{Vect}_{\Bbbk}})$$
$$f \longmapsto f|_V$$

Exterior algebras.

$$\Lambda(V) \overset{\text{def}}{=\!=} T(V)/J$$

$$J = \langle v \otimes v' + v' \otimes v \mid v, v' \in V \rangle$$

$$v_1 \wedge \cdots \wedge v_n \overset{\text{def}}{=\!=} v_1 \otimes \cdots \otimes v_n + J \in \Lambda(V).$$

We need to discuss graded algebras:

$$A = \bigoplus_{n=0}^{\infty} A_n$$

$$A_n A_m \subseteq A_{n+m}$$

$$1 \in A_0$$

A map of graded algebras $f: A \to B$ is an algebra map such that

$$f(A_n) \subseteq B_n \quad \forall n \geq 0.$$

A graded algebra A is graded commutative if $ab = (-1)^{|a| \cdot |b|} ba \ \forall a \in A_{|a|}, b \in A_{|b|}$

Graded ideals & quotients

$$A = \bigoplus_{n=0}^{\infty} A_n \quad \text{graded alg}$$

Ideal $I \subset A$ is graded ideal if $I = \bigoplus_{n=0}^{\infty} (I \cap A_n)$

Then $A/I \cong \bigoplus_{n=0}^{\infty} A_n/(I \cap A_n)$ becomes a graded algebra.

Exercise 1.1.12 If A is a graded algebra and $I \subseteq A$ is an ideal generated by homogeneous elements ($=$ elements of $\bigcup_{n=0}^{\infty} A_n$) then I is a graded ideal.

Example $T(V)$ is a graded alg. $S(V)$ & $\Lambda(V)$ are graded algs. $\Lambda(V)$ is graded commutative.

If $\dim V = n$ then
$\dim \Lambda^d(V) = \binom{n}{d}$
where $\Lambda^d(V) = T^d(V) / \mathcal{I} \cap T^d(V)$.

Basis for $\Lambda^d(V)$:

$$\{ v_{i_1} \wedge \cdots \wedge v_{i_d} \mid 1 \le i_1 < \cdots < i_d \le n \}$$

In particular $\dim \Lambda(V) = \sum_{d=0}^{n} \binom{n}{d} = 2^n$

T, S, Λ are <u>functors</u> meaning if $f : V \to W$ is a linear map we get a map of (in fact graded) algebras $T(f) : T(V) \to T(W)$ etc.
If $\dim V = n$ then $\Lambda^n(f) : \underset{\mathbb{k}}{\overset{\shortparallel}{\Lambda^n(V)}} \to \underset{\mathbb{k}}{\overset{\shortparallel}{\Lambda^n(V)}}$
is multiplication by $\det(f)$.