MATH 618 LECTURE 19

READ: §4.1.2
§4.2.1 – §4.2.3

HW19: Let V be an irrep of S_n. Show that

1) There exists a nonzero bilinear form $(\cdot,\cdot): V \times V \to \mathbb{k}$ such that
$$(g.v, g.w) = (v,w)$$
$\forall v,w \in V$ and $\forall g \in S_n$

2) Show that if $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ are two such forms then $\exists \alpha \in \mathbb{k} \setminus \{0\}$ such that
$$(v,w)_2 = \alpha \cdot (v,w)_1$$
for all $v,w \in V$.

<u>Hint</u>: Every irrep of S_n is self-dual.

Olshanskii's Thm implies the following description of $\mathcal{GZ}_n$.

<u>Corollary</u> $\mathcal{GZ}_n = \Bbbk[X_1, X_2, \ldots, X_n]$.

<u>Proof</u> We saw ($\supseteq$). We prove ($\subseteq$) by induction on n.

<u>$n=1$</u> $\mathcal{GZ}_1 = \Bbbk$ ok.

<u>$n>1$</u>: $\mathcal{GZ}_n = \Bbbk[\mathcal{GZ}_{n-1}, Z_n]$ by definition so by induction it suffices to show $Z_n \subseteq \Bbbk[\mathcal{GZ}_{n-1}, X_n]$.

We have

$$Z_n = (\Bbbk S_n)^{S_n}$$
$$\subseteq (\Bbbk S_n)^{S_{n-1}} \underset{\uparrow}{=} \Bbbk[Z_{n-1}, X_n] \subseteq \Bbbk[\mathcal{GZ}_{n-1}, X_n]$$

By Olshanskii's Thm

QED.

Multiplicity-free property

Let V be an irrep of S_n.
By Maschke's Thm, $V\downarrow_{S_{n-1}}$ can be decomposed into irreps of S_{n-1}.
Let $\{W_1, \ldots, W_s\}$ be the set of all irreps for S_{n-1} (up to equivalence).
Then

$$V\downarrow_{S_n} = \bigoplus_{i=1}^{s} W_i^{\oplus a_i}$$

for some $a_i \in \mathbb{Z}_{\geq 0}$ (called the multiplicity of W_i in $V\downarrow_{S_{n-1}}$)

Theorem $a_i \in \{0, 1\}$ for all i.

<u>Proof</u> By Wedderburn's Thm:

$$\Bbbk S_n \cong \prod_{V \in \operatorname{Irr} \Bbbk S_n} \operatorname{End}_{\Bbbk}(V)$$

$$\Rightarrow \Bbbk[S_n]^{S_{n-1}} \cong \prod_{V \in \operatorname{Irr} \Bbbk S_n} \operatorname{End}_{\Bbbk}(V)^{S_{n-1}} \cong$$

$$\cong \prod_{V \in \operatorname{Irr} \Bbbk S_n} \operatorname{End}_{S_{n-1}}(V{\downarrow}_{S_{n-1}})$$

For each $V \in \operatorname{Irr} \Bbbk S_n$ we have

$$V{\downarrow}_{S_{n-1}} = W_1^{\oplus a_1(V)} \oplus \cdots \oplus W_s^{\oplus a_s(V)}$$

for some $a_i(V) \in \mathbb{Z}_{\geq 0}$. By Schur's Lemma

$$\operatorname{End}_{S_{n-1}}(V{\downarrow}_{S_{n-1}}) \cong \prod_{i=1}^{s} \operatorname{Mat}_{a_i(V)}(\Bbbk)$$

On the other hand, by Olshanskii's Thm,

$$\Bbbk[S_n]^{S_{n-1}} = \Bbbk[Z_{n-1}, X_n] \subseteq GZ_n$$

hence is commutative.

This implies $a_i(V) = 0$ or 1 for all $i = 1, \dots, s$ and all $V \in \operatorname{Irr} \Bbbk S_n$.

QED.

Branching graph $\mathbb{B}$

Vertices: $\displaystyle\bigsqcup_{n\geq 1} \operatorname{Irr} \mathbb{k}S_n$

Directed edge $W \to V$

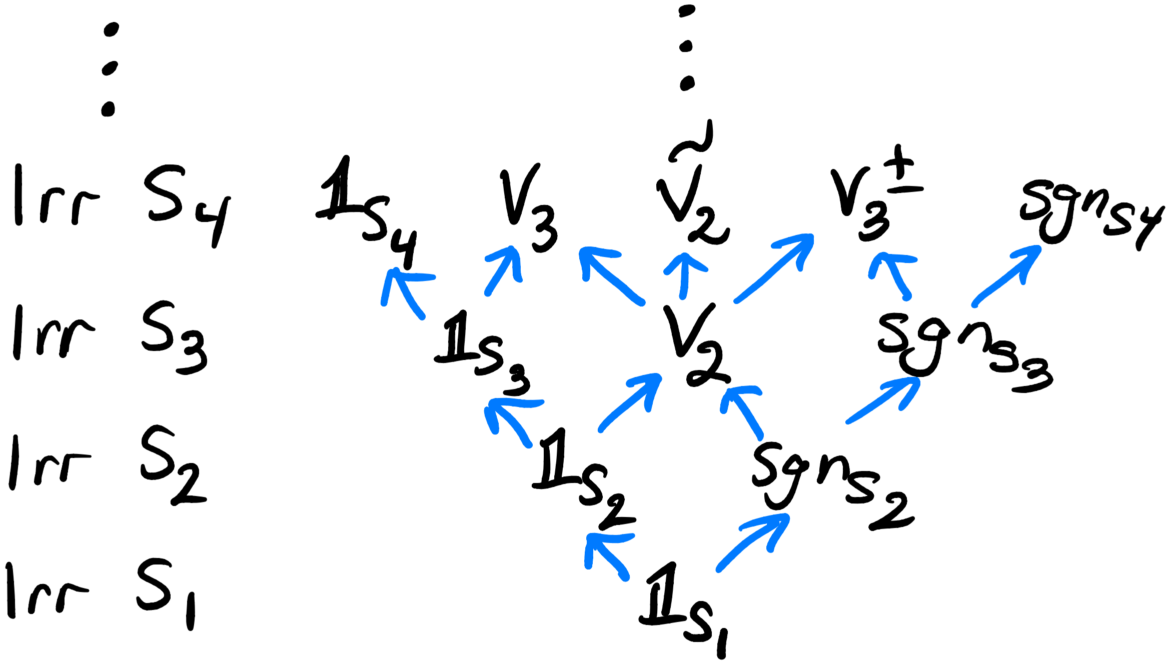
iff $W \in \operatorname{Irr} \mathbb{k}S_{n-1}$

$V \in \operatorname{Irr} \mathbb{k}S_n$

for some $n \geq 2$ and

$$\operatorname{Hom}_{S_{n-1}}(W, V\downarrow_{S_n}) \neq 0$$

i.e. W occurs in the decomp of $V\downarrow_{S_{n-1}}$ into irreps for S_{n-1}.



Bottom part of the branching graph $\mathbb{B}$.

The Gelfand-Zetlin bases.

Given $V \in \mathrm{Irr}\, \Bbbk S_n$ we have

$$V\big\downarrow_{S_{n-1}} = \bigoplus_{\substack{W \to V \\ \text{in } \mathbb{B}}} W.$$

We may further restrict to S_{n-2}:

$$V\big\downarrow_{S_{n-2}} = \bigoplus_{\substack{W \to V \\ \text{in } \mathbb{B}}} W\big\downarrow_{S_{n-2}} =$$

$$= \bigoplus_{\substack{W_{n-2} \to W_{n-1} \to V \\ \text{in } \mathbb{B}}} W_{n-2}$$

Continuing in this way we get

$$V = V\big\downarrow_{S_1} = \bigoplus_{\substack{\text{paths} \\ T : \mathbb{1}_{S_1} = W_1 \to W_2 \to \cdots \to W_n = V \\ \text{in } \mathbb{B}}} \Bbbk v_T$$

<u>Def</u> $\{v_T\}_T$ is called the Gelfand-Zetlin basis for V. T runs through the set of all paths in $\mathbb{B}$ from $\mathbb{1}_{S_1}$ to V.

<u>Remark</u> The vectors v_T are unique up to rescaling.

<u>Remark</u> The dimension of an irrep V of S_n equals the number of paths in $\mathbb{B}$ from $\mathbb{1}_{S_1}$ to V.