MATH 618 LECTURE 18

READ: - §4.1

- Introduction from [Okounkov-Vershik, 1996]

HW18: Prove that, in $\Bbbk S_n$,

$$\begin{cases} s_i X_i = X_{i+1} s_i - 1 & 1 \le i \le n-1 \\ s_i X_j = X_j s_i & \forall j \notin \{i, i+1\} \end{cases}$$

where $s_i = (i \;\; i+1)$ and X_i are the Jucys-Murphy elements. (Exercise 4.1.1 from the book.)

Representation Theory of Symmetric Groups.

We follow the modern approach of Okounkov-Vershik 1996, based on the Gelfand-Zetlin subalgebra and spectrum of Jucys-Murphy elements.

Consider the chain of subgroups:

$$S_1 \leq S_2 \leq \cdots \leq S_n$$

This gives a chain of subalgebras:

$$\Bbbk S_1 \subseteq \Bbbk S_2 \subseteq \cdots \subseteq \Bbbk S_n.$$

Put $Z_k = Z(\Bbbk S_k)$. (We don't have $Z_k \subseteq Z_{k+1}$)

<u>Def</u> The **Gelfand-Zetlin Subalgebra** of $\Bbbk S_n$ is the subalgebra generated by $Z_1 \cup Z_2 \cup \cdots \cup Z_n$:

$$GZ_n := \Bbbk[Z_1, Z_2, \dots, Z_n] \subseteq \Bbbk S_n$$

<u>Note</u> If $1 \le i \le j \le n$, then $Z_i \subseteq \Bbbk[S_j]$ hence $\alpha\beta = \beta\alpha \ \forall \alpha \in Z_i, \beta \in Z_j$.
So GZ_n is a commutative subalg of $\Bbbk S_n$.

Jucys-Murphy (JM) elements

For $1 \le k \le n$, define

$$X_k = (1\ k) + (2\ k) + \cdots + (k-1\ \ k) \in \Bbbk S_n$$

($X_1 = 0$, being an empty sum)

<u>Note</u>

$$X_k = \sum_{t \in S_k(1\ k)} t \in \Bbbk[S_n]^{S_{n-1}}$$

acts by conjugation.

and moreover

$$X_k = \underbrace{\begin{pmatrix}\text{sum of all} \\ \text{transpositions} \\ \text{in } S_k\end{pmatrix}}_{\in Z_k} - \underbrace{\begin{pmatrix}\text{sum of all} \\ \text{transpositions} \\ \text{in } S_{k-1}\end{pmatrix}}_{\in Z_{k-1}}.$$

Therefore $\Bbbk[X_1, \ldots, X_n] \subseteq GZ_n$.

Olshanskii's Theorem

Fix $1 \leq k \leq n$. Let S_k act on S_n by conjugation.
S_k-conjugacy classes in S_n are characterized by their marked cycle shape obtained by replacing $1, \dots, k$ by $*$ in the cycle decomposition of any element.

Ex. $k=5$, $n=7$

$$\sigma = (1\,6\,3)(2)(4\,7)(5)$$

$$\downarrow$$

$$(*\,6\,*)(*)(*\,7)(*)$$

The S_k-invariants of $\Bbbk S_n$

$$(\Bbbk S_n)^{S_k} = \{x \in S_n \mid \tau x \tau^{-1} = x \ \forall \tau \in S_k\}$$

is a subalgebra.

Theorem (Olshanskiĭ)

The subalgebra of S_k-invariants

$$(\Bbbk S_n)^{S_k} = \{ x \in \Bbbk S_n \mid \tau x \tau^{-1} = x \ \forall \tau \in S_k \}$$

is generated by

$$Z_k \cup S'_{n-k} \cup \{ X_1, X_2, \dots, X_k \}$$

where $S'_{n-k} = S_{\{k+1, k+2, \dots, n\}}$.

proof: Put

$$\mathcal{A} := \Bbbk[Z_k, S'_{n-k}, X_{k+1}, \dots, X_n]$$

$$\mathcal{B} := (\Bbbk S_n)^{S_k}$$

We have $Z_k = (\Bbbk S_k)^{S_k} \subseteq \mathcal{B}$

and for $i > k$: $X_i \in (\Bbbk S_i)^{S_{i-1}} \subseteq (\Bbbk S_i)^{S_k} \subseteq \mathcal{B}$

and clearly $S'_{n-k} \subseteq \mathcal{B}$.

Therefore $\mathcal{A} \subseteq \mathcal{B}$.

Conversely:

$\Bbbk S_n$ is a permutation rep of S_k (wrt conj.)

For $s \in S_n$, put $\sigma_s = \sum_{t \in {}^{S_k}s} t$.

Then $\{\sigma_s\}_{s \in S_n}$ spans $\mathcal{B}$.

Suffices to show $\sigma_s \in \mathcal{A}$ $\forall s \in S_n$.

Define ==length== of s as $\ell(s) = \{ i \in [\![1,n]\!] : s(i) \neq i \}$

(Different from usual def of length)

- ℓ is a class function $\ell(\tau s \tau^{-1}) = \ell(s)$ $\forall \tau, s \in S_n$
- $\ell(ss') \leq \ell(s) + \ell(s')$ with equality iff $\forall i : s(i) \neq i \iff s'(i) = i$ (they never move the same point)

$\mathcal{F}_\ell \subseteq \Bbbk S_n$, $\mathcal{F}_\ell := \mathrm{Span}\{ s \in S_n \mid \ell(s) \leq \ell \}$

$\Bbbk = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n = \Bbbk S_n$ and

$\mathcal{F}_\ell \mathcal{F}_{\ell'} \subseteq \mathcal{F}_{\ell + \ell'}$ ($\Rightarrow \{\mathcal{F}_\ell\}_{\ell=0}^{n}$ is a finite ==filtration== of $\Bbbk S_n$)

Moreover, each $\mathcal{F}_\ell$ is S_k-stable i.e. $\forall \tau \in S_k, \forall x \in \mathcal{F}_\ell, \tau x \tau^{-1} \in \mathcal{F}_\ell$.

Put $\mathcal{B}_\ell = \mathcal{B} \cap \mathcal{F}_\ell = (\mathcal{F}_\ell)^{S_k}$.

$\mathcal{B}_\ell$ is spanned by $\{\sigma_s \mid s \in S_n, \ell(s) \le \ell\}$.

We prove by induction on ℓ that

$\sigma_s \in \mathcal{A}$ $\forall s \in S_n$ with $\ell(s) = \ell$.

$\underline{\ell = 0}$: Then $s = (1)$ so $\sigma_s = 1_{\Bbbk S_n} \in \mathcal{A}$.

$\underline{\ell = 1}$: Impossible.

$\underline{\ell \geq 2}$: Let $s \in S_n$, $\ell(s) = \ell$. If $s = rt$, where $r \in S_k$, $t \in S'_{n-k}$ then $\sigma_s = \sigma_r \cdot t$.

Since $\sigma_r \in Z_k$ we have $\sigma_s \in Z_k \cdot S'_{n-k} \subseteq \mathcal{A}$.

Therefore we may assume the cycle decomposition of s involves a cycle of the form $(\dots\ i\ m)$ where $i, m \in [\![1, n]\!]$, $i \le k$, $m > k$.

If $\ell = 2$ then $s = (i\ m)$, hence

$$\sigma_s = (1\ m) + (2\ m) + \dots + (k\ m) =$$

$$= X_m \underbrace{- (k+1\ m) - (k+2\ m) - \dots - (m-1\ m)}_{\in \Bbbk S'_{n-k}}$$

$\Longrightarrow \sigma_s \in \mathcal{A}$.

Thus may assume $\ell > 2$.

By the induction hypothesis, $\mathcal{B}_{\ell-1} \subseteq \mathcal{A}_\ell$.

<u>Case 1</u>: Suppose $s = rt$ with $r, t \neq 1$ and $\ell(r) + \ell(t) = \ell$. Then $\sigma_r, \sigma_t \in \mathcal{A}$ by induction. Group the terms in $\sigma_r \sigma_t$ as follows:

$$\sigma_r \sigma_t = \sum_{\substack{r' \in {}^{S_k}r = \{\tau r \tau^{-1} \mid \tau \in S_k\} \\ r'' \in {}^{S_k}t \\ r', r'' \text{ move} \\ \text{a common point}}} r' r'' + \sum_{\substack{r' \in {}^{S_k}r \\ r'' \in {}^{S_k}t \\ r', r'' \text{ don't move any} \\ \text{common points.}}} r' r''$$

r', r'' move a common point $\Leftrightarrow \ell(r'r'') < \ell(r') + \ell(r'')$ and $\ell(r') + \ell(r'') = \ell(s) + \ell(t) = \ell$. Hence the left sum belongs to $\mathcal{A}$ by the induction hypothesis. If r', r'' don't move common points then s and $r'r''$ have the same marked cycle shape, hence are S_k-conjugate. Therefore the second sum is a nonzero multiple of σ_s.

$$\underset{\substack{\cap \\ \mathcal{A}}}{\sigma_r}\,\underset{\substack{\cap \\ \mathcal{A}}}{\sigma_t} = \underset{\substack{\cap \\ \mathcal{A}}}{(\text{1st sum})} + \underset{\substack{\cap \\ \mathbb{Z}_{>0}}}{d}\cdot\sigma_s \Longrightarrow \sigma_s \in \mathcal{A}.$$

<u>Case 2</u>: s cannot be factored as $r \cdot t$ where $r, t \neq 1$ are non-overlapping

Then s must be a single cycle:

$$s = (j_1, \dots, j_{\ell-2}, i, m) \qquad i \leq k,\ m > k.$$

Factor $s = rt$, $r = (i\ m)$, $t = (j_1, \dots, j_{\ell-1}, m)$

By induction $\sigma_r, \sigma_t \in \mathcal{A}$ hence

$$\mathcal{A} \ni \sigma_r \sigma_t = \sum_{\substack{1 \leq i' \leq k \\ t' \in {}^{S_k}t}} (i'\ m) \cdot t'$$

The sum of terms with $\ell((i'\ m)t') < \ell$ belongs to $\mathcal{A}$ by induction. The terms of length ℓ all are of the form

$$(i'\ m)(j_1', \dots, j_{\ell-2}', m), \quad i' \notin \{j_1', \dots, j_{\ell-2}'\}$$

which equals $(j_1', \dots, j_{\ell-2}', i', m) \in {}^{S_k}s$

hence those terms add up to $d \cdot \sigma_s$, $d \in \mathbb{Z}_{>0}$.

$\Longrightarrow \sigma_s \in \mathcal{A}$. <u>QED</u>.