MATH 618 LECTURE 17.

READ §3.6.1, §3.6.2

<u>HW17</u>: Let G, H be finite groups, V an irrep of G and W an irrep of H. Prove that $V \otimes W$ is an irrep of $G \times H$ w.r.t.

$$\rho_{V \otimes W}((g,h)) v \otimes w = \rho_V(g) v \otimes \rho_W(h) w.$$

Recall: Any central element acts by a scalar on a f.d. irrep (by Schur's Lemma).

$\Bbbk G$ case: $\rho_S(c) = c_S \cdot \mathrm{Id}_S$ Take trace:

$$\Rightarrow (\dim S) \cdot c_S = \chi_S(c) \quad c \in Z(\Bbbk G)$$

$$\sigma_x = \sum_{g \sim x} g \qquad \text{class sum}$$

$$(\sigma_x)_S = \frac{\chi_S(\sigma_x)}{\dim S} = \frac{|G_x| \cdot \chi_S(x)}{\dim S}$$

$\uparrow$
value of σ_x on an irr S

<u>Thm</u> (Frobenius' Divisibility Thm)
If S is an irrep of a finite group G then $\dim S \mid |G|$.

<u>Proof</u>

$$\frac{|G|}{\dim S} = \frac{|G|}{\dim S}(\chi_S, \chi_S) =$$

$$= \frac{1}{\dim S} \sum_{x} |C_x| \chi_S(x) \chi_S(x^{-1})$$

x in a set of class representatives

$$= \sum_{x} (\sigma_x)_S \cdot \chi_S(x^{-1})$$

$\sigma_x \in Z(\mathbb{Z}G)$ fin. gen. $\mathbb{Z}$-module

$\Rightarrow \sigma_x$ integral over $\mathbb{Z}$

$\Rightarrow (\sigma_x)_S \in \mathbb{k}$ integral over $\mathbb{Z}$

$\Rightarrow (\sigma_x)_S \in \mathbb{A}$

Also $\chi_S(x^{-1}) \in \mathbb{A}$

$\Rightarrow \dfrac{|G|}{\dim S} \in \mathbb{Q} \cap \mathbb{A} = \mathbb{Z}$

QED

Thm (Jordan's Density Thm)

If (V, ρ) is a fd irrep of an algebra A, then
$$\rho : A \to \mathrm{End}(V)$$
is **surjective**.

Pf: Future.

Cor (Burnside's Thm §1.4.6)
($A = \Bbbk G$ case.)

<u>Thm</u> (Schur's Divisibility Thm)
If V is an irrep of a finite group G, then

$$\dim V \mid [G : Z(G)]$$

<u>Proof</u> For any $m \in \mathbb{Z}_{>0}$,

$$\Bbbk G^m \cong (\Bbbk G)^{\otimes m} \twoheadrightarrow \operatorname{End}(V)^{\otimes m} \cong \operatorname{End}(V^{\otimes m})$$

Burnside's Thm

hence $V^{\otimes m}$ is an irrep of G^m. Consider

$$C = \{(c_1, c_2, \ldots, c_m) \in Z(G)^m \mid c_1 \cdots c_m = 1_G\}$$

C is a normal subgroup of G^m.
Moreover if $c \in C$ then
$c \cdot v = v \quad \forall v \in V^{\otimes m}$
$\Rightarrow V^{\otimes m}$ irrep of G^m/C.

By Frobenius Divisibility Thm,

$$\dim(V^{\otimes m}) \mid |G^m/C|$$

C has order $Z(G)^{m-1}$.

$$\Rightarrow (\dim V)^m \mid |Z(G)| \cdot [G:Z(G)]^m$$

Put $q = \dfrac{[G:Z(G)]}{\dim V}$.

Then $q^m \in \dfrac{1}{|Z(G)|}\mathbb{Z} \quad \forall m \in \mathbb{Z}_{>0}$.

$\Rightarrow q \in \mathbb{Z}$.

QED.