MATH 618 LECTURE 16

READ: §3.6.1 (Frobenius divisibility Thm)

<u>HW16</u>: Show that $\mathbb{A} \cap \mathbb{Q} = \mathbb{Z}$, where $\mathbb{A}$ is the ring of algebraic integers.

Integral dependence.

$R \subset S$ comm. rings.

="integer-like"

$s \in S$ is integral over R if $p(s)=0$ for some monic polynomial $p(x) \in R[x]$.

Lemma TFAE:

1) $s \in S$ is integral over R.
2) The subring $R[s]$ of S generated by $R \cup \{s\}$ is finitely generated as an R-module.
3) $R[s]$ is contained in a subring C of R which is finitely generated as an R-module.

Proof 1) $\Rightarrow$ 2): If $s^n + r_1 s^{n-1} + \cdots + r_n = 0$ where $r_i \in R$, then

$$R[s] = R1_R + Rs + Rs^2 + \cdots + Rs^{n-1}.$$

2) $\Rightarrow$ 3) Take $C = R[s]$. WLOG $x_1 = 1_C$

3) $\Rightarrow$ 1): Suppose $C = Rx_1 + \cdots + Rx_n$

Then
$$\begin{cases} s x_1 = a_{11} x_1 + \cdots + a_{1n} x_n \\ s x_2 = a_{21} x_1 + \cdots + a_{2n} x_n \\ \quad \vdots \\ s x_n = a_{n1} x_1 + \cdots + a_{nn} x_n \end{cases}$$
for some $a_{ij} \in R$.

$$\begin{bmatrix} s-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{12} & s-a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1n} & -a_{n2} & \cdots & s-a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Multiply from the left by the adjugate matrix $\Rightarrow \det(\delta_{ij} \cdot s - a_{ij})_{ij} \cdot x_k = 0 \ \forall k$

Take $k=1 \Rightarrow \det(\delta_{ij} \cdot s - a_{ij})_{ij} = 0$.

Expanding this shows that s is integral over R. QED

Theorem Given $R \subset S$, the set $\bar{R}$ of all $s \in S$ which are integral over R is a subring of S.

Proof If $s_1, s_2 \in \bar{R}$ then $R[s_1][s_2]$ is a finitely generated R-module.

$$R[s_1 + s_2] \subset R[s_1][s_2]$$

$$R[s_1 s_2] \subset R[s_1][s_2]$$

hence $s_1 + s_2 \in \bar{R}$ and $s_1 s_2 \in \bar{R}$.

QED

Def $\bar{R}$ is the integral closure of R in S.

Def The integral closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}$

field of algebraic numbers

is called the ring of algebraic integers and is denoted by $\mathbb{A}$.

Ex $\alpha = \frac{1+\sqrt{5}}{2}$ satisfies $\alpha^2 - \alpha - 1 = 0$, so $\alpha \in \mathbb{A}$.

$\beta = \sqrt[3]{7}$ satisfies $\beta^3 - 7 = 0$ so $\beta \in \mathbb{A}$. Since $\mathbb{A}$ is a ring for example

$$3\alpha\beta - 2\beta^2 \in \mathbb{A}$$

(but it's nontrivial to prove directly)

__Theorem__ Let G be a finite group and V a fin. dim'l rep of G. Then

$$\chi_V(g) \in \mathbb{A} \quad \forall g \in G$$

__Proof__ Let $g \in G$. Consider the cyclic subgroup $\langle g \rangle \leq G$. By Maschke's theorem

$$V\downarrow_{\langle g \rangle} = W_1 \oplus W_2 \oplus \cdots \oplus W_k$$

for some (not necessarily inequivalent) irreps W_i of $\langle g \rangle$. Since $\langle g \rangle$ is abelian, all W_i are 1-dim'l. Choose $w_i \in W_i \setminus \{0\}$ $\forall i = 1, \dots, k$. Then wrt the basis $\{w_1, \dots, w_k\}$ for V: $\rho_V(g) = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{bmatrix}$ for some scalars $\lambda_i \in \mathbb{k}$. Since g has finite order, say $g^n = 1$, we have $\lambda_i^n = 1$ $\forall i = 1, .., k$. Thus $\lambda_i \in \mathbb{A}$ $\forall i = 1, .., k$.

$\Rightarrow \chi_V(g) = \lambda_1 + \lambda_2 + \dots + \lambda_k \in \mathbb{A}$ since $\mathbb{A}$ is a ring. QED

<u>Ex.</u> All entries of all character tables we've seen have indeed been algebraic integers.

Central Characters.

Def A central character of an algebra A is an algebra map $\varphi : Z(A) \to \mathbb{k}$

Lemma Let (V, ρ) be an irreducible rep of an algebra A. Then there exists a central character φ of A such that

$$\rho(z) = \varphi(z)\, \mathrm{Id}_V \quad \forall z \in Z(A).$$

Proof Let $z \in Z(A)$ and put $T = \rho(z)$. Then $\forall a \in A$:

$$T \circ \rho(a) = \rho(za) = \rho(az) = \rho(a) \circ T.$$

Thus $T \in \mathrm{End}_A(V) = \mathbb{k}\, \mathrm{Id}_V$ by

Schur's Lemma. Therefore $\rho(z) = \varphi(z) \mathrm{Id}_V$ for some scalar $\varphi(z) \in \mathbb{k}$. It is easy to check that $z \mapsto \varphi(z)$ is an algebra map hence defines a character φ. QED

Application to groups

Pick $x \in G$, let $G_x = \{gxg^{-1} \mid g \in G\}$, and put

$$\sigma_x = \sum_{g \in G_x} g$$

This is a class sum (sum of all g in a conjugacy class). These form a basis for $Z(\mathbb{k}G)$. So if $S \in \mathrm{Irr}\,\mathbb{k}G$, then σ_x acts on S by a scalar.

Let $(\sigma_x)_S$ be this scalar.
That is:

$$\rho_S(\sigma_x) = (\sigma_x)_S \, \mathrm{Id}_S$$

Taking trace gives

$$\chi_S(\sigma_x) = (\sigma_x)_S \dim S$$

So $(\sigma_x)_S = \dfrac{\chi_S(\sigma_x)}{\dim S} =$

$$= \frac{\sum_{g \in G_x} \chi_S(g)}{\dim S} = \frac{|G_x| \, \chi_S(x)}{\dim S}$$

<u>Next time</u>: Will use this to prove $\dim S$ divides $|G|$.