MATH 618 LECTURE 15

READ: §3.6.4 (Clifford Theory).

HW15: Let G be the following subgroup of $GL_2(\mathbb{F}_p)$:

$$G = \left\{ \begin{bmatrix} 1 & 0 \\ a & x \end{bmatrix} : a \in \mathbb{F}_p, x \in \mathbb{F}_p, x \neq 0 \right\}$$

with respect to matrix multiplication. Find the character table of G.

<u>Hint:</u> $N = \left\{ \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} : a \in \mathbb{F}_p \right\} \trianglelefteq G$.

($\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is the field with p elements)

Automorphism-twisting.

Let G be a group, (V, ρ_V) a rep of G, $\theta : G \to G$ an automorphism of G.

The θ-twisted rep $({}^\theta V, \rho_{{}^\theta V})$ is defined by

$$ {}^\theta V = V \quad \text{(as vector spaces)} $$

$$ \rho_{{}^\theta V} = \rho \circ \theta^{-1} = (\theta^{-1})^* (\rho) $$

Thus ${}^\theta V$ is the pullback (=restriction) of the rep V along $\theta^{-1} : G \to G$.

$$ {}^{(\theta_1 \circ \theta_2)} V = {}^{\theta_1} ({}^{\theta_2} V) $$

$\forall \theta_1, \theta_2 \in \mathrm{Aut}(G)$, $\forall V \in \mathrm{Rep}\, G$.

If $T: V \to W$ is an intertwining operator:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\rho_V(g)"'] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[d, "\rho_W(g)"] \\ V \arrow[r, "T"] & W \end{tikzcd} \quad \forall g \in G$$

then the same map T is an intertwining operator

$$T: {}^{\theta}V \to {}^{\theta}W$$

Conclusion: $\boxed{\mathrm{Aut}(G) \text{ acts on } \mathrm{Rep}\, G}$

Assume $\dim V < \infty$.

The character of ${}^{\theta}V$ is

$$\chi_{{}^{\theta}V}(g) = \mathrm{Tr}\, \rho_{{}^{\theta}V}(g) =$$

$$= \mathrm{Tr}\, \rho_V(\theta^{-1}(g)) =$$

$$= \chi_V(\theta^{-1}(g)).$$

In particular $\forall V \in \operatorname{Rep} G$:

$$V \text{ irr} \iff {}^{\theta}V \text{ irr}.$$

The inner automorphisms are

$$\mathrm{Inn}(G) = \{\theta_x \mid x \in G\}$$

where $\theta_x : G \to G$, $\theta_x(g) = xgx^{-1}$

We have $\mathrm{Inn}(G) \cong G/Z(G)$

Also, $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$ and

$$\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G).$$

The point is, inner automorphisms θ_x act trivially on Rep V

$$\chi_{{}^{\theta_x}V}(g) = \chi_V(\theta_x^{-1}(g)) \quad \leftarrow \text{proof for fin. dim } V$$

$$= \chi_V(x^{-1}gx) = \chi_V(g)$$

$\forall g \in G$. Thus ${}^{\theta_x}V \cong V$.

Example. Let $N \trianglelefteq G$.

Then any $g \in G$ acts by conjugation on N.

$$G \to \mathrm{Aut}(N).$$
$$g \mapsto \theta_g|_N$$

In general $\theta_g|_N$ is not an inner automorphism of N and may act nontrivially on Rep N.

(When is $\theta_g|_N \in \mathrm{Inn}(N)$?)

Notation When $N \trianglelefteq G$ and $W \in \mathrm{Rep}\, N$, put

$${}^gW = {}^\theta W, \text{ where } \theta = \theta_g|_N.$$

Clifford's Theorem.

Let G be a finite group and $N \trianglelefteq G$ a normal subgroup. Let V be an irrep of G. Then there exists a subgroup H of G, containing N, and an irrep W of N such that

$$V\downarrow_N \cong \bigoplus_{gH \in G/H} \left({}^gW\right)^{\oplus m}, \text{ where}$$

1) $m \mid [H:N]$
2) ${}^gW \in \mathrm{Irr}\,\Bbbk N$ and ${}^gW \cong {}^{g'}W \Leftrightarrow gH = g'H$
3) $\dim {}^gW = \dim W \quad \forall g \in G$
4) $W^{\oplus m}$ is a subrep of $V\downarrow_H$ and $V \cong \mathrm{Ind}_H^G\left(W^{\oplus m}\right)$

<u>Proof</u> Let W be an irr subrep of $V\downarrow_N$. Then

$$V = \mathbb{k}G.W = \sum_{g\in G} g.W$$

<u>Claim</u>: $T: {}^gW \longrightarrow g.W$, $w \mapsto g.w$ is an equivalence of reps of N.

Indeed,

$$\left(T\circ\rho_{{}^gW}(n)\right)(w) = T\left(\rho_W(g^{-1}ng)(w)\right)$$

$$= g.\left(\rho_W(g^{-1}ng)(w)\right) =$$

$$= \rho_V(g)\rho_V(g^{-1}ng)(w) =$$

$$= \rho_V(n)\rho_V(g)(w) =$$

$$= \rho_W(n)(g.w) = \left(\rho_W(n)\circ T\right)(w)$$

and T is bijective, since

g acts invertibly.

Define H to be the "stabilizer" of $W \in \mathrm{Rep}\, N$:

$$H = \{ h \in G \mid {}^{h}W \cong W \}.$$

Then $N \leq H \leq G$ and

$${}^{g}W \cong {}^{g'}W \iff gH = g'H.$$

$$V\!\downarrow_N = \sum_{g \in G} g.W = \sum_{gH \in G/H} \; \sum_{h \in H} \overbrace{gh.W}^{\cong\, {}^{g}W \ \forall h \in H}$$

$$\cong \bigoplus_{gH \in G/H} \left({}^{g}W\right)^{\oplus m_g} \quad , \quad m_g \in \mathbb{Z}_{>0}$$

See remark after proof

But G permutes the terms bijectively:

$$\sum_{h\in H} gh.W \xrightarrow[\cong]{g_1\cdot} \sum_{h\in H} g_1gh.W$$

$$\begin{array}{ccc} \sum_{h\in H} gh.W & & \sum_{h\in H} g_1gh.W \\ \cong & & \cong \\ ({}^{g}W)^{\oplus m_g} & & ({}^{g_1g}W)^{\oplus m_{g_1g}} \end{array}$$

$\Rightarrow m_g \cdot \dim W = m_{g_1 g} \cdot \dim W$

$\Rightarrow m_g = m_{g_1\cdot g} \;\; \forall g, g_1 \in G$

$\Rightarrow m_g = m$ constant.

[We skip proof of 1)]

We always have ${}^{g}W = W$ as vector spaces,

$\Rightarrow \dim {}^{g}W = \dim W \;\; \forall g\in G.$

Lastly, $W^{\oplus m}$ came from $\sum_{h\in H} h.W$ hence is a subrep $V\downarrow_H$, and

$$V = \bigoplus_{gH\in G/H} g.\left(W^{\oplus m}\right) \cong \Bbbk G \otimes_{\Bbbk H} \left(W^{\oplus m}\right)$$
$$= \mathrm{Ind}_H^G\left(W^{\oplus m}\right)$$

<u>QED</u>.

<u>Remark</u> If a rep V of G is a sum of irr subreps $\{V_i\}_{i\in I}$, then there is a subset $J\subseteq I$ such that $V = \bigoplus_{i\in J} V_i$.

Exercise: Prove this for $\dim V<\infty$.