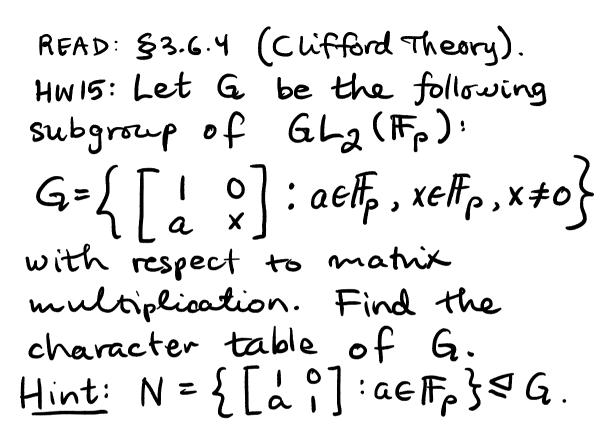
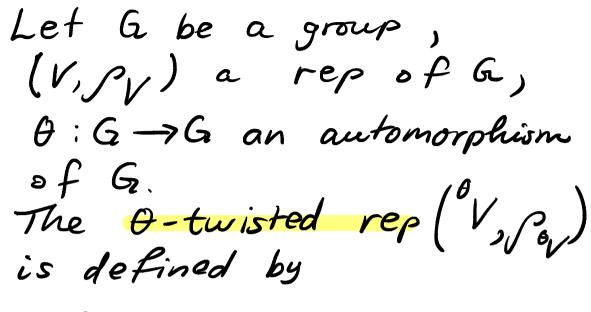
MATH 618 LECTURE 15



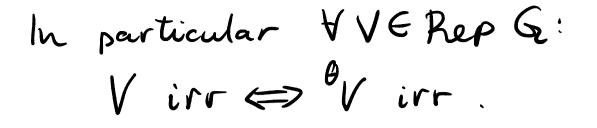
 $(\mathbb{F}_p = \mathbb{Z}_{p\mathbb{Z}})$ is the field with p elements)

Automorphism-twisting.



 $\theta = V$ (as vector spaces) $\mathcal{S}_{\theta_V} = \mathcal{S}_{\theta_V} = (\theta^{-1})^*(\rho)$ Thus "V is the pullback (= restriction) of the rep V along 0":G-G. $\begin{array}{c} (\theta_1 \circ \theta_2) \\ V = \theta_1 \left(\theta_2 \right) \\ \forall \theta_1, \theta_2 \in \operatorname{Aut}(G), \forall V \in \operatorname{Rep} G. \end{array}$

If T:V-W is an intertwining operator: $V \xrightarrow{T} W$ $\int V (g) V \xrightarrow{P} V (g) V = V (g) V = G$ $V \xrightarrow{V} W$ then the same map T is an intertwining operator $\mathcal{T}: ^{\circ}V \longrightarrow ^{\circ}W$ Conclusion: Aut (G) acts on Rep G Assume dimV<00. The character of V is $X_{\Theta_V}(g) = Tr \mathcal{P}_{\Theta_V}(g) =$ = Tr p (((g)) = $=\chi_V(\theta^{-1}(g)).$



The inner automorphisms are $lnn(G) = \{\theta_x \mid x \in G\}$ where $\theta_{x}: G \rightarrow G, \theta_{x}(g) = xgx^{-1}$ We have $Inn(G) \cong G/Z(G)$ Also, Inn(6) = Aut(6) and Out(G) := Aut(G)/Inn(G). The point is, inner automorphisms θ_X act trivially on Rep V $\chi_{\Theta_{X}}(g) = \chi_{V}(\Theta_{X}(g)) \notin fin.dim$ $=\chi_{V}(x^{-\prime}gx)=\chi_{V}(g)$ YgEG. Thus OxVIV.

Example. Let NSG. Then any geG acts by Conjugation on N. $G \rightarrow Aut(N).$ g ~ Og N In general $\theta_{g|N}$ is not an inner automorphism of N and may act nontrivially on Rep N. (When is $\theta_{g|N} \in Inn(N)?)$ Notation When NSG and WEREPN, put ${}^{9}W = {}^{0}W$, where $\Theta = \Theta_{g|N}$.

Clifford's Theorem.

Let G be a finite group and N=G a normal subgroup. Let V be an irrep of G. Then there exists a subgroup H of G, containing N, and an irrep W of N such that $V \downarrow_N \cong \bigoplus ({}^{g}W)^{\bigoplus}$, where $gH \in G_{H}$ i) m [H:N] ⁹W∈Irr IKN and ⁹W²⁹W ⇔ gH=g'H 3) $\dim^{9}W = \dim W \forall g \in G$ 4) $W^{\oplus m}$ is a subrep of VV_{H} and $V \cong \operatorname{Ind}_{H}^{G}(W^{\oplus m})$

Proof Let W be an irr subrep of VIN. Then $V = KG.W = \sum g.W$ JEG $\underline{Claim}: T: \overset{g}{\longrightarrow} M \longrightarrow g.W \longrightarrow w \longmapsto g.W$ is an equivalence of reps of N. Indeed, $(T \circ \int_{\mathcal{Y}}^{g} (n))(w) = T (\mathcal{V}_{W}(g'ng)(w))$ = g.(Pw(g⁻¹ng)(w)) = = ~ (g) ~ (g⁻¹ng) (w) = $= \mathcal{N}(n)\mathcal{N}(g)(w) =$ $= \mathcal{P}_{W}(n) (g.w) = \left(\mathcal{P}_{W}(n) \circ T \right) (w)$ and T is bijective, since

g acts invertibly. Define H to be the "stabilizer" of WERep N: $H = \chi h \in G \mid W \cong W$ Then N≤H≤G and $g_{W} \simeq g'_{W} \iff gH = g'H.$ $VV_N = \sum_{g \in G} g.W = \int_{M} g W \forall heH$ = $\sum_{gH \in G/H} \sum_{h \in H} (gh.W)^{=1}$ $\stackrel{\sim}{=} \bigoplus \left({}^{g}_{W} \right)^{\bigoplus m_{g}}$ $\int gH \in G_{H}$, mg e Z70 L see remark after proof

But G permutes the terms
bijectively:

$$\sum gh.W \xrightarrow{g_1} \sum g_1gh.W$$

 heH
 $2H$
 $(gW)^{Omg}$
 $= m_{g_1g} dim W$
 $= m_{g_1g} dim W$
 $= m_g = m_{g_1g} \forall g_2 g_1 \in G$
 $= m_g = m_{g_1g} \forall g_2 g_1 \in G$
 $= m_g = m_{g_1g} \text{ constant.}$
 $[We skip proof of 1)]$
 $We always have {}^{9}W = W as$
 $vector spaces,$
 $= dim {}^{9}W = dim W \forall geG.$

•

Lastly,
$$W^{\oplus m}$$
 came from
 Σ h.W hence is a
hEH
subrep VV_H , and
 $V = \bigoplus g. (W^{\oplus m}) \cong kG \otimes (W^{\oplus m})$
gHEG/H = $Ind_H^G (W^{\oplus m})$
 KH

Remark IF a rep V of G is a sum of irr subreps {Vi}iEI, then there is a subset JEI such that V= \bigoplus Vi. iEJ Exercise: Prove this for dinV<∞.