MATH 618 LECTURE 14

<u>HW14</u>: Find the character table for the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$

$$\begin{cases} ijk = i^2 = j^2 = k^2 = -1 \\ -1 \text{ is central, } (-1)^2 = 1 \end{cases}$$

S_5

Recall:

$$\chi_{\Lambda^2 V}(g) = \frac{1}{2}\left[\chi_V(g)^2 - \chi_V(g^2)\right]$$

$$\left.\begin{aligned} 5 &= 5 \\ &= 4+1 \\ &= 3+2 \\ &= 3+1+1 \\ &= 2+2+1 \\ &= 2+1+1+1 \\ &= 1+1+1+1+1 \end{aligned}\right\} \quad t = 7 \text{ conjugacy classes} \quad (\Rightarrow 7 \text{ irreps})$$

$$\chi_{V_4}(\sigma) = (\#\text{fix points of } \sigma) - 1 \quad \forall \sigma \in S_5$$

$\Rightarrow$ Can compute $\chi_{\Lambda^2 V_4}$ (see table).

$$(\chi_{\Lambda^2 V}, \chi_{\Lambda^2 V}) = \frac{1}{5!}\left(6^2 + 24 \cdot 1^2 + 15 \cdot (-2)^2\right) = 1$$

$\Rightarrow \Lambda^2 V$ is irreducible.

	(1)	(12)	(123)	(1234)	(12345)	$(12)(34)$	$(12)(345)$
$\lvert C_i \rvert$	1	$\binom{5}{2}=10$	$2\binom{5}{3}=20$		$4!=24$	$5\cdot 3=15$	$2\binom{5}{3}=20$
χ_{V_4}	4	2	1	0	-1	0	-1
$\chi_{\Lambda^2 V_4}$	6	0	0	0	$\frac{1}{2}(1-(-1)) = 1$	-2	0

$$(V_4 \otimes V_4)^{S_5} \cong (V_4 \otimes V_4^*)^{S_5}$$

$$\cong \mathrm{End}_{S_5}(V_4) \cong \mathbb{k}$$

$$(\chi_{V_4}, \chi_{V_4 \otimes V_4}) = \frac{1}{5!}\left(1\cdot 4^3 + 10\cdot 2^3 + 20\cdot 1^3 + 24\cdot(-1)^3 + 20\cdot(-1)^3\right) =$$

$$= \frac{1}{120}\left(64 + 80 \cancel{+20} - 24 - \cancel{20}\right) = 1$$

unknown 5-dim rep.

$$\Rightarrow V_4 \otimes V_4 \cong \mathbb{k} \oplus \Lambda^2 V_4 \oplus V_4 \oplus W$$

$$\text{dim:} \quad 16 = 1 + \binom{4}{2} + 4 + 5$$

$$\chi_W = (\chi_{V_4})^2 - 1 - \chi_{\Lambda^2 V_4} - \chi_{V_4}$$

S_5	(1)	(12)	(123)	(1234)	(12345)	$(12)(34)$	$(12)(345)$
$\lvert C_i \rvert$	1	$\binom{5}{2}=10$	$2\binom{5}{3}=20$	$5\cdot 3!=30$	$4!=24$	$5\cdot 3=15$	$2\binom{5}{3}=20$
$\mathbb{1}$	1	1	1	1	1	1	1
sgn	1	-1	1	-1	1	1	-1
χ_{V_4}	4	2	1	0	-1	0	-1
$\chi_{V_4^\pm}$	4	-2	1	0	-1	0	1
$\chi_{\Lambda^2 V_4}$	6	0	0	0	1	-2	0
χ_W	5	1	-1	-1	0	1	1
$\chi_{W^\pm}$	5	-1	-1	1	0	1	-1

$$(\chi_W, \chi_W) = \frac{1}{120}\left(5^2 + 10\cdot 1 + 20\cdot 1 + 30\cdot 1 + 15\cdot 1 + 20\cdot 1\right)$$

$$= \frac{1}{120}(25 + 60 + 35) =$$

$\Rightarrow$ W and $W^\pm$ are irreps.

Clifford's Theorem (index 2 case)

Let G be a group and H be a (normal) subgroup of index 2. Let V be a finite-dim'l irrep of G. Then:

set difference

1) If $\chi_V(g) = 0 \quad \forall g \in G \setminus H$, then $V\downarrow_H$ is a direct sum of two irreps of H, of equal dimension:

$$V\downarrow_H = W_1 \oplus W_2$$

$W_i \in \operatorname{Irr} \mathbb{k}H$ and $\dim W_1 = \dim W_2$

2) If $\chi_V(g) \neq 0$ for some $g \in G \setminus H$, then $V\downarrow_H$ is irreducible.

__Proof__ For any $x \in G \setminus H$, $G = H \cup xH$ and $xH = Hx$. Let W be an irreducible subrep of $V{\downarrow_H}$. Then

$$\begin{aligned} V &= \Bbbk G . W \quad \text{(since } W \neq 0 \text{ and } V \text{ irr.)} \\ &= (\Bbbk H + \Bbbk x H).W \quad \text{(since } G = H \cup xH\text{)} \\ &= W + x.W \quad \text{(since } H.W \subseteq W\text{)}. \end{aligned}$$

Since $Hx = xH$, $x.W$ is also a subrep of $V{\downarrow_H}$. In fact $x.W$ is an irrep of H, and $\dim(x.W) = \dim W$.
Thus, either $V{\downarrow_H} = W$, or $V{\downarrow_H} = W \oplus x.W$.

If $V{\downarrow_H} = W \oplus x.W$ then

$$\rho_V(x) = \begin{array}{c} \begin{array}{cc} W & x.W \end{array} \\ \left[\begin{array}{c|c} 0 & * \\ \hline * & 0 \end{array}\right] \end{array}$$

since $x.W \subseteq x.W$ and $x.(x.W) \subseteq W$
($G/H \cong C_2 \Rightarrow x^2 \in H$)

so $\chi_V(x) = 0$. This proves 2).

To prove 1), suppose $\chi_V(g)=0$ $\forall g \in G \setminus H$. Then:

$$(\chi_{V\downarrow_H}, \chi_{V\downarrow_H}) = \frac{1}{|H|}\sum_{h\in H} \chi_V(h)\chi_V(h^{-1}) =$$

$$= \frac{1}{|H|}\sum_{g\in G} \chi_V(g)\chi_V(g^{-1})$$

$$= \underbrace{\frac{|G|}{|H|}}_{2} \underbrace{(\chi_V, \chi_V)}_{1} = 2$$

Therfore $V\downarrow_H$ is not irreducible so $V\downarrow_H = W \oplus x.W$

QED.

Application to $A_5 \trianglelefteq S_5$

Conjugacy classes of A_n:
If $\sigma \in A_n$ has odd & distinct cycle lengths, then the conjugacy class ${}^{S_n}\sigma = \{\tau\sigma\tau^{-1} \mid \tau \in S_n\}$ splits into two A_n-classes of equal size.
Otherwise ${}^{S_n}\sigma = {}^{A_n}\sigma$.

A_5:

$$(1) \quad (123)\overbrace{(4)(5)}^{\text{same length}} \quad (12345) \quad (12354) \quad \overbrace{(12)(34)}^{\text{same}}(5)$$

Recall character table for S_5.
χ_{V_4} & χ_W are nonzero at $(12) \in S_5 \setminus A_5$

so $\quad V_4\downarrow_{A_5}$ and $W\downarrow_{A_5}$ are irreps.

$\chi_{\Lambda^2 V_4}$ vanishes on $S_5 \setminus A_5$.

$\Rightarrow \Lambda^2 V_4 \downarrow_{A_5} = X \oplus X'$

for some irreps X, X' of A_5

of dimension $\frac{1}{2} \dim \Lambda^2 V_4 = \frac{1}{2}\binom{4}{2} = 3$

A_5:

classes sizes	(1) 1	(123) 20	(12345) 12	(12354) 12	(12)(34) 15
$\mathbb{1}$	1	1	1	1	1
$\chi_{V_4 \downarrow_{A_5}}$	4	1	-1	-1	0
$\chi_{W \downarrow_{A_5}}$	5	-1	0	0	1
X	3	α	β	γ	δ
X'	3	$-\alpha$	$1-\beta$	$1-\gamma$	$-2-\delta$
$\chi_{\Lambda^2 V_4 \downarrow_{A_5}}$	6	0	1	1	-2

Column orthogonality:

$$\sum_{S \in \mathrm{Irr}\,\Bbbk G} \chi_S(g^{-1})\chi_S(g) = |C_G(g)|$$

$$= \frac{|G|}{|Cl_G(g)|}$$

2nd col.:

$$1^2 + 1^2 + (-1)^2 + 2\alpha^2 = \frac{60}{20} = 3 \Rightarrow \underline{\underline{\alpha = 0}}$$

3rd col:

$$1^2 + (-1)^2 + \beta^2 + (1-\beta)^2 = \frac{60}{12} = 5$$

$$\Rightarrow 2\beta^2 - 2\beta + 1 = 3$$

$$\beta^2 - \beta - 1 = 0$$

WLOG $\beta = (1+\sqrt{5})/2 \Rightarrow 1-\beta = (1-\sqrt{5})/2$

Similarly $\gamma = (1 \pm \sqrt{5})/2$. Orthogonality between 3rd & 4th col $\Rightarrow \gamma = (1-\sqrt{5})/2$

Finally,

5th col: $1^2+0^2+1^2+\delta^2+(-2-\delta)^2 = \frac{60}{15}$

$$2+2\delta^2+4\delta+\cancel{4} = \cancel{4}$$

$$\delta^2+2\delta+1=0 \quad (\delta+1)^2=0$$

$$\underline{\underline{\delta=-1}}$$

$$-2-\delta=-1$$

A_5:

classes / sizes	(1) 1	(123) 20	(12345) 12	(12354) 12	(12)(34) 15
$\mathbb{1}$	1	1	1	1	1
$\chi_{V_4\downarrow_{A_5}}$	4	1	-1	-1	0
$\chi_{W\downarrow_{A_5}}$	5	-1	0	0	1
X	3	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	-1
X'	3	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	-1