MATH 618 LECTURE 13

READ § 3.5

<u>HW13</u>: Prove that the standard representation V_{n-1} of S_n is irreducible for any $n \geq 2$.

Inflation via $S_4 \twoheadrightarrow S_3$

$a = (12)(34)$
$b = (13)(24)$
$c = (14)(23)$

S_4 acts on $\{a,b,c\}$ by conjugation giving $S_4 \longrightarrow S_{\{a,b,c\}} \cong S_3$

$$(12) \longmapsto (bc) \longmapsto (23)$$
$$(123) \longmapsto (acb) \longmapsto (132)$$
$$(12)(34) \longmapsto (a) \longmapsto (1)$$
$$(1234) \longmapsto (ac) \longmapsto (13)$$

Recall:

S_3	(1)	(12)	(123)
χ_{V_2}	2	0	-1

$$V_3 = \Bbbk v_1 \oplus \Bbbk v_2 \oplus \Bbbk v_3$$

$$v_1 = (1, -1, 0, 0)$$

$$v_2 = (0, 1, -1, 0)$$

$$v_3 = (0, 0, 1, -1)$$

all we really need to compute

$$\rho_{V_3}((12)) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \chi_{V_3}((12)) = 1$$

$$\rho_{V_3}((123)) = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow \chi_{V_3}((123)) = 0$$

$$\rho_{V_3}((12)(34)) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \Rightarrow \chi_{V_3}((12)(34)) = -1$$

Extra checks.

Column orthogonality relation.

For any two $g, h \in G$ (finite grp):

$$\sum_{S \in \operatorname{Irr} \Bbbk G} \chi_S(g^{-1})\chi_S(h) = \begin{cases} |C_G(g)| & \text{if } g \text{ and } h \text{ conjugate} \\ 0 & \text{otherwis} \end{cases}$$

Proof Let $\chi_1, \ldots, \chi_t$ be the characters of the irreps, and $C_1, C_2, \ldots, C_t$ be the conjugacy classes of G. By the row orthogonality relation,

$$\frac{1}{|G|} \sum_{g \in G} \chi_i(g)\chi_j(g^{-1}) = \delta_{ij}$$

Since characters are class functions, putting $d_i = |G|/|C_i| = |C_G(g_i)|$ for $g_i \in C_i$

$$\sum_{k=1}^{t} d_k^{-1} \chi_i(C_k)\chi_j(C_k^{-1}) = \delta_{ij} \quad (*)$$

Introducing the $t\times t$-matrices

$$A = \left(\chi_i(C_j)\right)_{i,j=1}^{t}, \quad A^* = \left(\chi_j(C_i^{-1})\right)_{i,j=1}^{t}$$

$$D = \operatorname{diag}(d_1, d_2, \ldots, d_t),$$

relation $(*)$ may be rewritten

$$A \cdot D^{-1} \cdot A^* = I_t$$

Taking the determinant

$$\det(A)\det(A^*) = d_1 d_2 \cdots d_t \neq 0$$

hence A, A^* are invertible.

Multiplying by A^*, A on left/right:

$$A^* A \cdot D^{-1} \cdot A^* A = A^* A$$

Since $A^* A$ is invertible,

$A^* A = D$. This is equivalent

to: $\displaystyle\sum_{k=1}^{t} \chi_k(C_i^{-1})\chi_k(C_j) = \delta_{ij} \cdot d_i$

QED.

S_4

classes	(1)	(12)	(123)	(12)(34)	(1234)
sizes	1	6	8	3	6
$\mathbb{1}$	1	1	1	1	1
sgn	1	-1	1	1	-1
$\chi_{\tilde{V}_2}$	2	0	-1	2	0
χ_{V_3}	3	1	$a=0$	$b=-1$	$c=-1$
$\chi_{V_3^{\pm}}$	3	-1	$a=0$	$b=-1$	$-c=1$

Column orthogonality:

$(1) \perp (123)$: $1\cdot 1 + 1\cdot 1 + 2\cdot(-1) + 3a + 3a = 0$

$\Rightarrow a = 0$

$(1) \perp (12)(34)$: $1 + 1 + 4 + 3b + 3b = 0 \Rightarrow b = -1$

$(12) \perp (1234)$: $1 + (-1)^2 + 0 + c + (-1)(-c) = 0$

$\Rightarrow c = -1$

Character of standard rep V_{n-1} of S_n

$$M_n = \Bbbk e_1 \oplus \Bbbk e_2 \oplus \cdots \oplus \Bbbk e_n$$ Standard permutation rep.

$\rho_{M_n}(\sigma)$ = permutation matrix

$\Rightarrow \chi_{M_n}(\sigma) = \mathrm{Tr}\,\rho_{M_n}(\sigma) =$

$= \#$ fixpoints for σ

Recall M_n decomposes into two subreps:

$$M_n = \underbrace{\Bbbk(e_1 + \cdots + e_n)}_{\cong \mathbb{1}} \oplus V_{n-1}$$

Thus $\chi_{M_n} = \mathbb{1} + \chi_{V_{n-1}}$, so

$$\boxed{\chi_{V_{n-1}}(\sigma) = (\#\text{fixpoints of } \sigma) - 1 \quad \forall \sigma \in S_n}$$

This gives a direct way to find $\chi_{V_{n-1}}$.

Irreducibility criterion for characters.

Lemma If V is a fin. dim'l rep of a finite group G, then V is irreducible $\Longleftrightarrow (\chi_V, \chi_V) = 1$.

Proof ($\Rightarrow$): By row orthogonality rel.

($\Leftarrow$): By Maschke's Thm,

$$V \cong V_1^{\oplus a_1} \oplus \cdots \oplus V_t^{\oplus a_t}$$

for some $a_i \in \mathbb{Z}_{\geq 0}$, where $\{V_i\}_{i=1}^{t}$ are the irreps of G. By additivity,

$$\chi_V = a_1 \chi_1 + \cdots + a_t \chi_t \quad , \quad \chi_i := \chi_{V_i}.$$

Thus

$$1 = (\chi_V, \chi_V) = a_1^2 + \cdots + a_t^2$$

hence $\exists i : a_i = 1$, $a_j = 0 \; \forall j \neq i$.

So $V \cong V_i$. QED

Wedge and symmetric square

$$V \otimes V \cong \mathrm{Sym}^2 V \oplus \Lambda^2 V$$

$$\chi_{V\otimes V} = \chi_{\mathrm{Sym}^2 V} + \chi_{\Lambda^2 V}$$

$$\chi_{\mathrm{Sym}^2 V}(g) = \mathrm{Tr}\, \rho_{\mathrm{Sym}^2 V}(g)$$

$$\mathrm{Sym}^2 V = \mathrm{Span}\, \{ v_i v_j \mid i \le j \},\ \underset{\text{basis}}{\{v_i\}}$$

Choose a basis of generalized eigenvectors for $\rho_V(g)$

$$\rho_V(g) v_i = \lambda_i v_i + \sum_{j<i} * v_j$$

$$\rho_{\mathrm{Sym}^2 V}(g)\, v_i v_j = \lambda_i \lambda_j v_i v_j + \text{lower terms}$$

$$\Rightarrow \mathrm{Tr}\, \rho_{\mathrm{Sym}^2 V}(g) = \sum_{i \le j} \lambda_i \lambda_j =$$

$$= \frac{1}{2}\left[(\lambda_1 + \dots + \lambda_n)^2 + \lambda_1^2 + \dots + \lambda_n^2 \right]$$

$$\left(\sum \lambda_i\right)^2 = \frac{1}{2}\left[\left(\sum \lambda_i\right)^2 + \sum \lambda_i^2\right] \leftarrow \mathrm{Sym}^2 V$$

$$+ \frac{1}{2}\left[\left(\sum \lambda_i\right)^2 - \sum \lambda_i^2\right] \leftarrow \Lambda^2 V$$

So $\forall g \in G$:

$$\boxed{\begin{aligned} \chi_{\mathrm{Sym}^2 V}(g) &= \frac{1}{2}\left[\chi_V(g)^2 + \chi_V(g^2)\right] \\ \chi_{\Lambda^2 V}(g) &= \frac{1}{2}\left[\chi_V(g)^2 - \chi_V(g^2)\right] \end{aligned}}$$