MATH 618 LECTURE 12

§ 3.5.1

§ 3.5.2

<u>HW12</u>: Prove that TFAE for a finite group G:

1) Every irrep is self-dual.
2) Every f.d. rep is self-dual.
3) Every $g \in G$ is conjugate to its inverse.

__Examples__ For the groups

$$G = S_n \ , \quad G = (C_2)^n \ , \quad G = D_n$$

cyclic grp, dihedral

every element is conjugate to its inverse, hence every fd (ir)rep is self-dual.

__Tensoring with 1-dim'l reps.__

If V is a 1-dim'l rep of a group G, and W is any irrep of G then $V \otimes W$ is also an irrep of G: The map

$$\{\text{Subreps of } W\} \longrightarrow \{\text{Subreps of } V\otimes W\}$$

$$U \longmapsto V \otimes U$$

is a bijection.

Application to symmetric group:

Def Let V be a fd rep of S_n. The corresponding sign-twisted rep is

$$V^{\pm} := V_{sgn} \otimes V$$

Note $\chi_{V^{\pm}} = \chi_{sgn} \cdot \chi_V$ and thus $V^{\pm} \cong V$ iff for every odd $\sigma \in S_n$ we have $\chi_V(\sigma) = 0$.

Ex

S_3	(1)	$\overset{\text{odd}}{(12)}$	(123)
χ_{triv}	1	1	1
χ_{sgn}	1	-1	1
χ_{V_2}	2	0	-1

$V_2^{\pm} \cong V_2$

Representation ring

The representation ring $R(G)$ of a finite group G is the free abelian group on the set of equivalence classes $[V]$ of fd reps of G modulo the subgroup generated by $[V \oplus W] - ([V] + [W])$ for all fd reps V, W. The multiplication in $R(G)$ is defined by

$$[V] \cdot [W] = [V \otimes W].$$

Example

S_3	(1)	(12)	(123)
χ_{triv}	1	1	1
χ_{sgn}	1	-1	1
χ_{V_2}	2	0	-1

Consider $V_2 \otimes V_2$. Its character is

$$\chi_{V_2 \otimes V_2} = \chi_{V_2} \cdot \chi_{V_2}$$ so its values are:

	(1)	(12)	(123)
$\chi_{V_2 \otimes V_2}$	4	0	1

Note! $\chi_{V_2 \otimes V_2} = \chi_{triv} + \chi_{sgn} + \chi_{V_2} =$

$$= \chi_{V_{triv} \oplus V_{sgn} \oplus V_2}$$

$$\Rightarrow V_2 \otimes V_2 \cong V_{triv} \oplus V_{sgn} \oplus V_2$$

$\Rightarrow [V_2] \cdot [V_2] = [V_{triv}] + [V_{sgn}] + [V_2]$ in $\mathcal{R}(S_3)$.

Inflation

Recall: Any group homom.
$\varphi: G \to H$ gives a restriction functor φ^*

$$\mathrm{Rep}\, H \xrightarrow{\varphi^*} \mathrm{Rep}\, G$$
$$(V, \rho) \longmapsto (V, \rho\circ\varphi)$$

$\rho\circ\varphi$ is just the composition.

$$G \xrightarrow{\varphi} H \xrightarrow{\rho} \mathrm{Aut}(V)$$

When φ is surjective we call this process inflation.

Main point: If (V, ρ) is an irrep of H, then $(V, \rho\circ\varphi)$ is an irrep of G! (Pf: Exercise.)