MATH 618 LECTURE 11

READ § 3.3.3 (A plethora of reps)
§ 3.4.2 (Orthogonality rels, Multiplicities)

HW11: Prove Lemma 3.20(a).

Unless otherwise stated, for the rest of the class we will assume that $\mathbb{k}$ is algebraically closed and has characteristic zero.

This implies that for any irreducible representation S we have $D(S) := \mathrm{End}_G(S) = \mathbb{k}$ (Schur's Lemma) and that every fd rep of G is completely reducible (Maschke's Thm).

V^*, $V \otimes W$, $\mathrm{Hom}(V,W)$, V^G

Let V be a representation of a group G.

The dual space $V^* = \mathrm{Hom}_{\mathbb{k}}(V, \mathbb{k})$ becomes a rep of G via the action

$$(g \cdot \xi)(v) = \xi(g^{-1} \cdot v)$$

$\forall g \in G$, $\xi \in V^*$, $v \in V$.

V^* is called the dual rep. to V.

V^* is irreducible iff V is

If $V \cong V^*$ we say that V is self-dual.

$V^{**} \cong V$ if V is fin. dim'l

Origin: $\exists$ algebra antiautomorphism

$$S : \mathbb{k}G \longrightarrow \mathbb{k}G$$
$$g \longmapsto g^{-1}$$

called the antipode.

Example. $G = C_n = \langle g \rangle$

$0 \le k \le n-1$: $V_k = \mathbb{k} v_k$, $g.v_k = \varepsilon^k v_k$

($\varepsilon \in \mathbb{k}^\times$ fixed primitive n:th root of 1
for ex. $\varepsilon = \exp(2\pi i/n)$ when $\mathbb{k} = \mathbb{C}$)

Then $V_k^* = \mathbb{k}\,\xi_k$, $\xi_k(v_k) = 1$

$$(g.\xi_k)(v_k) = \xi_k(g^{-1}.v_k) =$$

$$= \xi_k(\underbrace{\varepsilon^{-k}}_{= \varepsilon^{n-k}} v_k)$$

$$= (\varepsilon^{n-k}\xi_k)(v_k)$$

$$\Rightarrow V_k^* \cong V_{n-k} \quad \forall k = 0, 1, \ldots, n-1$$

Tensor product of reps.

V, W reps of G.

On $V \otimes W$ $(= V \otimes_{\Bbbk} W)$, define an action of $\Bbbk G$ by

$$g.(v \otimes w) = (g.v) \otimes (g.w)$$

$\forall g \in G$, $v \in V$, $w \in W$.

This makes $V \otimes W$ a representation of G called the tensor product (rep) of V and W.

Origin There is an alg map $\Delta : \Bbbk G \to \Bbbk G \otimes \Bbbk G$, given by $g \longmapsto g \otimes g$ called the comultiplication of $\Bbbk G$

$\underline{\underline{V^G}}$

$e := \frac{1}{|G|} \sum g \in kG$

symmetrizing idempotent

$e^2 = e$

$$e_V : V \longrightarrow V^G$$

$$v \longmapsto \frac{1}{|G|} \sum_{g \in G} g.v$$

averaging operator, $e_V^2 = e_V$

$$e_V \sim \left[\begin{array}{c|c} \mathrm{Id}_{V^G} & 0 \\ \hline 0 & 0 \end{array}\right]$$

$$\Rightarrow \boxed{\chi_V(e) = \mathrm{Tr}\, e_V = \dim(V^G)}$$

Again there is a structure on $\Bbbk G$ that allows for the construction of V^G:

$\exists$ alg map $\varepsilon : \Bbbk G \to \Bbbk$ given by $g \mapsto 1 \ \forall g \in G$

called the counit of $\Bbbk G$

Together, S, Δ, ε make $\Bbbk G$ into a Hopf algebra

$\underline{Hom_k(V,W)}$

If V, W are fd reps of G we can use the vector space isomorphism

$$Hom(V,W) \cong W \otimes V^*$$

to turn $Hom(V,W)$ into a rep of G. Explicitly,

$$(g \cdot L)(v) = g \cdot \left(L(g^{-1} \cdot v)\right)$$

$\forall L \in Hom(V,W)$, $g \in G$, $v \in V$.

Cool fact:

$$\mathrm{Hom}_G(V,W) = \mathrm{Hom}(V,W)^G$$

Pf $\text{RHS} = \{L : V \to W \mid g.L = L \ \forall g \in G\}$

$= \{L : V \to W \mid g.L(g^{-1}.v) = L(v) \ \forall v \in V \ \forall g \in G\}$

$\overset{v \mapsto g.v}{=} \{L : V \to W \mid g.L(v) = L(g.v) \ \forall v \in V \ \forall g \in G\}$

$= \text{LHS}.$

Lemma 3.20 (a) V fd rep of G

For all $g \in G$:

i) $\chi_{V^*}(g) = \chi_V(g^{-1})$

ii) $\chi_{V \otimes W}(g) = \chi_V(g)\chi_W(g)$

iii) $\chi_{Hom_k(V,W)}(g) = \chi_W(g)\chi_V(g^{-1})$

Proof HW 11

From this we can prove:

$$\dim Hom_G(V,W) = \frac{1}{|G|}\sum_{g \in G} \chi_W(g)\chi_V(g^{-1})$$

Indeed,

$\dim \mathrm{Hom}_G(V,W)$

$= \dim \mathrm{Hom}(V,W)^G$ since $\mathrm{Hom}_G(V,W) = \mathrm{Hom}(V,W)^G$

$= \chi_{\mathrm{Hom}(V,W)}(e)$ by formula for $\dim(V^G)$

$= \frac{1}{|G|} \sum_{g\in G} \chi_{\mathrm{Hom}(V,W)}(g)$ by def of e

$= \frac{1}{|G|} \sum_{g\in G} \chi_W(g)\chi_V(g^{-1})$ by Lem 3.20

Recall we defined

$$(\varphi,\psi) = \frac{1}{|G|} \sum_{g\in G} \varphi(g)\psi(g^{-1})$$

for class functions $\varphi, \psi : G \to \mathbb{k}$.

<u>Orthogonality</u> <u>Relations</u>.

For any irreps S, T of a finite group G:

$$(\chi_S, \chi_T) = \begin{cases} 1 & S \cong T \\ 0 & S \not\cong T \end{cases}$$

Pf Take dim in Schur's Lemma:

$$\dim \mathrm{Hom}_G(S, T) = \begin{cases} 1 & S \cong T \\ 0 & S \not\cong T \end{cases}$$

& use the dimension formula obtained earlier.

$\blacksquare$

Characters characterize reps!

Cor Let V, W be two finite-dim'l reps of a finite group G. Then V and W are equivalent iff $\chi_V = \chi_W$.

Proof Let $\{V_1, \dots, V_t\}$ be a complete set of representatives for the equivalence classes of irreps of G, and $\chi_i = \chi_{V_i}$.

By Maschke's Thm:

$$V \cong V_1^{\oplus a_1} \oplus \cdots \oplus V_t^{\oplus a_t}$$

$$W \cong V_1^{\oplus b_1} \oplus \cdots \oplus V_t^{\oplus b_t}$$

for some $a_i, b_i \in \mathbb{Z}_{\geq 0}$.

Take characters:

$$\chi_V = a_1\chi_1 + \cdots + a_t\chi_t$$

$$\chi_W = b_1\chi_1 + \cdots + b_t\chi_t$$

Thus if $\chi_V = \chi_W$ then

$a_i = (\chi_V, \chi_i)$ by orthogonality rel.

$= (\chi_W, \chi_i) = b_i \quad \forall i$

Conversely, if $V \cong W$

then $\exists T: V \to W$ st

$\rho_W(g) = T\rho_V(g)T^{-1} \quad \forall g \in G$

$\Rightarrow \chi_W = \chi_V$

$\blacksquare$