MATH 618 LECTURE 10

READ § 1.5.1 (Def. of character, additivity)

§ 3.1.5 (Class functions, Character tables)

§ 3.4.2 (Orthogonality relations).

§ 3.5.2 (Conjugacy classes of S_n)

HW 10 : Calculate the character table for the dihedral group D_4 of order 8.

Characters.

The character of a $\underbrace{\ }_{}$rep V (fin. dim.)
is the function $\chi_V : G \to \Bbbk$
given by

$$\chi_V(g) = \operatorname{Tr} \rho_V(g) \quad \forall g \in G.$$

Example. $G = C_3 = \{1, g, g^2\}$
$V = \Bbbk G$ regular representation.
$\Rightarrow \rho : G \to Aut(\Bbbk G) \cong GL_3(\Bbbk)$
is given by

$$\rho(1) = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad \rho(g) = \begin{array}{c} \begin{array}{ccc} 1, & g, & g^2 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}, \quad \rho(g^2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Therefore $\chi = \chi_V$ is the function
$\chi(1) = 3$, $\chi(g) = 0$, $\chi(g^2) = 0$.

Remark We always have
$$\chi_V(1) = \operatorname{Tr} \rho_V(1) = \operatorname{Tr} Id_V = \dim V$$

__Example.__ G any finite group, $V = \Bbbk G$ regular rep.

Then $\forall g \neq 1$, $\rho(g)$ is a permutation matrix with zero on the diagonal: There is no basis vector $h \in G$ for V s.t. $gh = h$. (since $g \neq 1$).

So $Tr(\rho(g)) = 0 \quad \forall g \neq 1$

So the character χ of the regular rep is

$$\chi(g) = \begin{cases} \dim V & , g = 1 \\ 0 & , g \neq 1 \end{cases}$$

Lemma For any finite-dim'l rep V of a group G we have

$$\chi_V(ghg^{-1}) = \chi_V(h) \quad \forall g,h \in G.$$

Proof $LHS = \mathrm{Tr}\,\rho_V(ghg^{-1}) =$

$$= \mathrm{Tr}\left[\rho_V(g)\rho_V(h)\rho_V(g)^{-1}\right]$$

$$= \mathrm{Tr}\left[\rho_V(h)\cancel{\rho_V(g)^{-1}\rho_V(g)}\right] = RHS$$

$\llcorner \mathrm{Tr}\,AB = \mathrm{Tr}\,BA$

Def Functions $\varphi : G \to \mathbb{k}$ that are constant on each conjugacy class in G are called class functions on G.

$$cf(G) = cf_{\mathbb{k}}(G) = \left\{ \begin{array}{c} \text{all class functions} \\ \text{on } G \end{array} \right\}$$

__Note__ The set $\mathbb{K}^G$ of __all__ functions $\varphi : G \to \mathbb{K}$ is a vector space with pointwise operations:

$$\begin{cases} (\varphi+\psi)(g) = \varphi(g)+\psi(g) & \forall g \in G \\ (\lambda\varphi)(g) = \lambda\,\varphi(g) & \forall \lambda \in \mathbb{K} \end{cases}$$

Moreover $cf(G)$ is a subspace of $\mathbb{K}^G$.

__Example.__ For any fin. dim. rep V of a group G,

$$\chi_V \in cf(G).$$

So for example if $\{V_i\}_{i=1}^t$ is the set of all (up to equivalence) irreps of G, then $\mathrm{Span}_{\mathbb{K}}\{\chi_{V_i}\}_{i=1}^t \subseteq cf(G)$.

__Lemma__ (Additivity)

If V and W are fin.-dim'l reps of a group G, then

$$\chi_{V\oplus W} = \chi_V + \chi_W$$

__Proof__

$$Tr\left(\rho_{V\oplus W}(g)\right) = Tr\left[\begin{array}{c|c} \rho_V(g) & 0 \\ \hline 0 & \rho_W(g) \end{array}\right]$$

$\blacksquare$

Example. For S_3 we have 3 irreps:

$V_{triv} = \Bbbk$, $\rho_{triv}(\sigma) = 1 \quad \forall \sigma \in S_3$

$V_{sgn} = \Bbbk$, $\rho_{sgn}(\sigma) = \operatorname{sgn} \sigma \quad \forall \sigma \in S_3$

$V_2 = \Bbbk\underbrace{(1,-1,0)}_{v_1} \oplus \Bbbk\underbrace{(0,1,-1)}_{v_2}$

$Cl(S_3) = \{ \{(1)\}, \{(12),(13),(23)\}, \{(123),(132)\} \}$

S_3	(1)	(12)	(123)
χ_{triv}	1	1	1
χ_{sgn}	1	-1	1
χ_2	2	$\underline{0}$	$\underline{-1}$

← class representatives

$(12).v_2 = (1,0,-1) = v_1 + v_2$

$\rho_2((12)) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$, $\rho_2((123)) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$

$\underline{Tr = 0}$ $\qquad$ $\underline{Tr = -1}$

$(123).v_2 = (-1,0,1) = -v_1 - v_2$

First row and column are always known:

G	$\{1\}=C_1$	C_2	$\cdots$	C_t
$\chi_1=\chi_{triv}$	$m_1=1$	1	$\cdots$	1
χ_2	m_2			
$\vdots$	$\vdots$			
χ_t	m_t			

because $\chi_i(1)=\dim V_i=m_i\ \forall i$ and $\chi_1(g)=1\ \forall g\in G$.

$t=\#\text{irreps}=\#\text{conjugacy classes}$

Example $G = C_n = \langle g \rangle$ cyclic.

Every conjugacy class is a singleton $\{g\}$.

Every irrep is one-dim'l. and thus is a group homomorphism

$$\rho : C_n \longrightarrow \mathbb{k}^{\times} \; (= GL_1(\mathbb{k}))$$

Since C_n is cyclic, ρ is uniquely determined by $\rho(g)$ which must satisfy $\rho(g)^n = 1$

Thus $\{\text{irreps}\} \longleftrightarrow \{n\text{:th roots of unity}\}$

$$\rho_k \longleftrightarrow \rho_k(g) = \varepsilon^k$$

ε fixed primitive n:th root of 1

C_n	g^0	g^1	g^2	$\cdots$	g^{n-1}
χ_0	1	1	1	$\cdots$	1
χ_1	1	ε	ε^2	$\cdots$	ε^{n-1}
χ_2	1	ε^2	ε^4	$\cdots$	$\varepsilon^{2(n-1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_{n-1}	1	ε^{n-1}	$\varepsilon^{(n-1)2}$		$\varepsilon^{(n-1)^2}$

$$\chi_k(g^\ell) = Tr\left[\rho_k(g)^\ell\right] = \varepsilon^{k\ell}$$

Orthogonality Relation

On the space $cf(G)$ of class functions, define a bilinear form:

$$(\varphi, \psi) = \frac{1}{|G|} \sum_{g \in G} \varphi(g)\, \psi(g^{-1})$$

Thm If V and W are irreps of a finite group G and $\operatorname{char} \Bbbk \nmid |G|$, $\overline{\Bbbk} = \Bbbk$, then

$$(\chi_V, \chi_W) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Next time.

Ex S_3 again

	(1)	(12)	(123)
$\lvert C_G(g)\rvert$	1	3	2
χ_{triv}	1	1	1
χ_{sgn}	1	-1	1
χ_2	2	0	-1

← sizes of conjugacy classes

account for 3 elements conjugate to (12)

$$(\chi_{triv}, \chi_{sgn}) = \frac{1}{6}\left(1\cdot(1\cdot 1) + 3\cdot(1\cdot(-1)) + 2\cdot 1\cdot 1\right)$$
$$= 0$$

$$(\chi_2, \chi_2) = \frac{1}{6}\left(1\cdot 2^2 + 3\cdot 0^2 + 2\cdot(-1)^2\right)$$
$$= 1$$

Remark Every $\sigma \in S_n$ is conjugate to its inverse. So $\psi(\sigma) = \psi(\sigma^{-1})$ for any class function ψ on S_n.

EX

$D_4 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$

$srs^{-1} = r^{-1} = r^3$

$rsr^{-1} = sr^2$

$r(sr)r^{-1} = sr^{-1}rr^{-1} = sr^3$

$\{1\}, \{r^2\}$

$\{r, r^3\}$

$\{s, sr^2\}$

$\{sr, sr^3\}$

t=5. $1^2 + 2^2 + 1^2 + 1^2 + 1^2 = 8$

HW 10 Calculate the character for D_4.