MATH 618 SPRING 2021

LECTURE 1

Book: Lorenz, A Tour of Representation Theory.

MAIN TOPICS:

Representations of	(weeks)
- associative algebras.	4
- finite groups	8
- Lie algebras.	2

Homework 1 (Due Fri Jan 29)

- Read in book:

  A.1. (categories)

A.2. (functors)

B.1.1 (tensor products)

§1.1.1, §1.1.2

- Solve problem 1.1.11(b) from the book: Prove that
$$Z(A\otimes B)\cong Z(A)\otimes Z(B)$$
where $Z(A)=\{x\in A \mid xa=ax\ \forall a\in A\}$ is the center of an algebra A.

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① Tensor products of vector spaces. $\Bbbk$ = field

$V, W \rightsquigarrow V\otimes W$ or $V\underset{\Bbbk}{\otimes} W$

$V\otimes W$ is a vector space with a canonical map
$$V\times W \longrightarrow V\otimes W$$
$$(v,w)\longmapsto v\otimes w$$

Satisfying
$$(u+v)\otimes w = u\otimes w + v\otimes w$$

$(u+v)\otimes$

$$u\otimes(v+w) = u\otimes v + u\otimes w$$

$$(\lambda v)\otimes w = \lambda(v\otimes w) = v\otimes(\lambda w)$$

<u>Universal property</u>

If T is any vector space with a bilinear map

$$b : V\times W \longrightarrow T$$

then there is a unique linear map $b' : V\otimes W \to T$ such that

$$\begin{tikzcd}
V\times W \arrow[r, "\otimes"] \arrow[dr, "b"'] & V\otimes W \arrow[d, dashed, "b'"] \\
& T
\end{tikzcd}$$

commutes i.e.

$$b(v,w) = b'(v\otimes w).$$

Existence

F vector space with basis $V \times W$. Notation: $e_{v,w} \in F$ basis vector corresponding to (v,w).

$$R = \mathrm{Span}\left\{ \begin{array}{l} e_{u+v,w} - e_{u,w} - e_{v,w} \\ e_{u,v+w} - e_{u,v} - e_{u,w} \\ e_{\lambda v,w} - \lambda e_{v,w} \\ e_{v,\lambda w} - \lambda e_{v,w} \end{array} \right\}$$

$V \otimes W := F/R$ with canonical map defined by

$$(v,w) \longmapsto v \otimes w := e_{v,w} + R$$

(coset mod R)

Properties

Prop.

$$U\otimes(V\otimes W)\cong(U\otimes V)\otimes W$$

$$\tau: U\otimes V\cong V\otimes U$$
$$u\otimes v\mapsto v\otimes u$$

$$\Bbbk\otimes V\cong V\cong V\otimes\Bbbk$$
$$\lambda\otimes v\mapsto v\leftarrow v\otimes\lambda$$

$$(U\oplus V)\otimes W\cong(U\otimes W)\oplus(V\otimes W)$$

If $f:V\to V'$, $g:W\to W'$ are linear maps, then there is a linear map

$$f\otimes g: V\otimes W\longrightarrow V'\otimes W'$$
$$\sum_i v_i\otimes w_i\longmapsto\sum_i f(v_i)\otimes g(w_i)$$

Example Any linear $\xi: V\to\Bbbk$ determines a map

$$\ldots\otimes W\cong W$$

$\xi \otimes \mathrm{id} : V \otimes W \longrightarrow \Bbbk \otimes W \cong W$

so if $\{v_i\}$ basis for V

let $\xi^i \in V^*$, $\xi^i(v_j) = \delta_{ij}$

Then

$\xi^i \otimes \mathrm{id} : V \otimes W \to W$

$$\sum_j v_j \otimes w_j \longmapsto w_i$$

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② Algebras $A = (A, m, u)$

A v.sp. with linear maps

$m = m_A : A \otimes A \to A$

$u = u_A : \Bbbk \longrightarrow A$

such that:

$$
\begin{tikzcd}
A\otimes A\otimes A \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & A\otimes A \arrow[d] \\
A\otimes A \arrow[r] & A
\end{tikzcd}
$$

$$
\begin{tikzcd}
& A\otimes A \arrow[dd, "m"] & \\
\mathbb{K}\otimes A \arrow[ur, "u\otimes\mathrm{Id}"] \arrow[dr, "\simeq"'] \arrow[r, phantom, "\circlearrowleft"] & {} & A\otimes\mathbb{K} \arrow[ul, "\mathrm{Id}\otimes u"'] \arrow[dl, "\simeq"] \arrow[l, phantom, "\circlearrowleft"] \\
& A &
\end{tikzcd}
$$

A is <u>commutative</u> if

$$
\begin{tikzcd}
A\otimes A \arrow[rr, "\tau"] \arrow[dr, "m"'] \arrow[rr, phantom, shift right=4ex, "\circlearrowright"] & & A\otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}
$$

A homomorphism $f : A \to B$ is a linear map s.t.

$$
\begin{tikzcd}
A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\circlearrowleft"] & B \otimes B \arrow[d, "m_B"] \\
A \arrow[r, "f"] & B
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[rr, "f"] & \arrow[d, phantom, "\circlearrowleft"] & B \\
& \mathbb{k} \arrow[ul, "u_A"] \arrow[ur, "u_B"'] &
\end{tikzcd}
$$

$Alg_{\mathbb{k}}$ – category of $\mathbb{k}$-algebras

$\mathrm{CommAlg}_{\mathbb{k}}$ – category of <u>commutative</u> $\mathbb{k}$-algebras.

<u>Tensor</u> <u>product</u> of <u>algebras</u>

Let (A, m_A, u_A) and (B, m_B, u_B) be two algebras. Then the vector space $A\otimes B$ becomes an algebra by defining

$$m_{A\otimes B} : A\otimes B\otimes A\otimes B \longrightarrow A\otimes B$$

to be the composition

$$\begin{array}{c} A\otimes B\otimes A\otimes B \\ \Big\downarrow \mathrm{Id}_A\otimes\tau_{B,A}\otimes\mathrm{Id}_B \\ A\otimes A\otimes B\otimes B \end{array}$$

$$A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B$$

and the unit $u_{A \otimes B}$ to be the composition

$$\Bbbk \cong \Bbbk \otimes \Bbbk \xrightarrow{u_A \otimes u_B} A \otimes B$$

Explicitly, in terms of elements, the product in $A \otimes B$ is given by

$$\Big(\sum_i a_i \otimes b_i\Big)\Big(\sum_j a_j' \otimes b_j'\Big) =$$

$$= \sum a_i a_j' \otimes b_i b_j'$$

$$= \sum_{i,j} a_i a_j \cdots$$

and the unit element in $A \otimes B$ is given by

$$1_{A \otimes B} = 1_A \otimes 1_B .$$

(3) Examples of algebras.

- $End(V)$, $Mat_n(\mathbb{k})$
- $\mathbb{k}\langle x_1, \dots, x_n \rangle$ free algebra on $\{x_1, \dots, x_n\}$
- $\mathbb{k}[x_1, \dots, x_n]$ polynomial

algebra (= free commutative algebra on $\{x_1, \ldots, x_n\}$).